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A NOTE ON COIN-TOSSING PROCESS

BY

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Abstract. Let $\{Y_n\}$ be an independent coin-tossing process such that $P(Y_n = 1) = p = 1 - P(Y_n = 0)$ for all $n \geq 1$, here p is a constant in $(0, 1)$. For each integer $m \geq 1$, let $\Omega_m = \{0, 1\} \times \ldots \times \{0, 1\}(m - \text{tuple})$, and $\Omega_m^k = \{(a_1, \ldots, a_m)|(a_1, \ldots, a_m) \in \Omega_m$ and $\sum_{j=1}^m a_j = k$ for all $k = 0, 1, \ldots, m\}$. In this paper, we obtain some interesting results about the first occurrence of elements in Ω_m and in Ω_m^k with respect to the stochastic process $\{Y_n\}$.

Let $\{Y_n\}_{n\geq 1}$ be an independent coin-tossing process such that

$$P(Y_n = 1) = 1 - P(Y_n = 0) = p,$$

for all $n \geq 1$, here p is a constant in $(0, 1)$. For each integer $m \geq 1$, let

$$\Omega_m = \{0, 1\} \times \ldots \times \{0, 1\}(m - \text{tuple}),$$

and

$$\Omega_m^k = \{(a_1, \ldots, a_m)|(a_1, \ldots, a_m) \in \Omega_m \text{ and } \sum_{j=1}^{m} a_j = k \text{ for all } k = 0, 1, \ldots, m\}.$$

For each A in Ω_m, let T_A be the first occurrence time of A (with respect to the process $\{Y_n\}_{n\geq 1}$) defined by

$$T_A(Y_1, Y_2, \ldots) = \begin{cases} \inf\{n|A = (Y_{n-m+1}, \ldots, Y_n)\} \\ \infty \text{ if no such } n \text{ exists,} \end{cases}$$

and let $E(T_A)$ be the expectation of T_A. For each pair

$$A = (a_1, \ldots, a_m) \neq B = (b_1, \ldots, b_m) \text{ in } \Omega_m,$$

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let T_B^A be the conditional first occurence time of B given A (with respect to the process $\{Y_n\}_{n\geq 1}$) defined by $T_B^A(Y_1, Y_2, \ldots) = \inf\{n | B$ is a subsequence of consecutive terms in $(a_1, \ldots, a_m, Y_1, \ldots, Y_n)\}$, $= \infty$ if no such n exists and let $E(T_B^A)$ be the expectation of T_B^A.

For $1/2 < p < 1, A = (1,1,0)$, and $B = (1,0,0)$, then

$$P(T_B < T_A)/P(T_A < T_B) = (1-p)^2/p < 1,$$

and we intuitively expect this fact since

$$P\{(Y_n, Y_{n+1}, Y_{n+2}) = A\} > P\{(Y_n, Y_{n+1}, Y_{n+2}) = B\},$$

for all $n \geq 1$ and $E(T_A) < E(T_B)$. However, if $0.51609 < p < 0.64780$, $A = (1,0,1,1,1)$, and $B = (0,1,0,1,1)$, then

$$P\{(Y_n, Y_{n+1}, \ldots, Y_{n+4}) = A\} > P\{(Y_n, Y_{n+1}, \ldots, Y_{n+4}) = B\},$$

for all $n \geq 1$ and $E(T_A) < E(T_B)$, but $P(T_A < T_B) < P(T_B < T_A)$, i.e., if $0.51609 < p < 0.64780$, B will more likely occur before A does even though

$$P\{(Y_n, Y_{n+1}, \ldots, Y_{n+4}) = A\} > P\{(Y_n, Y_{n+1}, \ldots, Y_{n+4}) = B\},$$

for all $n \geq 1$ and $E(T_A) < E(T_B)$. This fact is quite surprising and contradicts to our intuition and the study of this paper is motivated by this surprising fact. The study of this paper might provide us with a better and deeper understanding of the independent coin-tossing process.

Chen and Zame (1979) proved that if $p = 1/2$ and $m \geq 3$, then for each A in Ω_m, there is a B in Ω_m such that $P(T_A < T_B) < P(T_B < T_A)$. Chen and Lin (1984) sharpened, among other results, this result to the subclass Ω_m^k, i.e., they proved that if $p = 1/2$, $m \geq 4$, and $1 \leq k \leq m-1$, then for each A in Ω_m^k, there is a B also in Ω_m^k such that $P(T_A < T_B) < P(T_B < T_A)$. In this paper, we will extend both results, among other results, to the case with an arbitrary p in (0, 1). We start with the following notation and lemmas.

