A discussion of 2–critical sets in Abelian 2-groups

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Abstract

In this paper we take the latin square which corresponds to the abelian 2-group of order 2^n and attempt to identify subsets which uniquely determine this square and are minimal with respect to this property. In particular we are interested in the size of such subsets. We give some general results and state some open questions. We also present a characterisation of the problem in terms of colourings of the complete bipartite graph $K_{2^n,2^n}$.

1 2-critical sets

In this paper we focus on the latin square which corresponds to the abelian 2-group of order 2^n. To this end we define a *latin square* L of order v to be a $v \times v$ array with entries chosen from the set $V = \{1, \ldots, v\}$ in such a way that each element of V occurs precisely once in each row and precisely once in each column of the array. The latin squares corresponding to the abelian 2-groups of order 2^1 and 2^2 are given below.

L^1 =

1	2
2	1

L^2 =

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

In general, we construct the latin square L^n, $n \geq 2$, recursively by taking the direct product of L^1 with a latin square L^{n-1}. To explain this construction formally we will use the notation L_1^{n-1} to denote a copy of L^{n-1} where symbol z, $1 \leq z \leq 2^{n-1}$, is relabelled $z + 2^{n-1}$. Then we construct L^n by

- replacing symbol 1 in cell $(1, 1)$ of L^1 by a copy of L^{n-1};
- replacing symbol 2 in cell $(1, 2)$ of L^1 by a copy of L_1^{n-1};
- replacing symbol 2 in cell $(2, 1)$ of L^1 by a copy of L_1^{n-1};

- replacing symbol 1 in cell $(2,2)$ of L^1 by a copy of L^{n-1}.

Thus we have constructed a $2^n \times 2^n$ array which forms a latin square in which each of the symbols $1, \ldots, 2^n$ occurs precisely once in each row and once in each column. The array can be partitioned as follows.

L^{n-1}	L_1^{n-1}
L_1^{n-1}	L^{n-1}

It will be useful to number the rows and columns of a latin square and then used this notation to identify certain subarrays in the latin square. So at times we will adjoin a head line and side line to the latin square and the entries in this additional row and column will denote, respectively, the column number and the row number. So for L^1 and L^2 above we have:

L^1 :

	1	2
1	1	2
2	2	1

L^2 :

	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	1	2
4	4	3	2	1

Fix row r (column c) of L^n and assume that symbol z occurs in cell (r,j), for some j. For all symbols $z' \in \{1, \ldots, 2^n\} \setminus \{z\}$ there exists $j' \neq j$ such that symbol z' occurs in cell (r,j') of L^n. Further, for some $r' \neq r$ symbol z' occurs in cell (r',j) and symbol z occurs in cell (r',j') of L^n. So in rows r and r' and columns j and j' we have:

	j	j'
r	z	z'
r'	z'	z

This set of four cells $(r,j), (r,j'), (r',j)$ and (r',j') in L^n constitutes a copy of L^1 based on the set of symbols $\{z, z'\}$. Such a latin subsquare is termed an *intercalate*. Thus for each entry of the latin square L^n there exists $2^n - 1$ intercalates which contain that entry.

We are interested in identifying a subset of entries of L^n which uniquely determines L^n and is minimal with respect to this property.

So we define a *partial latin square* P of order v to be a $v \times v$ array with entries chosen from the set $V = \{1, \ldots, v\}$ in such a way that each element of V occurs at most once in each row and at most once in each column of the array. The *size* of the partial latin square is the number of non–empty cells in P. We are specifically interested in identifying partial latin squares which are contained in precisely one latin square, L^n. In addition we require the property that when any entry of the partial latin square is removed what remains is contained in at least two latin squares, L^n and M, where L^n and M differ in an intercalate. Such partial latin squares are termed 2–*critical sets*.

Stinson and van Rees **[2]** were the first to formally identify examples of 2-critical sets. They showed that the partial latin square P^2 given below is contained in precisely one latin square of order 4, that is, in L^2. In addition they showed that if any entry is removed then the reduced partial latin square is contained in at least two latin squares L^2 and M, where L^2 and M differ in an intercalate. Note that the partial latin square P^1 given below is a trivial 2-critical set in L^1.

