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# NUMBERS OF COMMON WEIGHTS FOR EXTENDED TRIPLE SYSTEMS

BY

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**Abstract**. Let $K_v$ be the complete graph on $v$ vertices and $K_v^+$ the graph obtained by attaching a loop to each vertex of $K_v$. An extended triple system of order $v$ is a pair $(V, B)$, where $V$ is a $v$-set and $B$ is a collection of non-ordered triples of elements in $V$ (each triple may have repeated elements), such that every pair of elements of $V$ (not necessarily distinct) belongs to exactly one triple. It has been established that an extended triple system of order $v$ corresponds to a decomposition of edges of $K_v^+$ into triangles, lollipops, and loops. In this paper the decomposition of $K_v^+$ is used to construct two extended triple systems of order $v$ with each prescribed intersection numbers in the following set:
(1) $\{0, 1, 2, \ldots, m-6, m-5, m-3, m\}$, for even $v \geq 8$, and
(2) $\{0, 1, 2, \ldots, m-11, m-10, m-8, m-6, m\}$, for odd $v \geq 11$, where $m = v(v+1)/2$.

## 1. Introduction

The concept of an extended triple system was introduced by D. M. Johnson and N. S. Mendelsohn [4]. An extended triple system of order $v$ (ETS($v$)) is a pair $(V, B)$, where $V$ is a $v$-set and $B$ is a collection of non-ordered triples of elements in $V$ (each triple may have repeated elements), such that every pair of elements of $V$ (not necessarily distinct) belongs to exactly one triple. An element of $B$ is called a block. There are three types of blocks: (1) $\{x, x, x\}$ (2) $\{y, y, z\}$ (3) $\{a, b, c\}$ (we write the blocks as $xxx$, $yyz$ and $abc$ for brevity).

We want to characterize extended triple systems similar to Steiner triple systems (which are equivelent to a $C_3$-decompositions of the complete graph) by graph decompositions. Just of all we will introduce some graph terminology.

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A loop is an edge whose two ends are the same. A link is an edge whose two ends are distinct. A graph on two vertices which consists of one loop and one link is called a lollipop. As usual, $K_v$ denotes the complete graph on $v$ vertices, while $K_v^+$ denotes the graph obtained by attaching a loop to each vertex of $K_v$. $K_3$ is called a triangle. Thus an extended triple system of order $v$ can be regarded as a partition of the edges of $K_v^+$ into triangles, lollipops, and loops.

The necessary and sufficient conditions for the existence of an extended triple system of order $v$ with no idempotent is $v \equiv 0 \pmod 3$. This design has $s_v = v(v+3)/6$ blocks. Lo Faro [7] constructed two non-idempotent extended triple systems of order $v$ with intersection numbers in following set: (i) $\{0, 1, \ldots, s_v - 3, s_v\}$ for $v \equiv 0 \pmod 3$, $v \neq 9$ and (ii) $\{0, 1, 2, \ldots, 12, 14, 15, 18\}$ for $v = 9$.

For extended triple systems, we define an intersection in the following sense.

Let $(V, T_1)$ and $(V, T_2)$ be two ETSs with the same point set $V$. The two systems have a common weight $k$ if $k = \sum \omega(B)$, where $B$ is a common block in $T_1$ and $T_2$, and

$$\omega(B) = \begin{cases} 1 & \text{if } B = xxx \\ 2 & \text{if } B = yyz \\ 3 & \text{if } B = abc\ . \end{cases}$$

In this case, we write $|T_1 \cap T_2| = k$.

Let $J[v]$ be the set of all integers $k$ such that there exists a pair of extended triple systems of order $v$ which have a common weight $k$. Let $I_e[v] = \{0, 1, 2, \ldots, m-6, m-5, m-3, m\}$ and $I_o[v] = \{0, 1, 2, \ldots, m-11, m-10, m-8, m-6, m\}$, where $m = v(v+1)/2$.

**Main Theorem.** *For even $v$, $J[v] = I_e[v]$ if $v \geq 8$; and for odd $v$, $J[v] = I_o[v]$ if $v \geq 11$.*

Let $A$ and $B$ be two sets of integers and $k$ a positive integer. We define $A + B = \{a + b \mid a \in A, b \in B\}$, $k + A = \{k\} + A$, and $kA = \{k \cdot a | a \in A\}$.

## 2. Auxiliary Constructions of ETS

In order to count the common weights, we need some special embedding constructions. Therefore, let $(V_1, B)$ be an ETS$(v)$, where $V_1 = \{a_1, a_2, \ldots, a_v\}$.

**(1) $v$ to $2v$, $v$ even**

Let $\mathcal{F} = \{F_i \mid i = 1, 2, \ldots, v-1\}$ be a 1-factorization of $K_v$ on $V_2 = \{x_1, x_2, \ldots, x_v\}$. Let $S = V_1 \cup V_2$ and $T = B \cup C \cup D$, where $C = \{a_i xy \mid xy \in F_i, i = 1, 2, \ldots, v-1\}$ and $D = \{a_v xx \mid \text{for each } x \in V_2\}$. Then $(S, T)$ is an ETS$(2v)$ denoted by $(V_1 \cup V_2, (B, \mathcal{F}))$.

**(2) $v$ to $2v+2$, $v$ even**

Let $\mathcal{F} = \{F_i \mid i = 1, 2, \ldots, v+1\}$ be a 1-factorization of $K_{v+2}$ on $V_2 = \{x_1, x_2, \ldots, x_{v+2}\}$. Let $S = V_1 \cup V_2$ and $T = B \cup C \cup D$, where $C = \{a_i xy \mid xy \in F_i, i = 1, 2, \ldots, v\}$ and $D = \{xxy, yyy \mid \text{for each } xy \in F_{v+1}\}$. Then $(S, T)$ is an ETS$(2v+2)$.

Let $\mathcal{F} = \{F_i \mid i = 1, 2, \ldots, 2v-1\}$ be a 1-factorization of $K_{2v}$ on $N = \{1, 2, \ldots, 2v\}$. If $F_a, F_b \in \mathcal{F}$, the notation $F_a \cdot F_b$ ([7]) will denote the following set of blocks: $\{11x_{i_2},\ x_{i_2}x_{i_2}x_{i_3},\ \ldots,\ x_{i_r}x_{i_r}1\} \cup \{x_{j_1}x_{j_1}x_{j_2},\ x_{j_2}x_{j_2}x_{j_3},\ \ldots,\ x_{j_s}x_{j_s}x_{j_1}\} \cup \ldots \cup \{x_{p_1}x_{p_1}x_{p_2},\ x_{p_2}x_{p_2}x_{p_3},\ \ldots,\ x_{p_t}x_{p_t}x_{p_1}\} \cup \{x_{q_1}x_{q_1}x_{q_2},\ x_{q_2}x_{q_2}x_{q_3},\ \ldots,\ x_{q_m}x_{q_m}x_{q_1}\}$ where $x_{j_1} = min(N \backslash \{1, x_{i_2}, x_{i_3}, \ldots, x_{i_r}\}), \ldots, x_{q_1} = min(N \backslash \{1, x_{i_2}, x_{i_3}, \ldots, x_{i_r}, x_{j_1}, x_{j_2}, x_{j_3}, \ldots, x_{j_s}, \ldots, x_{p_1}, x_{p_2}, x_{p_3}, \ldots, x_{p_t}\})$; $F_a = \{1x_{i_2},\ x_{i_3}x_{i_4},\ \ldots,\ x_{i_{r-1}}x_{i_r},\ x_{j_1}x_{j_2},\ x_{j_3}x_{j_4},\ \ldots,\ x_{j_{s-1}}x_{j_s},\ \ldots,\ x_{p_1}x_{p_2},\ x_{p_3}x_{p_4},\ \ldots,\ x_{p_{t-1}}x_{p_t},\ x_{q_1}x_{q_2},\ x_{q_3}x_{q_4},\ \ldots,\ x_{q_{m-1}}x_{q_m}\}$ and $F_b = \{x_{i_2}x_{i_3},\ x_{i_4}x_{i_5},\ \ldots,\ x_{i_r}1,\ x_{j_2}x_{j_3},\ x_{j_4}x_{j_5},\ \ldots,\ x_{j_s}x_{j_1},\ \ldots,\ x_{p_2}x_{p_3},\ x_{p_4}x_{p_5},\ \ldots,\ x_{p_t}x_{p_1},\ x_{q_2}x_{q_3},\ x_{q_4}x_{q_5},\ \ldots,\ x_{q_m}x_{q_1}\}$.

