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A GENERAL PRINCIPLE FOR ISHIKAWA ITERATIONS FOR MULTI-VALUED MAPPINGS

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We prove a generic theorem for the Ishikawa iterates of a pair of multi-valued maps, and then cite some results that are special cases of this theorem.

Key Words : Ishikawa Iterations; Multi-valued Mappings; Fixed Point Theorems

There are a number of theorems in the literature of the following type. T is a selfmap of a Banach space satisfying a contractive condition that may or may not be strong enough to guarantee convergence of the ordinary iterates of T to a fixed point. It is then assumed that the Mann iterates of T converge to some point p, and then it is proved that p is a fixed point of T.

One of the first theorems in this direction was Theorem 4 of Kannan[1]. In a recent paper the third author (Rhoades[2]) showed that there is a generic theorem of this type, which includes most of the other theorems on this topic in the literature. More recently, the author (Rhoades[3]) extended this generic theorem to the Ishikawa iteration process. In that paper he indicated that the theorem could be extended to pairs of maps.

It is the purpose of this paper to establish a generic theorem for the Ishikawa iterates of a pair of multi-valued mappings on a Banach space, and then show that the result has a number of corollaries.

Let (X, d) be a metric space, $CB(X)$ the collection of closed, nonempty, bounded subsets of X, and $H(A, B)$ the Hausdorff metric on X. We shall need the following result.

Lemma 1 (Nadler[4]) — If $A, B \in CB(X)$ and $a \in A$, then, for each $\varepsilon > 0$ there exists a $b \in B$ such that $d(a, b) \leq H(A, B) + \varepsilon$.

Let K be a nonempty subset of X. The Ishikawa iteration scheme associated with two multi-valued mappings $S, T : K \to CB(K)$ is defined as follows :

$$x_0 \in K,\ y_n = (1-\beta_n)x_n + \beta_n a_n,\ \ a_n \in Tx_n,$$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n b_n,\ \ b_n \in Sy_n, \qquad \text{... (1)}$$

where (i) $0 \le \alpha_n, \beta_n \le 1$ for all n. Additional conditions will be placed on $\{\alpha_n\}$ and $\{\beta_n\}$ as needed.

As a consequence of Lemma 1, we can select the a_n and b_n to satisfy the inequality

$$\| a_n - b_n \| \le H(Tx_n, Sy_n) + \varepsilon_n, \text{ with } \lim \varepsilon_n = 0. \qquad \text{... (2)}$$

Theorem 1 — *Let X be a Banach space, K a closed convex subset of X, S, T multi-valued mappings from K into $CB(X)$. Suppose that the Ishikawa scheme (1), with $\{\alpha_n\}$ satisfying (i) and (ii) $\liminf \alpha_n = d > 0$, and $\{a_n\}$, $\{b_n\}$ satisfying (2), converges to a point p. If there exist nonnegative numbers $\alpha, \beta, \gamma, \delta$ with $\beta < 1$ such that, for all n sufficiently large, S and T satisfy*

$$H(Tx_n, Sy_n) \le \alpha \| x_n - b_n \| + \beta \| x_n - a_n \| \qquad \text{... (3)}$$

and

$$H(Sp, Tx_n) \le \alpha \| x_n - p \| + \gamma d(x_n, Tx_n) + \delta d(p, Tx_n) + \beta \max\{d(p, Sp), d(x_n, Sp)\}, \qquad \text{... (4)}$$

then p is a fixed point of S.

If also,

$$H(Sp, Tp) \le \beta[d(p, Tp) + d(p, Sp)], \qquad \text{... (5)}$$

then p is a common fixed point of S and T.

PROOF : Use (1) and note that $\| x_{n+1} - x_n \| = \alpha_n \| x_n - b_n \|$. Since $\lim x_n = p$, $\lim \| x_{n+1} - x_n \| = 0$. Condition (ii) then forces $\lim \| x_n - b_n \| = 0$; i.e., $\lim b_n = p$.

Using (2) and (3),

$$\| a_n - b_n \| \le H(Tx_n, Sy_n) + \varepsilon_n \le \alpha \| x_n - b_n \| + \beta \| x_n - a_n \| + \varepsilon_n.$$

Taking the lim sup as $n \to \infty$ yields

$$\limsup \| a_n - p \| \le \beta \limsup \| p - a_n \| ,$$

which implies that $\lim a_n = p$.