For each $A = (a_1, \ldots, a_m)$ in Ω_m, let

$$\epsilon_j = \begin{cases} 1, (a_j, \ldots, a_m) = (a_1, \ldots, a_{m-j+1}), \\ 0, (a_j, \ldots, a_m) \neq (a_1, \ldots, a_{m-j+1}), \ j = 1, 2, \ldots, m, \end{cases}$$

and

$$A \circ A = \sum_{j=1}^{m} \epsilon_j \Big(\prod_{i=1}^{m-j+1} P(Y_1 = a_i) \Big)^{-1}.$$

For each pair $A = (a_1, \ldots, a_m)$ and $B = (b_1, \ldots, b_m)$ in Ω_m, let

$$A \circ B = \begin{cases} (a_i, \ldots, a_m) \circ (b_1, \ldots, b_{m-i+1}), \text{ if } (a_i, \ldots, a_m) = (b_1, \ldots, b_{m-i+1}) \text{ and} \\ \qquad\qquad\qquad\qquad\qquad\qquad (a_j, \ldots, a_m) \neq (b_1, \ldots, b_{m-j+1}) \ \forall j < i, \\ 0, \qquad\qquad\qquad\qquad\qquad \text{otherwise.} \end{cases}$$

The following lemmas have been proved in Chen and Zame (1979) and we state them here for the sake of completeness.

Lemma 1. *For any A in Ω_m,*

$$E(T_A) = A \circ A. \tag{1}$$

Lemma 2. *For each pair A and B in Ω_m,*

$$E(T_B^A) = B \circ B - A \circ B. \tag{2}$$

Lemma 3. *Let A and B be two distinct elements in Ω_m, then*

$$P(T_A < T_B)(A \circ A - A \circ B) = P(T_B < T_A)(B \circ B - B \circ A). \tag{3}$$

Remark. If $0 < p < 1$, $m \geq 1$, and A, B are two distinct elements in Ω_m, then $A \circ A - A \circ B > 0$ and $B \circ B - B \circ A > 0$.

We say that $A = (a_1, \ldots, a_m)$ is alternating if $a_i \neq a_{i+1}$ for all $i = 1, \ldots, m-1$. For any two elements A and B in Ω_m, we write $A < B$ if $P(T_A < T_B) < P(T_B < T_A)$, $B < A$ if $P(T_A < T_B) > P(T_B < T_A)$, and $A \cong B$ if $P(T_A < T_B) = P(T_B < T_A)$. Now we state and prove our main results.

Theorem 1. *For each $j = 1, \ldots, m$, let $A_j = (a_1, \ldots, a_j, \ldots, a_m)$ such that*

$$a_i = \begin{cases} 0 & \text{if } i = j, \\ 1 & \text{if } i \neq j, \end{cases}$$

and $B_j = (b_1, \ldots, b_j, \ldots, b_m)$ *such that*

$$b_i = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Then, for $m \geq 4$ *we have*

$$A_1 < A_2 < \ldots < A_m < A_1 \text{ if } p^{m-1} < 1/2, \tag{4}$$

and

$$B_1 < B_2 < \ldots < B_m < B_1 \text{ if } (1-p)^{m-1} < 1/2. \tag{5}$$

Proof. By a direct computation using Lemma 3.

Theorem 2. *If* p *is in* $(0,1)$, $m \geq 4$, *and* $A = (a_1, a_2, \ldots, a_m)$ *is an alternating element in* Ω_m^k, *then there is a* B *also in* Ω_m^k *such that*

$$A < B. \tag{6}$$

Proof. Since A is an alternating element in Ω_m^k, $m = 2k-1$, or $2k$, or $2k+1$. If $m = 2k$, then we can choose $B = (a_2, a_1, a_3, \ldots, a_m)$. If $m = 2k-1$ (or $2k+1$), then we can choose $B = (a_m, a_1, a_2, \ldots, a_{m-1})$.

Theorem 3. *For any* p *in* $(0,1)$, $k \geq 2$, *and* $A = (a_1, \ldots, a_{2k})$ *an alternating element in* Ω_{2k}^k, *then for any* B *in* $\Omega_{2k}^k - \{A, A_1, A_2\}$,

$$A < B, \tag{7}$$

here $A_1 = (a_{2k}, a_1, \ldots, a_{2k-1})$ *and* $A_2 = (a_1, \ldots, a_{2k-2}, a_{2k}, a_{2k-1})$.

Proof. By a direct computation using Lemma 3.

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