**(3) $v$ to $2v+3$, $v$ odd**

Let $\mathcal{F} = \{F_i \mid i = 1, 2, \ldots, v+2\}$ be a 1-factorization of $K_{v+3}$ on $V_2 = \{x_1, x_2, \ldots, x_{v+3}\}$. Let $S = V_1 \cup V_2$ and $T = B \cup C \cup D$, where $C = \{a_i xy \mid xy \in F_i, i = 1, 2, \ldots, v\}$ and $D = F_{v+1} \cdot F_{v+2}$. Then $(S, T)$ is an ETS$(2v+3)$.

**(4) $v$ to $2v+1$, $v$ odd**

Let $\mathcal{F} = \{F_i \mid i = 1,\ 2,\ \ldots, v\}$ be a 1-factorization of $K_{v+1}$ on $V_2 = \{x_1,\ x_2,\ \ldots,\ x_{v+1}\}$. Let $S = V_1 \cup V_2$ and $T = B \cup C \cup D$, where $C = \{a_i xy \mid xy \in F_i,\ i = 1,\ 2,\ \ldots,\ v\}$ and $D = \{x_i x_i x_i \mid i = 1,\ 2,\ \cdots,\ v+1\}$. Then $(S, T)$ is an ETS$(2v+1)$.

Let $K_{2v}$ be a complete graph on $2v$ vertices$(2v \geq 8)$. The edges of $K_{2v}$ fall into $v$ disjoint classes $P_1,\ P_2,\ \ldots,\ P_v$ with $\{i, k\} \in P_j$ if and only if $i - k \equiv j$ mod$(2v)$. R. G. Stanton and I. P. Goulden [8] have proved that:

**P1.** If $2x + 1 < v$ then $P_{2x} \cup P_{2x+1}$ splits into four one-factors.

**P2.** The graph $K_{2v}$ may be factored into a set of $2v$ triangles covering $P_1$, $P_{2j}$, $P_{2j+1}$ $(2j+1<v)$ and a set of $2v-7$ one factors covering the other $P_i$.

**(5) $v$ to $2v+9$, $v$ odd**

Using the above description we factor the complete graph $K_{v+9}$ on $V_2 = \{x_1, x_2, \ldots, x_{v+9}\}$. Let $L$ be the set of $v+9$ triangles and $\mathcal{F} = \{F_i \mid i = 1, 2, \ldots, v+2\}$ the set of 1-factors. Put $S = V_1 \cup V_2$ and $T = B \cup L \cup C \cup D$, where $C = \{a_i xy \mid xy \in F_i, i = 1, 2, \ldots, v\}$ and $D = F_{v+1} \cdot F_{v+2}$. Then $(S, T)$ is an ETS$(2v+9)$.

## 3. Proof of the Main Theorem for Even $v$

It has been established that the class of all extended triple systems is coextensive with the variety of quasigroups satisfying the identities $x(xy) = y$, $(yx)x = y$ (It is called a totally symmetric quasigroup). In 1980, Hilton and Rodger [2] proved that if $v$ is odd then those lollipops ignoring loops form a vertex-disjoint union of cycles, and if $v$ is even, they form a vertex-disjoint union of unicycles with trees each of whose degree is odd.

The smallest possible mutually balanced subgraphs of $K_v^+$ are $\{xxx,\ xyy\}$ or $\{xxx, xyy, yzz\}$, (which can be changed to $\{yyy, yxx\}$ or $\{xxy,\ yyz,\ zzz\}$, respectively), $J[v] \subseteq I_e[v] = \{0,\ 1,\ 2,\ \ldots,\ m-6,\ m-5,\ m-3,\ m\}$, where $m = v(v+1)/2$.

Using exhaustive computer checking for $v = 2$, 4 and 6 the following results were obtained:
$J[2] = \{0, 3\}$,
$J[4] = \{0, 1, 2, 3, 5, 7, 10\}$,
$J[6] = \{0, 1, 2, \ldots, 14, 15, 18, 21\}$.

When we talk about the intersection of two one-factorizations of $K_v$, where $v$ is even, we have to order the one-factors and consider their intersections. Let $\mathcal{F} = \{F_1, F_2, \ldots, F_{v-1}\}$ and $\mathcal{G} = \{G_1, G_2, \ldots, G_{v-1}\}$ be two one-factorizations of $K_v$, where the $F_i$ and $G_i$ are one-factors, we define

$$|\mathcal{F} \cap \mathcal{G}| = \sum_{i=1}^{v-1} |F_i \cap G_i|, \text{ where } F_i \in \mathcal{F} \text{ and } G_i \in \mathcal{G}.$$

Let $J_F(v)$ be a set of $k$ such that there exist pairs of 1-factorizations of $K_v$ having $k$ common edges. In [6], C. C. Lindner and W. D. Wallis showed that $J_F(2) = \{1\}$, $J_F(6) = \{0,1,2,3,5,6,7,9,15\}$ and $J_F(v) = \{0,1,2,\ldots,\binom{v}{2} = t\} \setminus \{t-1, t-2, t-3, t-5\}$ for $v = 4$ or $v \geq 8$.

**Lemma 3.1.** *If $v$ is even, $v \geq 8$, and $J[v] = I_e[v]$ then $J[2v] = I_e[2v]$.*

**Proof.** Let $(V_1, B_1)$ and $(V_1, B_2)$ be two ETS$(v)$ which have a common weight $k$. Also let $\mathcal{F}$ and $\mathcal{G}$ be two 1-factorizations of $K_v$ on $V_2 = \{x_1, x_2, \ldots, x_v\}$ such that $h = \sum_{i=1}^{v-1} |F_i \cap G_i|$. We can see that $(V_1 \cup V_2, (B_1, \mathcal{F}))$ and $(V_1 \cup V_2, (B_2, \mathcal{G}))$ are two ETS$(2v)$ with a common weight $k + 2v + 3h$. Therefore

$$J[2v] \supseteq J[v] + 2v + 3J_F(v).$$

Since $J[v] = I_e[v]$,

$$J[2v] \supseteq I_e[v] + 2v + 3J_F(v) = I_e[2v] \setminus \{0, 1, \cdots, 2v-1\}.$$

For the remaining data, let $(V_1, B_1)$ and $(V_1, B_2)$ be two ETS$(v)$ with a common weight $k$, $k \in \{0, 1, \cdots 2v-1\}$, and $T = (B_1, \mathcal{F})$. If $T^*$ is the union of $B_2$, $\{a_{i+1}xy \mid xy \in F_i,\ i = 1,\ 2,\ \cdots,\ v-1\}$ and $\{a_1xx \mid x \in V_2\}$, then $|T \cap T^*| = k$. This implies that $J[2v] = I_e[2v]$.