Using (4),

$$\begin{aligned} d(p, Sp) &\le \| x_n - p \| + d(x_n, Tx_n) + H(Sp, Tx_n) \\ &\le \| x_n - p \| + \| x_n - a_n \| + \alpha \| x_n - p \| + \gamma d(x_n, Tx_n) + \delta d(p, Tx_n) \\ &\qquad + \beta \max\{d(p, Sp), d(x_n, Sp)\} \\ &\le (1+\alpha) \| x_n - p \| + (1+\gamma) \| x_n - a_n \| + \delta \| p - a_n \| \end{aligned}$$

$$+ \beta \max \{d(p, Sp), \| x_n - p \| + d(p, Sp)\}.$$

Taking the limit as $n \to \infty$ yields $d(p, Sp) \le \beta d(p, Sp)$, which implies that $p \in Sp$.

Using (5),

$$d(p, Tp) \le H(Sp, Tp) \le \beta[d(p, Sp) + d(p, Tp)] = \beta d(p, Tp),$$

and $p \in Tp$.

Corollary 1 — Let X be a normed space, K a closed convex subset of X. Let S, T be two multi-valued mappings from K into $CB(K)$ satisfying

$$H(Tx, Sy) \le q \max \{k \| x - y \|, d(x, Tx) + d(y, Sy), d(x, Sy) + d(y, Tx)\} \quad \ldots (6)$$

for all x, y in K, where $k \ge 0$ and $0 < q < 1$.

If there exists a point $x_0 \in K$ such that $\{x_n\}$, satisfying (1), (2), (i), (ii), and (iv) $\lim \beta_n = 0$, converges to a point p, then p is a common fixed point of S and T.

PROOF : It is sufficient to show that S and T satisfy conditions (3) - (5).

From (6),

$$H(Tx_n, Sy_n) \le q \max \{k \| x_n - y_n \|, d(x_n, Tx_n) + d(y_n, Sy_n), d(x_n, Sy_n) + d(y_n, Tx_n)\}. \quad \ldots (7)$$

From (1), $\| x_n - y_n \| = \beta_n \| x_n - b_n \|$, $d(y_n, Tx_n) \le \| y_n - a_n \| = (1 - \beta_n) \| x_n - a_n \|$, and $d(y_n, Sy_n) \le \| y_n - b_n \| \le \| x_n - y_n \| + \| x_n - b_n \| \le (1 + \beta_n) \| x_n - b_n \|$. Also, $d(x_n, Tx_n) \le \| x_n - a_n \|$ and $d(x_n, Sy_n) \le \| x_n - b_n \|$. Substituing into (7) gives

$$H(Tx_n, Sy_n) \le q \max \{k\beta_n \| x_n - b_n \|, \| x_n - a_n \| + (1 + \beta_n) \| x_n - b_n \|, \| x_n - b_n \| + (1 - \beta_n) \| x_n - a_n \| \}$$

$$\le \max \{qk\beta_n, q(1 + \beta_n)\} \| x_n - \beta_n \| + q \| x_n - a_n \|,$$

and (3) is satisfied.

Again from (6),

$$H(Tx_n, Sp) \le q \max \{k \| x_n - p \|, d(x_n, Tx_n) + d(p, Sp), d(x_n, Sp) + d(p, Tx_n)\}$$

$$\le qk \| x_n - p \| + qd(x_n, Tx_n) + qd(p, Tx_n) + q \max \{d(p, Sp), d(x_n, Sp)\},$$

and (4) is satisfied.

Using (6),

$$H(Tp, Sp) \le q \max \{0, d(p, Tp) + d(p, Sp), d(p, Sp) + d(p, Tp)\},$$

and (5) is satisfied.

Theorem 1 of Beg and Azam[5] is a special case of Corollary 1, since their contractive definition is a special case of (6).