$P^1 =$

1	

$P^2 =$

1	2	3	
2	1		
3		1	

Note that P^2 is a partial latin square of order 4 and size 7. Stinson and van Rees **[2]** were able to generalise this construction. To explain their construction we introduce the notation P_1^{n-1} to represent a copy of P^{n-1}, where each symbol $z \in \{1, \ldots, 2^{n-1}\}$ has been relabelled $z + 2^{n-1}$. The partial latin square P^n, $n \geq 2$, is obtained recursively by taking L^1 and

- replacing symbol 1 in cell $(1, 1)$ of L^1 by a copy of L^{n-1};
- replacing symbol 2 in cell $(1, 2)$ of L^1 by a copy of P_1^{n-1};
- replacing symbol 2 in cell $(2, 1)$ of L^1 by a copy of P_1^{n-1};
- replacing symbol 1 in cell $(2, 2)$ of L^1 by a copy of P^{n-1}.

Thus we have constructed a $2^n \times 2^n$ array which forms a partial latin square where each of the symbols $1, \ldots, 2^n$ occurs at most once in every row and at most once in every column. The array can be partitioned as follows.

L^{n-1}	P_1^{n-1}
P_1^{n-1}	P^{n-1}

Since the cells corresponding to the intersection of rows 1 to 2^{n-1} with columns 1 to 2^{n-1} contain a complete copy of L^{n-1}, and since P^1 is contained in the unique latin square L^1, one can use induction to prove that the partial latin square P^n is contained in a unique latin square L^n. In addition, Stinson and van Rees showed that if any entry of P^n is removed, the reduced partial latin square is contained in a latin square of order 2^n which differs from L^n in an intercalate.

Stinson and van Rees work raises many question. For instance:

Question 1: If P is a 2–critical set contained in L^n, what is the size of P?

Using the Stinson and van Rees construction we can see that it is possible to construct a 2-critical set P^n of size $4^n - 3^n$.

Question 2: Does there exist a 2–critical set P in L^n, of order 2^n, where the size of P is greater than $4^n - 3^n$?

Since L^n can be partitioned into 2^{2n-2} intercalates and any 2-critical set must intersect every intercalate in L^n, we see that all 2-critical sets must be of size greater than or equal to $2^{2n-2} = 4^{n-1}$.

Question 3: For what values $t \geq 2^{2n-2}$ does there exist a 2–critical set P of size t, in the latin square L^n, of order 2^n?

2 A graphical representation

It is interesting to note that the above combinatorial configuration may be represented as an edge colouring of the complete bipartite graph $K_{v,v}$. We present the details of this representation in the hope that it may shed some light on the above questions.

A latin square L may be represented as a v–colouring of the edges of the complete bipartite graph $K_{v,v}$. That is, if $V_1 \cup V_2$ represents the vertex set of $K_{v,v}$ where $V_1 = \{r_1, \ldots, r_v\}$ and $V_2 = \{c_1, \ldots, c_v\}$, then for $1 \leq i, j \leq v$, the edge $\{r_i, c_j\}$ is coloured e_k, $1 \leq k \leq v$, if symbol k occurs in cell (i, j) of L.

The latin square L^1 corresponds to bipartite graph $K_{2,2}$, where $V_1 = \{r_1, r_2\}$ and $V_2 = \{c_1, c_2\}$. Further, edges $\{r_1, c_1\}$ and $\{r_2, c_2\}$ are coloured e_1 and edges $\{r_1, c_2\}$ and $\{r_2, c_1\}$ are coloured e_2. Denote this graph by GL^1.

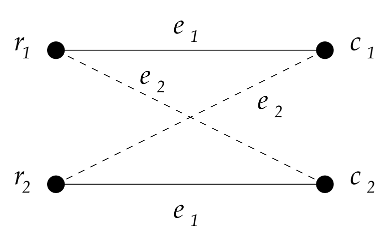

Figure 1: The graph GL^1

For $n \geq 2$, the bipartite graph GL^n corresponding to L^n is obtained from GL^1 by:

- replacing vertices r_1 and c_1 by the sets of vertices $\{r_1, \ldots, r_{2^{n-1}}\}$ and $\{c_1, \ldots, c_{2^{n-1}}\}$ respectively, and by replacing vertices r_2 and c_2 by the sets of vertices $\{r_{2^{n-1}+1}, \ldots, r_{2^n}\}$ and $\{c_{2^{n-1}+1}, \ldots, c_{2^n}\}$ respectively;

- replacing edge $\{r_1, c_1\}$ in GL^1 by a copy of GL^{n-1} on vertex set $\{r_1, \ldots, r_{2^{n-1}}\} \cup \{c_1, \ldots, c_{2^{n-1}}\}$ and using colours $e_1, \ldots, e_{2^{n-1}}$;