**Lemma 3.2.** *If $v$ is even, $v \geq 8$, and $J[v] = I_e[v]$ then $J[2v+2] = I_e[2v+2]$.*

**Proof.** Let $(V_1, B_1)$ and $(V_1, B_2)$ be two ETS$(v)$ with a common weight $k$ and $\mathcal{F}$ a 1-factorization of $K_{v+2}$ on $V_2 = \{x_1,\ x_2,\ \ldots,\ x_{v+2}\}$. Let $C = \{a_ixy \mid xy \in F_i,\ i = 1, 2, \ldots, v\}$ and $C_\alpha = \{a_ixy \mid xy \in F_{\alpha(i)},\ i = 1, 2, \ldots, v\}$, where $\alpha$ is a permutation of $\{1, 2, \ldots, v\}$ with exactly $p$ elements fixed ($\alpha$ exists for $p = 0, 1, 2, \ldots, v-2, v$). Then, $C$ and $C_\alpha$ have $p(v+2)/2$ blocks in common. Let $D = \{xxy,\ yyy \mid \text{for each } xy \in F_{v+1}\}$ and $D^i$ is obtained by $D$ replacing the first $i$ pairs $xxy$ and $yyy$ with $yyx$ and $xxx$ for $i = 0, 1, 2, \ldots, (v+2)/2$. We can see that $(V_1 \cup V_2, B_1 \cup C \cup D)$ and $(V_1 \cup V_2, B_2 \cup C_\alpha \cup D^i)$ are two ETS$(2v)$ with a common weight $k + 3p(v+2)/2 + 3((v+2)/2 - i)$. Therefore

$$J[2v+2] \supseteq J[v] + 3\frac{v+2}{2}\{0, 1, 2, \ldots, v-2, v\} + 3\{0, 1, 2, \ldots, \frac{v+2}{2}\}.$$

Since $J[v] = I_e[v]$, then

$$J[2v+2] \supseteq I_e[v] + 3\frac{v+2}{2}\{0,1,2,\dots,v-2,v\} + 3\{0,1,2,\dots,\frac{v+2}{2}\} = I_e[2v+2].$$

Next, in order to solve for small $v$ such that $J[v] = I_e[v]$, we need the results obtained in reference [6] as follows:

**Lemma 3.3.** *Let $v$ be positive integer, $v \geq 4$. $J[3v] \supseteq J[v]+J[v]+J[v]+3S_v$, where $S_v = \{0,1,2,\dots,v^2-7,v^2-6,v^2-4,v^2\}$ for $v \geq 5$ and $S_4 = \{0,2,4,5,6,$ 8, 9, 12, 16$\}$.*

**Proof.** $K_{3v}^+$ can be partitioned into three vertex-disjoint $K_v^+$ and a complete tripartite graph $K_{v,v,v}$. The partition of $K_{v,v,v}$ into $v^2$ edge-disjoint triangles can be constructed by a latin square of order $v$. Using different ETS$(v)$ in each $K_v^*$ and different latin squares of order $v$, we have $J[3v] \supseteq J[v] + J[v] + J[v] + 3S_v$ where $S_v = \{0,1,2,\dots,v^2-7,v^2-6,v^2-4,v^2\}$ for $v \geq 5$ and $S_4 = \{0,2,4,5,6,8,9,12,16\}$ (see [1]).

We start from $v = 8$ in order to obtain $J[v] = I_e[v]$.

$\boldsymbol{v = 8}$. Applying the method of Lemma 3.1 to $J[4]$, we have $J[8] \supseteq J_e[8] \setminus \{4,6,12,20,22,23,25,30\}$. For some unsolvable data, using a similar argument to that in Lemma 3.1, let $(V_1, B_1)$ and $(V_1, B_2)$ be two ETS(4) with a common weight of 0 or 10, and $\mathcal{F} = \{F_1, F_2, F_3\}$ a 1-factorization of $K_4$ on $V_2 = \{x_1, x_2, x_3, x_4\}$. Let $T_1 = (B_1, \mathcal{F})$ and $T_2 = B_2 \cup C \cup D$, where

$$C = \{a_1xy \mid xy \in F_1\} \cup \{a_{i+1}xy \mid xy \in F_i, i = 2,3\}$$
$$D = \{a_2xx \mid x \in V_2\}$$

or

$$C = \{a_ixy \mid xy \in F_i, i = 1,2\} \cup \{a_4xy \mid xy \in F_3\}$$
$$D = \{a_3xx \mid x \in V_2\},$$

then $\{0,10\} + \{6,12\} \subseteq |T_1 \cap T_2|$. Thus $J[8] \supseteq I_e[8] \setminus \{4,20,23,25,30\}$.

Let $E_1, E_2, E_3$ be the following ETS(8): $E_1 = \{111, 122, 133, 144, 155, 167,$ $188, 234, 256, 278, 357, 368, 458, 466, 477\}$, $E_2 = \{111, 122, 133, 144, 155, 166, 178,$

$234, 256, 277, 288, 357, 368, 458, 467\}$, $E_3 = \{112, 135, 147, 168, 222, 238, 246, 257, 334, 367, 444, 458, 556, 666, 777, 788\}$.

Consider the isomorphic designs obtained from $E_1$ and $E_2$ by permuting elements: $N_1 = (12654)(387)E_1$, $N_2 = (23)E_2$. Now, $N_3$ comes from $E_1$ with $167, 357, 368, 458, 466, 477$ replaced by $166, 177, 358, 367, 457, 468$. $N_4$ comes from $E_3$ with $112, 222, 334, 444$ replaced by $221, 111, 443, 333$. $N_5$ comes from $E_3$ with $112, 222, 334, 444, 367, 666, 777$ replaced by $111, 122, 336, 667, 773, 344$.

Therefore $|E_1 \cap N_1| = 4$, $|E_2 \cap N_2| = 23$, $|E_1 \cap N_3| = 20$, $|E_3 \cap N_4| = 30$ and $|E_3 \cap N_5| = 25$. Thus, we have $J[8] = I_e[8]$.

$\boldsymbol{v} = \mathbf{12}$. Applying the method of Lemma 3.1 to $J[6]$, we have $J[12] \supseteq J_e[12] \setminus \{55, 56, 73\}$. The values remaining are handled by Lemma 3.3, so we have $J[12] = I_e[12]$.

From now on, for convenience, we will write $t_i$ for $10 + i$ and $t$ for 10.

$\boldsymbol{v} = \mathbf{10}$. Using the method of Lemma 3.2, we have $J[10] \supseteq J[4] + 9\{0, 1, 2, 4\} + 3\{0, 1, 2, 3\}$. That is $J[10] \supseteq I_e[10] \setminus \{33, 35\}$.

Let $E_1$ be the following ETS(10): $E_1 = \{111, 122, 134, 156, 17t, 189, 233, 244, 257, 268, 29t, 358, 36t, 379, 45t, 469, 478, 555, 599, 666, 677, 888, 8tt\}$. Now, $N_1$ comes from $E_1$ with $111, 122, 233, 45t, 469, 478, 599, 677, 8tt$ replaced by $112, 223, 333, 459, 467, 48t, 5tt, 699, 778$. $N_2$ comes from $N_1$ with $778, 888, 5tt, 555$ replaced by $55t, ttt, 887, 777$. Therefore $|E_1 \cap N_1| = 35$ and $|E_1 \cap N_2| = 33$. Thus, we have $J[10] = I_e[10]$.

$\boldsymbol{v} = \mathbf{14}$. Using the method of Lemma 3.2, we have $J[14] \supseteq I_e[14] \setminus \{100\}$. From the existence of Steiner triple system of order 13, we can give a $C_3$-decomposition of $K_{13}$ based on $V_1 = \{a_1, a_2, \ldots, a_{13}\}$, denoted by $B_1$. Let $V_2 = V_1 \cup \{a_{14}\}$ and $B_2 = \{a_i a_i a_{14} \mid i = 2, 3, \ldots, 13\} \cup \{a_{14} a_{14} a_1\} \cup \{a_1 a_1 a_1\}$, then $(V_2, B_1 \cup B_2)$ is an ETS(14). Now $B_3$ comes from $B_1 \cup B_2$ by replacing $a_2 a_2 a_{14}$, $a_{14} a_{14} a_1$, $a_1 a_1 a_1$ with $a_1 a_1 a_{14}$, $a_{14} a_{14} a_2$, $a_2 a_2 a_2$. It is shown that $|B_3 \cap (B_1 \cup B_2)| = 100$. We have $J[14] = I_e[14]$.

**Lemma 3.4.** *$J[8] = I_e[8]$, $J[10] = I_e[10]$, $J[12] = I_e[12]$, and $J[14] = I_e[14]$.*

Applying the results in Lemma 3.4 to Lemma 3.1 and 3.2 recursively, we obtained the following result.