Corollary 2 — Let K be a nonempty closed convex subset of a Banach space E and $T, S : K \to CB(K)$ satisfy

$$H(Tx, Sy) \leq \max \{\|x-y\|, [d(x, Tx)+d(y, Sy)]/2, [d(x, Sy)+d(y, Tx)]/2\} \quad \ldots (8)$$

for all x, y in K. If there exists a point $x_0 \in K$ such that $\{x_n\}$, satisfying (1), (2), (i), (ii) and (iii) lim $\sup\beta_n = \beta < 1$ converges to a point p, then p is a common fixed point of S and T.

PROOF : From (8),

$$H(Tx_n, Sy_n) \leq \max \{\|x_n - y_n\|, [d(x_n, Tx_n)+d(y_n, Sy_n)]/2 + d(x_n, Sy_n)+d(y_n, Tx_n)]/2\}. \quad \ldots (9)$$

But, from (1), $\|x_n - y_n\| = \beta_n \|x_n - a_n\|$, $d(y_n, Sy_n) \leq \|y_n - b_n\| \leq \|x_n - y_n\| + \|x_n - b_n\| = \beta_n \|x_n - a_n\| + \|x_n - b_n\|$, $d(x_n, Sy_n) \leq \|x_n - b_n\|$, and $d(y_n, Tx_n) \leq \|y_n - a_n\| = (1-\beta_n)\|x_n - a_n\|$. Substituting into (9) yields

$$H(Tx_n, Sy_n) \leq \max \{\beta_n \|x_n - a_n\|, [\|x_n - a_n\| + \beta_n \|x_n - a_n\| + \|x_n - b_n\|]/2, [\|x_n - b_n\| + (1-\beta_n)\|x_n - a_n\|]/2\}$$

$$\leq \max \{\beta_n, (1+\beta_n)/2\} \|x_n - a_n\| + \|x_n - b_n\|/2,$$

and condition (3) is satisfied for all n sufficiently large.

From (8),

$$H(Tx_n, Sp) \leq \max \{\|x_n - p\|, [d(x_n, Tx_n)+d(p, Sp)]/2, [d(x_n, Sp)+d(p, Tx_n)]/2\}$$

$$\leq \|x_n - p\| + d(x_n, Tx_n)/2 + d(p, Tx_n)/2 + (1/2) \max \{d(p, Sp), d(x_n, Sp)\},$$

and condition (4) is satisfied.

Finally, from (8),

$$H(Tp, Sp) \leq \max \{0, [d(p, Tp)+d(p, Sp)]/2, [d(p, Sp)+d(p, Tp)]/2\}$$

$$= [d(p, Sp)+d(p, Tp)]/2,$$

and condition (5) is satisfied.

We now prove a result involving a rational contractive condition.

Corollary 3 — Let X, K be as in Corollary 1, S, $T : K \to CB(K)$ satisfying

$$H(Tx, Sy) \le q \max \left\{ \|x-y\|, \frac{d(y, Sy)[1+d(x, Tx)]}{1+\|x-y\|}, \frac{d(x, Sy)[1+d(x, Tx)+d(y, Tx)]}{2[1+\|x-y\|]} \right\} \quad \ldots (10)$$

for all $x, y \in K$, where $0 < q < 1$. If there exists an $x_0 \in K$ such that a sequence $\{x_n\}$ satisfying (1), (2), (i), (ii) and (iii) $\lim \beta_n = 0$ converges to a point p, then p is a common fixed point of S and T.

PROOF : We need to show that S and T satisy (3) — (5). Using (10) and the inequalities from the proof of Corollary 1,

$$H(Tx_n, Sy_n) \le q \max \left\{ \|x_n - y_n\|, \frac{\|y_n - b_n\|[1+\|x_n - a_n\|]}{1+\beta_n\|x_n - a_n\|}, \frac{\|x_n - b_n\|[1+\|x_n - a_n\|+\|y_n - a_n\|]}{2[1+\beta_n\|x_n - a_n\|]} \right\}$$

From (1),

$$\begin{aligned} \|y_n - b_n\| &\le (1-\beta_n)\|x_n - b_n\| + \beta_n\|a_n - b_n\| \\ &\le (1-\beta_n)\|x_n - b_n\| + \beta_n[\|x_n - b_n\| + \|x_n - a_n\|] \\ &= \|x_n - b_n\| + \beta_n\|x_n - a_n\| \end{aligned}$$

Again from (1), $\|x_{n+1} - x_n\| = \alpha_n\|x_n - b_n\|$. From (ii) it follows that, since $\{x_n\}$ converges, $\lim \|x_{n+1} - x_n\| = 0$ and thus $\lim \|x_n - b_n\| = 0$. Therefore, for all n sufficiently large, $\|x_n - b_n\| + \beta_n\|x_n - a_n\| \le 1 + \beta_n\|x_n - a_n\|$.