- replacing edge $\{r_2, c_2\}$ in GL^1 by a copy of GL^{n-1} on vertex set $\{r_{2^{n-1}+1}, \ldots, r_{2^n}\} \cup \{c_{2^{n-1}+1}, \ldots, c_{2^n}\}$ and using colours $e_1, \ldots, e_{2^{n-1}}$;

- replacing edge $\{r_1, c_2\}$ in GL^1 by a copy of GL^{n-1} on vertex set $\{r_1, \ldots, r_{2^{n-1}}\} \cup \{c_{2^{n-1}+1}, \ldots, c_{2^n}\}$ and using colours $e_{2^{n-1}+1}, \ldots, e_{2^n}$;

- replacing edge $\{r_2, c_1\}$ of GL^1 by a copy of GL^{n-1} on vertex set $\{r_{2^{n-1}+1}, \ldots, r_{2^n}\} \cup \{c_1, \ldots, c_{2^{n-1}}\}$ and using colours $e_{2^{n-1}+1}, \ldots, e_{2^n}$.

The bipartite graph GP^2 corresponding to P^2 is a subgraph of GL^2 and can be obtained from GL^1 by:

- replacing each of the vertices r_1 and c_1 by the sets of vertices $\{r_1, r_2\}$ and $\{c_1, c_2\}$, respectively, and replacing each of the vertices r_2 and c_2 by the sets of vertices $\{r_3, r_4\}$ and $\{c_3, c_4\}$, respectively;

- the edge $\{r_1, c_1\}$ in GL^1 is replaced by a copy of GL^1 on the vertex set $\{r_1, r_2\} \cup \{c_1, c_2\}$ using colours e_1 and e_2; and

- the edges $\{r_1, c_2\}$, $\{r_2, c_1\}$ and $\{r_2, c_2\}$ of GL^1 are replaced by the edges $\{r_1, c_3\}$, $\{r_3, c_1\}$, and $\{r_3, c_3\}$, coloured e_3, e_3 and e_1 respectively.

Roughly speaking, for $n \geq 3$, GP^n can be obtained from GL^1 by replacing each vertex by 2^{n-1} new vertices and replacing edge $\{r_1, c_1\}$ by a copy of GL^{n-1} using colours $e_1, \ldots, e_{2^{n-1}}$, and replacing edges $\{r_1, c_2\}$, $\{r_2, c_1\}$ and $\{r_2, c_2\}$ by copies of GP^{n-1} using colours $e_{2^{n-1}+1}, \ldots, e_{2^n}$, $e_{2^{n-1}+1}, \ldots, e_{2^n}$ and $e_1, \ldots, e_{2^{n-1}}$, respectively.

We know that the subgraph GP^n can be completed to precisely one edge colouring of the complete bipartite graph $K_{2^n,2^n}$, namely GL^n, and if we remove any edge $\{r_x, c_y\}$ coloured e_z the reduced subgraph can be completed to at least two distinct edge colourings of the bipartite graph $K_{2^n,2^n}$. That is, for some $r_{x'}, c_{y'}, e_{z'}$, the edges $\{r_x, c_y\}$ and $\{r_{x'}, c_{y'}\}$ of GL^n are coloured e_z while edges $\{r_x, c_{y'}\}$ and $\{r_{x'}, c_y\}$ of GL^n are coloured $e_{z'}$ and in the second colouring these colours have been reversed.

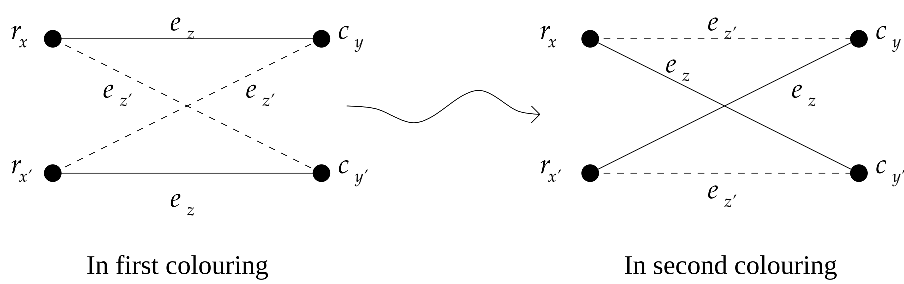

Figure 2

In this context Questions 1, 2, and 3 correspond to finding subgraphs of GL^n which fix GL^n and are minimal with respect to this property.