**Theorem 3.5.** *$J[v] = I_e[v]$ for even $v$, $v \geq 8$.*

## 4. Proof of the Main Theorem for Odd $v$

Since $v$ is odd, the smallest cycle contains 3 edges. Thus two distinct ETS($v$) contain at least 6 different weights. After the cycle of length 3 is a cycle of length 4 and length 5, two ETS($v$) containing 7 or 9 different weights do not exist. Therefore, $J[v] \subseteq I_o[v] = \{0,\ 1,\ 2,\ \ldots,\ m-10,\ m-8,\ m-6,\ m\}$, where $m = v(v+1)/2$.

Exhaustive computer checking for $v = 3$, 5 and 7 produced the following results:
$J[3] = \{0, 6\}$,
$J[5] = \{0, 2, 3, 7, 15\}$,
$J[7] = \{0, 1, \ldots, 14, 16, 17, 22, 28\}$.

**Lemma 4.1.** *If $v$ is odd, $v \geq 9$, and $J[v] = I_o[v]$ then $J[2v+3] = I_o[2v+3]$.*

**Proof.** Let $(V_1, B_1)$ and $(V_1, B_2)$ be two ETS($v$) with a common weight $k$ and $\mathcal{F}$ a 1-factorization of $K_{v+3}$ on $V_2 = \{x_1, x_2, \ldots, x_{v+3}\}$. Let $C = \{a_i xy \mid xy \in F_i, i = 1, 2, \ldots, v\}$ and $C_\alpha = \{a_i xy \mid xy \in F_{\alpha(i)}, i = 1, 2, \ldots, v\}$, where $\alpha$ is a permutation of $\{1, 2, \ldots, v\}$ with exactly $p$ elements fixed. Then, $C$ and $C_\alpha$ have $p(v+3)/2$ blocks in common. It is easy to see that $(V_1 \cup V_2, B_1 \cup C \cup F_{v+1} \cdot F_{v+2})$ is an ETS($2v+3$). If we replace $B_1$ with $B_2$, $C$ with $C_\alpha$ or $(F_{v+1} \cdot F_{v+2})$ with $(F_{v+2} \cdot F_{v+1})$, $((F_{v+1} \cdot F_{v+2}) \cap (F_{v+2} \cdot F_{v+1}) = \emptyset)$, then the two ETS($2v+3$) produced have a common weight $k + 3p(v+3)/2 + 2q(v+3)$, where $p \in \{0, 1, 2, \ldots, v-2, v\}$ and $q \in \{0, 1\}$. Therefore

$$J[2v+3] \supseteq J[v] + 3\frac{v+3}{2}\{0, 1, 2, \ldots, v-2, v\} + 2(v+3)\{0, 1\}.$$

Since $J[v] = I_o[v]$,

$$J[2v+3] \supseteq I_o[v] + 3\frac{v+3}{2}\{0, 1, 2, \ldots, v-2, v\} + 2(v+3)\{0, 1\} = I_o[2v+3].$$

This implies that $J[2v+3] = I_o[2v+3]$.

**Lemma 4.2.** *If $v$ is odd, $v \geq 11$, and $J[v] = I_o[v]$ then $J[2v+1] \supseteq I_o[2v+1] \setminus \{0, 1, 2, ..., v\}$.*

**Proof.** Let $(V_1, B_1)$ and $(V_1, B_2)$ be two ETS$(v)$ with a common weight $k$ and $\mathcal{F}$ a 1-factorization of $K_{v+1}$ on $V_2 = \{x_1, x_2, \ldots, x_{v+1}\}$. Let $D = \{x_i x_i x_i \mid i = 1, 2, \cdots, v+1\}$, $C = \{a_i xy \mid xy \in F_i,\ i = 1, 2, \ldots, v\}$ and $C_\alpha = \{a_i xy \mid xy \in F_{\alpha(i)}, i = 1, 2, \ldots, v\}$, where $\alpha$ is a permutation of $\{1, 2, \ldots, v\}$ with exactly $p$ elements fixed. Then, $C$ and $C_\alpha$ have $p(v+1)/2$ blocks in common. It is easy to see that $(V_1 \cup V_2, B_1 \cup C \cup D)$ is an ETS$(2v+1)$. If we replace $B_1$ with $B_2$ or $C$ with $C_\alpha$, then the two ETS$(2v+1)$ produced have a common weight $k + 3p(v+1)/2 + (v+1)$, where $p \in \{0, 1, 2, \ldots, v-2, v\}$. Therefore

$$J[2v+1] \supseteq J[v] + 3\frac{v+1}{2}\{0, 1, 2, \ldots, v-2, v\} + \{v+1\}.$$

Since $J[v] = I_o[v]$,

$$J[2v+1] \supseteq I_o[v] + 3\frac{v+1}{2}\{0, 1, 2, \ldots, v-2, v\} + \{v+1\} = I_o[2v+1] \backslash \{0, 1, 2, \cdots, v\}.$$

For proof of $\{0, 1, 2, \cdots, v\} \subset J[2v+1]$, we need to embed an ETS$(v)$ into an ETS$(2v+9)$, for odd $v$.

**Lemma 4.3.** *If $v$ is odd, $v \geq 11$, and $\{0,\ 1,\ \ldots,\ v\} \subset J[v-4]$ then $\{0,\ 1,\ 2,\ \ldots,\ v\} \subset J[2v+1]$.*

**Proof.** Let $(V_1, B_1)$ and $(V_1, B_2)$ be two ETS$(v-4)$ with a common weight $k$, where $k \in \{0, 1, 2, \cdots, v\}$. From construction 5 in section 2, an ETS$(v-4)$ can be embedded in an ETS$(2v+1)$. Let $K_{v+5}$ be the complete graph on vertex set $V_2 = \{x_1, x_2, \cdots, x_{v+5}\}$. Set $L_1 = \{x_i x_{i+1} x_{i+3} \mid i = 1, 2, \cdots, v+5\}$ and $L_2 = \{x_i x_{i+4} x_{i+5} \mid i = 1, 2, \cdots, v+5\}$. From **P1**, $P_4 \cup P_5$ splits into four 1-factors $F_1$, $F_2$, $F_3$, $F_4$ and $P_2 \cup P_3$ splits into four 1-factors $G_1$, $G_2$, $G_3$, $G_4$. From **P2**, we have two sets of one-factors $\{F_i \mid i = 1, 2, \cdots, v-2\}$ covering all $P_j$ with $j = 4, 5, \cdots, (v+5)/2$ and $\{G_i \mid i = 1, 2, \cdots, v-2\}$ covering all $P_j$ with $j = 2, 3, 6, 7, \cdots, (v+5)/2$. We can assume that $F_i = G_i$, for $i = 5, 6, \cdots, v-2$.

Let $\alpha$ be a permutation of $\{1, 2, \ldots, v-4\}$ with 0 elements fixed, $C = \{a_i xy \mid xy \in F_i, i = 1, 2, \ldots, v-4\}$ and $C_\alpha = \{a_i xy \mid xy \in F_{\alpha(i)}, i = 1, 2, \ldots, v-4\}$, then $(B_1 \cup C \cup L_1 \cup F_{v-3} \cdot F_{v-2})$ and $(B_2 \cup C_\alpha \cup L_2 \cup F_{v-2} \cdot F_{v-3})$ have exactly a common weight $k$. Thus $\{0, 1, 2, \cdots, v\} \subset J[2v+1]$.

**Lemma 4.4.** *Let $v$ be odd and $v \geq 11$. If $J[v] = I_o[v]$ and $\{0, 1, \ldots, v\} \subset J[v-4]$ then $J[2v+1] = I_o[2v+1]$.*

**Proof.** It follows from Lemmas 4.2 and 4.3.

For small $v$, we start from $v = 9$.

$\boldsymbol{v = 9}$**.** Using a similar argument to Lemma 4.1, we have $J[9] \supseteq J[3] + 9\{0,1,3\} + 12\{0,1\} = \{0,6,9,12,15,18,21,27,33,39,45\}$. Let $E_1 = \{111,$ 123, 145, 169, 178, 222, 246, 257, 289, 333, 347, 359, 368, 448, 499, 556, 588, 667, 779$\}$. $E_2 = A \cup B$, where $A = \{111,$ 123, 146, 157, 189, 222, 247, 259, 268, 333, 356, 669, 677$\}$ and $B = \{349,$ 378, 444, 458, 555, 799, 888$\}$. $E_3 = \{118,$ 122, 136, 147, 159, 233, 249, 258, 267, 344, 357, 389, 455, 468, 566, 699, 779, 788$\}$.

Now, $N_1$ comes from $E_2$ by replacing $B$ with $\{348,$ 379, 445, 499, 558, 788$\}$. $N_2$ comes from $E_2$ by replacing $\{669,$ 997, 776, 349, 333, 444$\}$ with $\{334,$ 449, 993, 679, 666, 777$\}$. $N_3$ comes from $E_3$ by replacing $\{118,$ 136, 344, 468$\}$ with $\{113,$ 168, 346, 448$\}$. $N_4$ comes from $E_1$ by replacing $\{145,$ 448, 885, 111$\}$ with $\{458,$ 441, 115, 888$\}$.

**Table 1.**

| Intersection | Size | Intersection | Size |
|---|---|---|---|
| $E_1 \cap (1347)(26)(58)E_1$ | 1 | $E_1 \cap (154)(289)(367)E_1$ | 20 |
| $E_1 \cap (162534)E_1$ | 2 | $E_1 \cap (15)(36)E_1$ | 22 |
| $E_1 \cap (163452)E_1$ | 3 | $E_1 \cap (14)(36)E_1$ | 23 |
| $E_1 \cap (156432)E_1$ | 4 | $E_1 \cap (12)(4857694)E_1$ | 24 |
| $E_1 \cap (153624)E_1$ | 5 | $E_1 \cap (14)E_1$ | 25 |
| $E_1 \cap (2654)E_1$ | 7 | $E_1 \cap (23)(476)(589)E_1$ | 26 |
| $E_1 \cap (165)(23)E_1$ | 8 | $E_1 \cap (123)(4789)E_1$ | 28 |
| $E_1 \cap (13654)E_1$ | 10 | $E_1 \cap (24)(35)E_1$ | 29 |
| $E_1 \cap (132)(465)E_1$ | 11 | $E_3 \cap (25)(36)(49)E_3$ | 30 |
| $E_1 \cap (23)(46)E_1$ | 13 | $E_2 \cap N_1$ | 31 |
| $E_1 \cap (12563)E_1$ | 14 | $E_2 \cap N_2$ | 34 |
| $E_1 \cap (243)E_1$ | 16 | $E_3 \cap N_3$ | 35 |
| $E_1 \cap (1256)E_1$ | 17 | $E_1 \cap N_4$ | 37 |
| $E_1 \cap (13)(46)E_1$ | 19 | | |

Thus, $J[9] \supseteq I_o[9] \setminus \{32\}$. If $32 \in J[19]$, then the only possible mutually balanced subgraph on $K_9^+$ is $A = \{a_1a_1a_2,\ a_2a_2a_3,\ a_3a_3a_4,\ a_1a_3a_7,\ a_2a_4a_7,\ a_7a_7a_7\}$

which can be changed to $\{a_1a_1a_7,\ a_7a_7a_2,\ a_2a_2a_4,\ a_1a_2a_3,\ a_3a_4a_7,\ a_3a_3a_3\}$. Since the lollipops of the ETS(9) ignoring loops are cycles of length 3, 6, or 9, and the only possible cycle is of length 6, we can add blocks $B = \{a_4a_4a_5,\ a_5a_5a_6,\ a_6a_6a_4\}$ to the partial ETS(9) $A$. Using a computer program showed that the partial ETS(9) containing $A \cup B$ can not be completed to an ETS(9). Thus $32 \notin J[9]$. So $J[9] = I_o[9] \setminus \{32\}$.

$\boldsymbol{v = 13}$, $\boldsymbol{v = 17}$ or $\boldsymbol{v = 21}$. First, we used a similar argument to Lemma 4.1.

When $v = 13$, Lindner and Rosa [5] showed that for each $k \in A = \{0, 1, 2, \cdots, 14, 16, 18, 20, 22, 26\}$, there exists a pair of Steiner triple systems of order 13 (the structure with 13 loops forms ETS(13)) intersecting in $k$ triples. Therefore, $3A + 13 \subseteq J[13]$. The missing data give the following:

Let $E_1 = \{111,\ 124,\ 135,\ 16t_3,\ 179,\ 18t,\ 1t_1t_2,\ 223,\ 255,\ 267,\ 289,\ 2tt_1,\ 2t_2t_3,\ 334,\ 368,\ 37t_1,\ 39t_3,\ 3tt_2,\ 445,\ 46t_1,\ 47t_2,\ 48t_3,\ 49t,\ 56t_2,\ 578,\ 59t_1,\ 5tt_3,\ 66t,\ 699,\ 77t_3,\ 7tt,\ 88t_2,\ 8t_1t_1,\ 9t_2t_2,\ t_1t_3t_3\}$. $E_2 = \{11t_1,\ 124,\ 137,\ 156,\ 188,\ 19t,\ 1t_2t_3,\ 228,\ 235,\ 267,\ 299,\ 2tt_3,\ 2t_1t_2,\ 33t_3,\ 346,\ 38t_2,\ 39t_1,\ 3tt,\ 44t_2,\ 457,\ 48t,\ 49t_3,\ 4t_1t_1,\ 559,\ 58t_3,\ 5tt_1,\ 5t_2t_2,\ 66t,\ 68t_1,\ 69t_2,\ 6t_3t_3,\ 777,\ 789,\ 7tt_2,\ 7t_1t_3\}$. $E_3 = A \cup B$, where $A = \{111,\ 123,\ 145,\ 17t_2,\ 189,\ 225,\ 244,\ 269,\ 2t_2t_3,\ 334,\ 355,\ 38t_1,\ 3tt_3,\ 49t,\ 568,\ 57t_3,\ 59t_1,\ 5tt_2, 7tt\}$ and $B = \{16t,\ 1t_1t_3,\ 278,\ 2tt_1,\ 367,\ 39t_2,\ 46t_2,\ 47t_1,\ 48t_3,\ 66t_1,\ 6t_3t_3,\ 779,\ 88t,\ 8t_2t_2,\ 99t_3,\ t_1t_1t_2\}$. $E_4 = \{118,\ 122,\ 137,\ 14t_3,\ 156,\ 19t,\ 1t_1t_2,\ 235,\ 249,\ 267,\ 288,\ 2tt_2,\ 2t_1t_3,\ 33t_1,\ 346,\ 38t_3,\ 39t_2,\ 3tt,\ 44t_2,\ 457,\ 48t,\ 4t_1t_1,\ 559,\ 58t_1,\ 5tt_3,\ 5t_2t_2,\ 66t,\ 68t_2,\ 69t_1,\ 6t_3t_3,\ 777,\ 789,\ 7tt_1,\ 7t_2t_3,\ 99t_3\}$. $E_5 = \{112,\ 137,\ 14t_3,\ 156,\ 188,\ 19t,\ 1t_1t_2,\ 228,\ 235,\ 249,\ 267,\ 2tt_2,\ 2t_1t_3,\ 33t,\ 346,\ 38t_3,\ 39t_2,\ 3t_1t_1,\ 44t_1,\ 457,\ 48t,\ 4t_2t_2,\ 55t_2,\ 58t_1,\ 599,\ 5tt_3,\ 66t_3,\ 68t_2,\ 69t_1,\ 6tt,\ 777,\ 789,\ 7tt_1,\ 7t_2t_3, 9t_3t_3\}$. $E_6 = C \cup D$, where $C = \{111,\ 123,\ 145,\ 16t,\ 17t_1,\ 18t_2,\ 19t_3,\ 224,\ 255,\ 26t_1,\ 28t_3,\ 335,\ 344,\ 36t_2,\ 37t_3,\ 38t,\ 39t_1,\ 46t_3,\ 48t_1,\ 568,\ 5t_1t_3\}$ and $D = \{279,\ 2tt_2,\ 47t_2,\ 49t,\ 57t,\ 59t_2,\ 669,\ 677,\ 788,\ 899,\ ttt_3,\ tt_1t_1,\ t_1t_2t_2,\ t_2t_3t_3\}$. $E_7 = E \cup F$, where $E = \{16t,\ 18t_2,\ 26t_1,\ 27t_2,\ 28t_3,\ 29t,\ 36t_3,\ 37t_3,\ 38t,\ 39t_1,\ 46t_3,\ 47t,\ 48t_1,\ 49t_2,\ 568,\ 5tt_2,\ ttt_1,\ tt_3t_3,\ t_1t_1t_2,\ t_2t_2t_3\}$ and $F = \{111,\ 124,\ 135,\ 17t_1,\ 19t_3,\ 223,\ 255,\ 334,\ 445,\ 579,\ 5t_1t_3,\ 667,\ 699,\ 778,\ 889\}$. $E_8 = \{112,\ 133,\ 145,\ 167,\ 189,\ 1tt_1,\ 1t_2t_3,\ 223,\ 246,\ 257,\ 28t,\ 29t_2,\ 2t_1t_3,\ 348,\ 35t_2,\ 36t_3,\ 37t_1,\ 39t,\ 444,\ 479,\ 4tt_3,\ 4t_1t_2,\ 555,\ 56t,\ 58t_1,\ 59t_3,\ 666,\ 68t_2,\ 69t_1,\ 77t,\ 78t_3,\ 7t_2t_2,\ 888,\ 999,\ ttt_2,\ t_1t_1t_1,\ t_3t_3t_3\}$.