3 Theoretical results

The Stinson and van Rees result **[2]** proved that, for $n \geq 1$, L^n contains a 2-critical set of size $4^n - 3^n$. In this section we explore the other possible sizes of 2-critical sets in L^n and establish some partial results.

Using the computational results given in Section 4 we see that there exists a 2-critical set of size 6 in the latin square L^2. Combining this with Stinson and van Rees original method of construction we obtain the following result.

LEMMA 1 *For $n \geq 2$, let L^n be the latin square of order 2^n as defined in Section 1. Then L^n contains a 2-critical set of size $4^n - 3^n - 3^{n-2}$.*

Proof: Let Q^2 be the partial latin square:

1		3	
2	1		
3			2

Construct a partial latin square Q^n, $n \geq 3$ recursively by taking L^1 and

- replacing symbol 1 in cell $(1,1)$ of L^1 by a copy of L^{n-1},
- replacing symbol 2 in cell $(1,2)$ of L^1 by a copy of Q_1^{n-1},
- replacing symbol 2 in cell $(2,1)$ of L^1 by a copy of Q_1^{n-1}, and
- replacing symbol 1 in cell $(2,2)$ of L^1 by a copy of Q^{n-1}.

Then this array can be partitioned as follows (note Q_1^{n-1} is defined appropriately).

L^{n-1}	Q_1^{n-1}
Q_1^{n-1}	Q^{n-1}

Since the cells corresponding to the intersection of rows 1 to 2^{n-1} with columns 1 to 2^{n-1} contain a complete copy of L^{n-1}, and since P^1, as defined in Section 1, is contained in the unique latin square L^1, one can use induction to prove that the partial latin square Q^n is contained in a unique latin square L^n. One can also use induction to prove if we remove any entry from Q^n there are at least two completions which differ in an intercalate. Hence Q^n is a 2-critical set of size $4^n - 3^n - 3^{n-2}$.

LEMMA 2 *For $n \geq 2$, let L^n be the latin square of order 2^n as defined in Section 1. Then L^n contains a 2–critical set of size $4^n - 3^n - 2^{n-1} + 1$.*

Proof: The partial latin square P^n, as defined in Section 1, is a 2-critical set of size $4^n - 3^n$. Let Q^n be a partial latin square obtained from P^n by removing the symbol $2m+1$ from cell $(1, 2m+1)$, for $m = 0, \ldots, 2^{n-1} - 1$, and inserting the symbol 2^n in cell $(1, 2^n)$. Thus all odd numbers have been removed from the first row and the symbol 2^n has been inserted in the last cell of the first row. We will prove that Q^n is a 2-critical set. First we shall show that L^n is the only latin square which contains Q^n. Second we shall show that for each entry x in Q^n there exists an intercalate contained in L^n which intersects Q^n in the entry x alone.

Let c be odd and $1 \leq c \leq 2^n$. We see that each odd number $2m+1$, where $0 \leq m \leq 2^{n-1}-1$, occurs in column c of P^n. So the only odd number which does not occur in column c of Q^n is symbol c. In addition, all of the even numbers occur in row 1. Thus when completing Q^n we must place symbol $2m+1$, $0 \leq m \leq 2^{n-1}-1$ in cell $(1, 2m+1)$. Hence we obtain a superset of P^n which is contained in precisely one latin square, L^n. Consequently Q^n has a unique completion to L^n.

Now we see that if symbol 2^n is removed from cell $(1, 2^n)$ of Q^n, then the reduced partial latin square has two completions and these two completions differ in the subarrays shown below:

	2^n-1	2^n
1	2^n-1	2^n
2	2^n	2^n-1

Subarray in L^n

	2^n-1	2^n
1	2^n	2^n-1
2	2^n-1	2^n

Corresponding subarray in 2nd completion

In addition, for $1 \leq m \leq 2^{n-1}$, symbol $2m$ does not occur in column 2^n-1 of Q^n and symbol 2^n-1 does not occur in column $2m$. It is now immediate that if any symbol of the form $2m$, $1 \leq m \leq 2^{n-1}$, is removed from row 1, there will be two completions. These two completions differ in the subarray indicated below.

	$2m$	2^n-1
1	$2m$	2^n-1
2^n-2m+2	2^n-1	$2m$

Subarray in L^n

	$2m$	2^n-1
1	2^n-1	$2m$
2^n-2m+2	$2m$	2^n-1

Corresponding subarray in the 2nd completion

The fact that Q^n is 2-critical now follows from the Stinson and van Rees result.