Now, $N_1$ comes from $E_4$ by replacing {118, 122, $14t_3$, 249, 288, $33t_1$, 346, $3tt$, $4t_1t_1$, $66t$, $99t_3$} with {$11t_3$, 124, 188, 228, 299, 333, $34t_1$, $36t$, 466, $49t_3$, $ttt$, $t_1t_1t_1$}. $N_2$ comes from $E_7$ by replacing $F$ with {112, 133, 145, 179, $1t_1t_3$, 224, 235, 344, 555, $57t_1$, $59t_3$, 669, 677, 788, 899}. $N_3$ comes from $E_3$ by replacing $B$ with {$16t_3$, $1tt_1$, $27t_1$, $28t$, $36t_2$, 379, 467, $48t_2$, $4t_1t_3$, $66t$, $6t_1t_1$, 778, $88t_3$, $99t_2$, $9t_3t_3$, $t_1t_2t_2$}. $N_4$ comes from $E_4$ by replacing {118, 122, 137, $14t_3$, 156, 235, 249, 267, 288, $99t_3$} with {$11t_3$, 124, 135, 167, 188, 228, 237, 256, 299, $49t_3$}. $N_5$ comes from $E_6$ by replacing $D$ with {$27t_2$, $29t$, $47t$, $49t_2$, 579, $5tt_2$, 667, 699, 778, 889, $ttt_1$, $tt_3t_3$, $t_1t_1t_2$, $t_2t_2t_3$}. $N_6$ comes from $E_4$ by replacing {118, 882, 221} with {112, 228, 881}. $N_7$ comes from $E_2$ by replacing {137, 156, 235, 267, $33t_3$, $3tt$, $66t$, $6t_3t_3$} with {135, 167, 237, 256, $33t$, $3t_3t_3$, $66t_3$, $6tt$}. $N_8$ comes from $E_2$ by replacing {$11t_1$, 124, 188, 228, 299, $44t_2$, $4t_1t_1$, 559, $5t_2t_2$} with {111, 128, $14t_1$, 229, 244, $4t_2t_2$, $55t_2$, 599, 888, $t_1t_1t_1$}. $N_9$ comes from $E_8$ by replacing {112, 133, 145, 223, 444, 555, $77t$, $7t_2t_2$, $ttt_2$} with {114, 123, 155, 222, 333, 445, 777, $7tt_2$, $ttt$, $t_2t_2t_2$}. $N_{10}$ comes from $E_4$ by replacing {118, 122, $14t_3$, 249, 288, $99t_3$} with {$11t_3$, 124, 188, 228, 299, $49t_3$}.

**Table 2.**

| Intersection | Size | Intersection | Size |
|---|---|---|---|
| $E_1 \cap (2t)(38t_2t_14759t_36)E_1$ | 1 | $E_5 \cap N_2$ | 48 |
| $E_1 \cap (138)(267459)E_1$ | 4 | $E_3 \cap N_3$ | 50 |
| $E_1 \cap (174692538)E_1$ | 5 | $E_5 \cap N_4$ | 53 |
| $E_1 \cap (29486)(357)E_1$ | 6 | $E_7 \cap N_2$ | 56 |
| $E_1 \cap (1759368)E_1$ | 8 | $E_6 \cap N_5$ | 57 |
| $E_1 \cap (27463)(598)E_1$ | 9 | $E_4 \cap N_4$ | 65 |
| $E_1 \cap (127589)(34)E_1$ | 10 | $E_4 \cap N_1$ | 66 |
| $E_1 \cap (15)(286)(3479)E_1$ | 11 | $N_1 \cap N_4$ | 68 |
| $E_1 \cap (384756)E_1$ | 17 | $N_4 \cap N_6$ | 69 |
| $E_1 \cap (13948)(567)E_1$ | 20 | $E_2 \cap N_7$ | 71 |
| $E_1 \cap (34689)E_1$ | 21 | $E_2 \cap N_8$ | 72 |
| $E_1 \cap (1369)(47)E_1$ | 29 | $E_8 \cap N_9$ | 74 |
| $E_1 \cap (4586)E_1$ | 32 | $E_4 \cap N_{10}$ | 77 |
| $E_1 \cap (3457)E_1$ | 33 | $N_1 \cap N_{10}$ | 80 |
| $E_1 \cap (39)(46)E_1$ | 41 | $N_6 \cap N_{10}$ | 81 |
| $E_1 \cap (78)(9t)E_1$ | 44 | $E_4 \cap N_6$ | 85 |
| $E_2 \cap N_1$ | 45 | | |

We have $J[13] = I_o[13]$.

When $v = 17$, we have $J[17] \supseteq I_o[17] \setminus \{120,\ 124,\ 140,\ 143,\ 145\}$. Let $E_1 = \{113,\ 12t_1,\ 14t_3,\ 15t_6,\ 16t_5,\ 17t_7,\ 18t_4,\ 199,\ 1tt_2,\ 224,\ 23t_2,\ 25t_4,\ 26t_7,\ 27t_6,\ 28t_5,\ 29t_3,\ 2tt,\ 335,\ 34t_1,\ 36t_3,\ 37t_5,\ 38t_7,\ 39t_6,\ 3tt_4,\ 446,\ 45t_2,\ 47t_4,\ 48t_6,\ 49t_7,\ 4tt_5,\ 557,\ 56t_1,\ 58t_3,\ 59t_5,\ 5tt_7,\ 668,\ 67t_2,\ 69t_4,\ 6tt_6, 779, 78t_1,\ 7tt_3,\ 88t,\ 89t_2, 9tt_1,\ t_1t_1t_1,\ t_1t_2t_3,\ t_1t_4t_6,\ t_1t_5t_7,\ t_2t_2t_2,\ t_2t_4t_5,\ t_2t_6t_7,\ t_3t_3t_4,\ t_3t_5t_6,\ t_3t_7t_7,\ t_4t_4t_7,\ t_5t_5t_5,\ t_6t_6t_6\}$, $E_2 = \{11t,\ 124,\ 139,\ 157,\ 168,\ 1t_1t_1,\ 1t_2t_7,\ 1t_3t_6,\ 1t_4t_5,\ 22t_1,\ 235,\ 269,\ 278,\ 2tt_2,\ 2t_3t_3,\ 2t_4t_7,\ 2t_5t_6,\ 33t_5,\ 348,\ 367,\ 3tt_4,\ 3t_1t_3, 3t_2t_2,\ 3t_6t_7,\ 44t_3,\ 456,\ 479,\ 4tt_6,\ 4t_1t_5,\ 4t_2t_4,\ 4t_7t_7,\ 55t_4,\ 589,\ 5tt,\ 5t_1t_7,\ 5t_2t_6,\ 5t_3t_5,\ 66t_2,\ 6tt_1,\ 6t_3t_7,\ 6t_4t_6,\ 6t_5t_5,\ 77t_6,\ 7tt_3,\ 7t_1t_2,\ 7t_4t_4,\ 7t_5t_7,\ 88t_7,\ 8tt_5,\ 8t_1t_4,\ 8t_2t_3,\ 8t_6t_6,\ 999,\ 9tt_7,\ 9t_1t_6,\ 9t_2t_5,\ 9t_3t_4\}$ and $E_3 = \{111,\ 123,\ 14t_3,\ 159,\ 16t_5,\ 17t_7,\ 18t_6,\ 1tt_4,\ 1t_1t_2,\ 224,\ 25t_3,\ 26t, 27t_6,\ 28t_7,\ 29t_5,\ 2t_1t_4,\ 2t_2t_2,\ 333,\ 345,\ 36t_3,\ 37t_1,\ 38t_5,\ 39t_7,\ 3tt_6,\ 3t_2t_4,\ 446,\ 47t_4, 48t_2, 49t_6, 4tt_7, 4t_1t_5,\ 555,\ 567,\ 58t_4,\ 5tt_5, 5t_1t_7, 5t_2t_6,\ 668,\ 69t_4,\ 6t_1t_6,\ 6t_2t_7,\ 777,\ 789,\ 7tt_3,\ 7t_2t_5,\ 88t,\ 8t_1t_3,\ 999,\ 9tt_1,\ 9t_2t_3,\ ttt_2,\ t_1t_1t_1,\ t_3t_3t_4,\ t_3t_5t_5,\ t_3t_6t_7,\ t_4t_4t_6,\ t_4t_5t_7,\ t_5t_6t_6,\ t_7t_7t_7\}$.

Now, $N_1$ comes from $E_1$ by removing the blocks $\{113,\ 199,\ 335,\ 557,\ 779,\ t_1t_4t_6, t_1t_5t_7, t_2t_4t_5, t_2t_6t_7, t_3t_3t_4, t_3t_5t_6, t_3t_7t_7, t_4t_4t_7, t_5t_5t_5, t_6t_6t_6\}$ and replacing them with $\{119,\ 133,\ 355,\ 577,\ 799,\ t_1t_4t_5,\ t_1t_6t_7,\ t_2t_4t_6,\ t_2t_5t_7,\ t_3t_3t_5,\ t_3t_4t_7,\ t_3t_6t_6,\ t_4t_4t_4,\ t_5t_5t_6,\ t_7t_7t_7\}$. $N_2$ comes from $E_1$ by removing the blocks $\{113,\ 199,\ 335,\ 557,\ 779,\ t_1t_4t_6,\ t_1t_5t_7,\ t_2t_2t_2,\ t_2t_6t_7,\ t_3t_5t_6,\ t_3t_7t_7,\ t_4t_4t_7,\ t_5t_5t_5,\ t_6t_6t_6\}$ and replacing them with $\{119,\ 133,\ 355,\ 577,\ 799,\ t_1t_4t_7,\ t_1t_5t_6,\ t_2t_2t_7,\ t_2t_6t_6,\ t_3t_5t_5,\ t_3t_6t_7,\ t_4t_4t_6,\ t_5t_7t_7\}$. $N_3$ comes from $E_2$ by removing the blocks $\{278,\ 2t_4t_7,\ 7t_4t_4,\ 88t_7\}$ and replacing them with $\{27t_4,\ 28t_7,\ 788,\ t_4t_4t_7\}$. $N_4$ comes from $E_3$ by removing the blocks $\{t_3t_3t_4,\ t_3t_5t_5,\ t_3t_6t_7,\ t_4t_4t_6,\ t_4t_5t_7,\ t_7t_7t_7\}$ and replacing them with $\{t_3t_3t_3,\ t_3t_4t_6,\ t_3t_5t_7,\ t_4t_4t_7,\ t_4t_5t_5,\ t_6t_7t_7\}$. $N_5$ comes from $E_3$ by removing the blocks $\{t_3t_3t_4,\ t_3t_5t_5,\ t_4t_4t_6,\ t_5t_6t_6\}$ and replacing them with $\{t_3t_3t_5,\ t_3t_4t_4,\ t_4t_6t_6,\ t_5t_5t_6\}$.

Then $|E_1 \cap N_1| = 120$, $|E_1 \cap N_2| = 124$, $|E_2 \cap N_3| = 143$, $|E_3 \cap N_4| = 140$ and $|E_3 \cap N_5| = 145$. Thus, $J[17] = I_o[17]$.

When $v = 21$, the only missing data is 218, we can embed ETS(5) into ETS(21) ([3]). Thus we have $J[21] = I_o[21]$.

$\boldsymbol{v = 11, v = 15}$ or $\boldsymbol{v = 19}$. First, we use a similar argument to Lemma 4.2.

When $v = 11$, we can embed ETS(3) into ETS(11) as follows. Given a ETS(3) $(V_1, B_1)$, where $V_1 = \{a_1, a_2, a_3\}$, we can decompose the graph $K_8^+$ (based on $V_2 = \{x_1, x_2, \cdots, x_8\}$) into three 1-factors $\mathcal{F} = \{F_1, F_2, F_3\}$, triangles $T$, lollipops $L_1$, and loops $L_2$, where $F_1 = \{x_1x_5,\ x_2x_6,\ x_3x_7,\ x_4x_8\}$, $F_2 = \{x_1x_4,\ x_2x_7,\ x_5x_8,\ x_3x_6\}$, $F_3 = \{x_4x_7,\ x_2x_5,\ x_3x_8,\ x_1x_6\}$, $T = \{x_1x_2x_3,\ x_3x_4x_5,\ x_5x_6x_7,\ x_7x_8x_1\}$, $L_1 = \{x_2x_2x_4,\ x_4x_4x_6,\ x_6x_6x_8,\ x_8x_8x_2\}$ and $L_2 = \{x_1x_1x_1,\ x_3x_3x_3,\ x_5x_5x_5,\ x_7x_7x_7\}$. Let $C = \{a_ixy \mid xy \in F_i, i = 1, 2, 3\}$, then $(V_1 \cup V_2, B_1 \cup C \cup T \cup L_1 \cup L_2)$ is an ETS(11). Replacing the blocks in ETS(3) on $V_1$ and changing $C$ by $\{a_{\alpha(i)}xy \mid xy \in F_i, i = 1, 2, 3\}$ with $\alpha = (23)$ or $(123)$, $L_1$ by $\{x_2x_2x_8,\ x_8x_8x_6,\ x_6x_6x_4,\ x_4x_4x_2\}$, or $T \cup L_1 \cup L_2$ by $\{x_2x_3x_4,\ x_4x_5x_6,\ x_6x_7x_8,\ x_8x_1x_2,\ x_1x_1x_3,\ x_3x_3x_5,\ x_5x_5x_7,\ x_7x_7x_1,\ x_2x_2x_2,\ x_4x_4x_4,\ x_6x_6x_6,\ x_8x_8x_8\}$, we have $J[11] \supseteq \{0, 6\} + \{0, 12, 36\} + \{0, 16, 24\} \supseteq \{0,\ 6,\ 18,\ 22,\ 24,\ 30,\ 36,\ 58\}$. The missing data gives the following.