From row one of P^n we have removed 2^{n-1} entries and added one entry. Thus the size of Q^n is $4^n - 3^n - 2^{n-1} + 1$.

Combining this with Stinson and van Rees original method of construction we obtain the following corollary.

COROLLARY 3 *For $n \geq 2$, let L^n be the latin square of order 2^n as defined in Section 1. Then L^n contains a 2–critical set of size $4^n - 3^n - 3.2^{n-2} + 3$.*

This result can be generalised as follows.

LEMMA 4 *For $n \geq 2$ let L^n be the latin square of order 2^n as defined in Section 1. Then L^n contains a 2–critical set of size $4^n - 3^n - 2^{n-i-1} - 2^{i-1} + 1$, where $1 \leq i \leq n-1$.*

Proof: The partial latin square P^n, as defined in Section 1, is a 2-critical set of size $4^n - 3^n$. Fix i, $1 \leq i \leq n-1$. Construct a partial latin square R^n from P^n by adding symbol 2^i in cell $(2^n - 2^i + 1, 2^n)$, removing the symbol $2m+1$ from cells $(2^n - 2^i + 1, 2^n - 2^i + 2m + 1)$, $m = 0, \ldots, 2^{i-1}-1$, and removing the symbol 2^i from cells $(1 + \alpha 2^{i+1}, 2^i + \alpha 2^{i+1})$, for $0 \leq \alpha \leq 2^{n-i-1}-1$. So we have removed 2^{i-1} odd numbers from row $2^n - 2^i + 1$. These symbols occurred in columns $2^n - 2^i + 1$ to 2^n. In addition from 2^{n-i-1} cells we have removed the symbol 2^i, but at the same time we have added symbol 2^i in column 2^n. Once again we will show that L^n is the only latin square of order 2^n containing R^n and that for each entry x in R^n there exists an intercalate contained in L^n which intersects R^n in the entry x alone.

For $0 \leq \alpha \leq 2^{n-i-1}-1$, in row $1+\alpha 2^{i+1}$ of R^n if cell $(1+\alpha 2^{i+1}, j)$ is empty, then $j = 2^i + \alpha 2^{i+1}$ or column j already contains symbol 2^i. Hence in any completion of R^n, symbol 2^i must be placed in cell $(1+\alpha 2^{i+1}, 2^i + \alpha 2^{i+1})$ for $0 \leq \alpha \leq 2^{n-i-1} - 1$. Once this is done we note that the intersection of rows $2^n - 2^i + 1$ to 2^n with columns $2^n - 2^i + 1$ to 2^n contains a copy of the partial latin subsquare Q^i, as defined in the proof of Lemma 2. And once cells $(1+\alpha 2^{i+1}, 2^i + \alpha 2^{i+1})$ for $0 \leq \alpha \leq 2^{n-i-1} - 1$ have been filled, each of the symbols $1, \ldots, 2^i$ occurs once in row r, $1 \leq r \leq 2^n - 2^i$, and once in column c, $1 \leq c \leq 2^n - 2^i$. Hence all cells in the intersection of rows $2^n - 2^i + 1$ to 2^n with columns $2^n - 2^i + 1$ to 2^n must contain symbols chosen from the set $\{1, \ldots, 2^i\}$. By Lemma 2 these cells must contain a copy of L^i. This gives a superset of P^n which has unique completion to L^n and so R^n has unique completion to L^n.

The fact that R^n is a 2-critical set follows, in the main part, from the Stinson and van Rees result and Lemma 2. However we do have to check that for each occurrence of symbol 2^i in cells $(2^i + \alpha 2^{i+1} + 1, 2^{i+1} + \alpha 2^{i+1})$, for $0 \leq \alpha \leq 2^{n-i-1} - 1$ there exists an intercalate which intersects R^n in that cell alone. But we note that, for $0 \leq \alpha \leq 2^{n-i-1} - 1$, in L^n the subsquare

	$2^{i+1} + \alpha 2^{i+1}$	$2^n - 2^i$
$2^i + \alpha 2^{i+1} + 1$	2^i	$2^n - \alpha 2^{i+1}$
$2^n - 2^{i+1} + 1$	$2^n - \alpha 2^{i+1}$	2^i

contains at most one entry from R^n and that entry is symbol 2^i in the cell $(2^i + \alpha 2^{i+1} + 1, 2^{i+1} + \alpha 2^{i+1})$. Hence if this entry is removed from R^n there are at least two completions, L^n and a latin square which differs from L^n in the subsquare:

	$2^{i+1} + \alpha 2^{i+1}$	$2^n - 2^i$
$2^i + \alpha 2^{i+1} + 1$	$2^n - \alpha 2^{i+1}$	2^i
$2^n - 2^{i+1} + 1$	2^i	$2^n - \alpha 2^{i+1}$

We have removed 2^{i-1} entries from row $2^n - 2^i + 1$, 2^{n-i-1} entries containing the symbol 2^i and added one entry containing symbol 2^i. Thus the size of R^n is $4^n - 3^n - 2^{n-i-1} - 2^{i-1} + 1$, where $1 \leq i \leq n-1$.

Combining this with Stinson and van Rees original method of construction we obtain the following corollary.

COROLLARY 5 *For $n \geq 2$, let L^n be the latin square of order 2^n as defined in Section 1. Then L^n contains a 2-critical set of size $4^n - 3^n - 3.(2^{n-i-2} - 2^{i-1}) + 3$, where $1 \leq i \leq n-2$.*

4 Computational results

We give some partial answers to Questions 1, 2 and 3 for small values of n. In particular, we focus on searching for 2-critical sets of order 2^n and sizes between 4^{n-1} and $4^n - 3^n$.

For $n = 2$, $4^{n-1} = 4$ and $4^n - 3^n = 7$. It is well-known that the size of any critical set contained in L^2 must have at least 5 filled cells. Hence any 2-critical set must contain at least 5 entries. Exhaustive searches verify that there do not exist 2-critical sets of size 5 in L^2. However they do exist for sizes 7 (7 is obtained from the Stinson and van Rees work) and 6 .

1	2	3	
2	1		
3		1	

Size 7

1		3	
2	1		
3			2

Size 6

For $n = 3$, $4^{n-1} = 16$ and $4^n - 3^n = 37$. In [**1**] it was shown that the size of any critical set contained in L^3 must have at least 25 filled cells. Hence any 2-critical set must contain at least 25 entries. To date we have been able to construct 2-critical sets of sizes $37, 35, 34, \ldots, 27$ and 26. (Note size 37 is obtained from the Stinson and van Rees work.)

1	2	3	4	5	6	7	
2	1	4	3	6	5		
3	4	1	2	7		5	
4	3	2	1				
5	6	7		1	2	3	
6	5			2	1		
7		5		3		1	

Size 37

1	2	3	4	5		7	
2	1	4	3	6	5		
3	4		2	7			6
4	3	2	1				
5	6	7		1	2	3	
6	5			2	1		
7		5		3		1	

Size 35

1	2	3	4	5		7	
2	1	4	3	6	5		
3	4		2	7			6
4	3	2	1				
5	6	7		1	2	3	
	5			2	1		
7				3		1	
		6					

Size 34

1		3	4	5		7	
2	1	4	3	6	5		
3	4		2	7			6
4	3	2	1				
5	6	7		1		3	
6	5			2	1		
7		5		3			2

Size 33

1		3	4	5		7	
2	1	4	3	6	5		
3	4	1	2	7			6
4	3		1				
5	6	7		1		3	
	5			2	1		
7				3			2
		6					

Size 32

1	2		4		6		8
2	1	4	3	6	5		
	4		2	7		5	
4	3	2	1				
	6	7			2	3	
6	5			2	1		
		5		3			
8							1

Size 31

	2		4		6		8
2	1		3	6	5		
	4	1	2	7		5	
4	3	2	1				
	6	7		1	2		
6	5				1	4	
		5		3		1	
8							

Size 30

	2		4		6		8
2	1	4	3	6	5		
	4	1			8	5	
4	3	2	1				
			8	1		3	
6	5			2	1		
7				3			2
		6					

Size 29

	2		4		6		8
2	1		3	6			
	4	1	2			5	
4		2	1		7		
	6	7		1	2		
6	5				1	4	
		5		3		1	
8							

Size 28

1	2		4		6		8
2	1		3	6			
	4		2			5	
4		2	1		7		
	6	7			2		
6	5				1	4	
		5		3			
8							1

Size 27

				5	6	7	8
	1		3	6			
	4	1	2			5	
4		2	1		7		
	6	7			2		
6	5					4	
		5		3			2
8							1

Size 26

References

[1] A. Khodkar, *On smallest critical sets for the elementary abelian* 2*-group*, Utilitas Mathematica **54** (1998), 45–50.

[2] D.R. Stinson and G.H.J. van Rees, *Some large critical sets*, Congressus Numerantium, **34**, 1982, 441–456.