Let $E_1 = \{111,\ 123,\ 145,\ 167,\ 18t_1,\ 19t,\ 224,\ 255,\ 268,\ 279,\ 2tt_1,\ 335,\ 344,\ 369,\ 37t_1,\ 38t,\ 46t,\ 478,\ 49t_1,\ 56t_1,\ 57t,\ 589,\ 666,\ 777,\ 888,\ 999,\ ttt,\ t_1t_1t_1\}$. $E_2 = \{116,\ 123,\ 145,\ 177,\ 18t_1,\ 19t,\ 225,\ 244,\ 268,\ 279,\ 2tt_1,\ 334,\ 355,\ 369,\ 37t_1,\ 38t,\ 46t,\ 478,\ 49t_1,\ 56t_1,\ 57t,\ 589,\ 667,\ 888,\ 999,\ ttt,\ t_1t_1t_1\}$. $E_3 = \{112,\ 133,\ 145,\ 167,\ 18t_1,\ 19t,\ 224,\ 235,\ 268,\ 279,\ 2tt_1,\ 344,\ 369,\ 37t_1,\ 38t,\ 46t,\ 478,\ 49t_1,\ 556,\ 57t,\ 589,\ 5t_1t_1,\ 66t_1,\ 777,\ 888,\ 999,\ ttt\}$. $E_4 = \{118,\ 123,\ 145,\ 166,\ 179,\ 1tt_1,\ 225,\ 244,\ 267,\ 28t_1,\ 29t,\ 334,\ 355,\ 369,\ 37t_1,\ 38t,\ 46t,\ 478,\ 49t_1,\ 56t_1,\ 57t,\ 589,\ 688,\ 777,\ 999,\ ttt,\ t_1t_1t_1\}$. $E_5 = \{112,\ 13t_1,\ 144,\ 156,\ 179,\ 18t,\ 22t,\ 237,\ 249,\ 25t_1,\ 268,\ 336,\ 34t,\ 358,\ 399,\ 457,\ 466,\ 48t_1,\ 555,\ 59t,\ 67t,\ 69t_1,\ 778,\ 7t_1t_1,\ 889,\ ttt_1\}$.

Now, $N_1$ comes from $E_2$ by removing the blocks $\{116,\ 123,\ 145,\ 177,\ 18t_1,\ 225,\ 244,\ 355,\ 667,\ 888,\ t_1t_1t_1\}$ and replacing them with $\{118,\ 124,\ 135,\ 167,\ 1t_1t_1,\ 223,\ 255,\ 445,\ 666,\ 777,\ 88t_1\}$. $N_2$ comes from $E_4$ by removing the blocks $\{166,\ 179,\ 1tt_1,\ 267,\ 28t_1,\ 29t,\ 688,\ t_1t_1t_1\}$ and replacing them with $\{167,\ 19t,\ 1t_1t_1,\ 268,\ 279,\ 2tt_1,\ 666,\ 88t_1\}$. $N_3$ comes from $E_2$ by removing the blocks $\{116,\ 177,\ 18t_1,\ 225,\ 244,\ 334,\ 355,\ 667, 888,\ t_1t_1t_1\}$ and replacing them with $\{118,\ 167,\ 1t_1t_1,\ 224,\ 255,\ 335,\ 344,\ 666,\ 777,\ 88t_1\}$. $N_4$ comes from $E_3$ by removing the blocks $\{112,\ 133,\ 167,\ 235,\ 556,\ 5t_1t_1,\ 66t_1,\ 777\}$ and replacing them with $\{116,\ 123,\ 177,\ 255,\ 335,\ 56t_1,\ 667,\ t_1t_1t_1\}$. $N_5$ comes from $E_5$ by removing the blocks $\{112,\ 144,\ 22t,\ 237,\ 336,\ 466,\ 67t\}$ and replacing them with $\{114,\ 122,\ 233,\ 27t,\ 367,\ 446,\ 66t\}$. $N_6$ comes from $E_5$ by removing the blocks $\{466,\ 889,\ 48t_1,\ 69t_1\}$

and replacing them with {669, 884, 46$t_1$, 89$t_1$}. $N_7$ comes from $E_3$ by removing the blocks {112, 133, 167, 224, 235, 344 556, 5$t_1t_1$, 66$t_1$, 777} and replacing them with {116, 123, 177, 225, 244, 334, 355 56$t_1$, 667, $t_1t_1t_1$}.

**Table 3.**

| Intersection | Size | Intersection | Size |
|---|---|---|---|
| $E_5 \cap (14)(23)(6t_17t8)E_5$ | 1 | $E_1 \cap (46)(57)E_1$ | 32 |
| $E_5 \cap (3865t_14t79)E_5$ | 2 | $E_1 \cap E_4$ | 37 |
| $E_1 \cap (13764)(2958)E_1$ | 3 | $E_1 \cap (26)(37)E_1$ | 38 |
| $E_1 \cap (1263)(475)E_1$ | 4 | $E_2 \cap N_1$ | 43 |
| $E_1 \cap (135624)E_1$ | 5 | $E_2 \cap E_4$ | 44 |
| $E_1 \cap (156)(2734)E_1$ | 7 | $E_3 \cap N_7$ | 45 |
| $E_1 \cap (14)(2736)E_1$ | 10 | $E_4 \cap N_2$ | 46 |
| $E_1 \cap (157426)E_1$ | 11 | $E_2 \cap N_3$ | 47 |
| $E_1 \cap (234756)E_1$ | 14 | $E_3 \cap N_4$ | 49 |
| $E_1 \cap (14)(36)(57)E_1$ | 19 | $E_5 \cap N_5$ | 50 |
| $E_1 \cap (2734)E_1$ | 20 | $E_2 \cap N_2$ | 55 |
| $E_1 \cap (265)(47)E_1$ | 23 | $E_5 \cap N_6$ | 56 |
| $E_1 \cap (257634)E_1$ | 25 | | |

We have $J[11] = I_o[11]$.

When $v = 15$, we can obtain $J[15] \supseteq I_o[15] \setminus \{1, 83\}$ using Lemma 3.3. Let $E_1 = A \cup B$, where $A$ = {123, 1$tt_5$, 1$t_1t_4$, 1$t_2t_3$, 28$t$, 29$t_1$, 2$t_3t_4$, 335, 38$t_1$, 39$t_3$, 3$tt_2$, 3$t_4t_5$, 48$t_2$, 49$t_5$, 4$tt_4$, 4$t_1t_3$, 58$t_3$, 59$t$, 5$t_1t_5$, 5$t_2t_4$, 68$t_4$, 69$t_2$, 6$t_3t_5$, 777, 78$t_5$, 79$t_4$, 7$tt_3$, 7$t_1t_2$, $t_3t_3t_3$, $t_4t_4t_4$} and $B$ = {118, 146, 157, 199, 22$t_2$, 247, 256, 2$t_5t_5$, 344, 367, 455, 66$t$, 6$t_1t_1$, 889, $ttt_1$, $t_2t_2t_5$}. $N_1$ is obtained from $E_1$ by removing the blocks $B$ and replacing them with {111, 145, 167, 189, 222, 246, 257, 2$t_2t_5$, 347, 366, 444, 556, 6$tt_1$, 888, 999, $ttt$, $t_1t_1t_1$, $t_2t_2t_2$, $t_5t_5t_5$}. Then $|E_1 \cap N_1| = 83$ and $|(1529t_24)(3t_1)(6tt_4)(7t_38t_5)E_1 \cap E_1| = 1$. Thus we have $J[15] = I_o[15]$.

For $v = 19$, the only missing data is 177, we can embed ETS(5) into ETS(19) ([3]). Thus we have $J[19] = I_o[19]$.

**Lemma 4.5.** $J[9] = I_o[9] \setminus \{32\}$ *and* $J[v] = I_o[v]$ *for* $v = 11, 13, 15, 17, 19, 21$.

Applying Lemma 4.5 to Lemmas 4.1 and 4.4 recursively, we obtained the following result.

**Theorem 4.6.** *$J[v] = I_o[v]$ for odd $v$, $v \geq 11$.*

## 5. Conclusions.

By Theorems 3.5 and 4.6, we obtained the following results:

**Main Theorem.** *For even $v$, $J[v] = I_e[v]$ if $v \geq 8$; and for odd $v$, $J[v] = I_o[v]$ if $v \geq 11$.*

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