Thus, for all n sufficiently large,

$$\begin{aligned} \frac{\|y_n - b_n\|[1+\|x_n - a_n\|]}{1+\beta_n\|x_n - a_n\|} &\le \|y_n - b_n\| + \frac{\|y_n - b_n\|\,\|x_n - a_n\|}{1+\beta_n\|x_n - a_n\|} \\ &\le \|y_n - b_n\| + \|x_n - a_n\| \\ &\le (1+\beta_n)\|x_n - a_n\| + \|x_n - b_n\| \end{aligned}$$

Since $\|y_n - a_n\| = (1-\beta_n)\|x_n - a_n\|$,

$$\begin{aligned} \frac{\|x_n - b_n\|[1+\|x_n - a_n\|+\|y_n - a_n\|]}{2[1+\beta_n\|x_n - a_n\|]} &= \frac{\|x_n - b_n\|[1+(2-\beta_n)\|x_n - a_n\|]}{2[1+\beta_n\|x_n - a_n\|]} \\ &\le \frac{1}{2}[\|x_n - b_n\| + (2-\beta_n)\|x_n - a_n\|]. \end{aligned}$$

Then, for all n sufficiently large,

$$H(Tx_n, Sy_n) \le q \max \{\beta_n\|x_n - a_n\|, (1+\beta_n)\|x_n - a_n\| + \|x_n - b_n\|,$$

$$\frac{1}{2}[\| x_n - b_n \| + (2-\beta_n) \| x_n - a_n \|]\}$$

$$\leq \max \{q\beta_n, q(1+\beta_n), q(2-\beta_n)/2\} \| x_n - a_n \| + q \| x_n - b_n \|,$$

and (3) is satisfied, since, by (iii), $\lim \beta_n = 0$.

Again from (10),

$$H(Tx_n, Sy) \leq q \max$$

$$\left\{\| x_n - p \|, \frac{d(p, Sp)[1 + \| x_n - a_n \|]}{1 + \| x_n - p \|}, \frac{d(x_n, Sp)[1 + \| x_n - a_n \| + \| p - a_n \|]}{2[1 + \| x_n - p \|]}\right\}$$

$$\leq q \| x_n - p \| + qt_n \max \{d(p, Sp), d(x_n, Sp)\},$$

where

$$t_n = \max\left\{\frac{1 + \| x_n - a_n \|}{1 + \| x_n - p \|}, \frac{1 + \| x_n - a_n \| + \| p - a_n \|}{2[1 + \| x_n - p \|}\right\}.$$

Since (3) is satisfied,

$$\| x_n - a_n \| \leq \| x_n - b_n \| + \| b_n - a_n \| \leq \| x_n - b_n \| + H(Tx_n, Sy_n) + \varepsilon_n$$

$$\leq \| x_n - b_n \| + \alpha \| x_n - b_n \| + \beta \| x_n - a_n \| + \varepsilon_n.$$

Since $\lim \| x_n - b_n \| = 0$, we obtain $\lim\sup \| x_n - a_n \| \leq \beta \lim\sup \| x_n - a_n \|$, which implies, since $0 \leq \beta < 1$, that $\lim \| x_n - a_n \| = 0$. Since $\| p - a_n \| \leq \| p - x_n \| + \| x_n - a_n \|$, it follows that $\lim t_n = \max\{1, 1/2\} = 1$. Therefore, for all n sufficiently large, (4) is satisfied.

Since (3) and (4) are satisfied, by Theorem 1, p is a fixed point of S. From (10),

$$H(Sp, Tp) \leq q \max$$

$$\left\{0, d(p, Sp)[1 + d(p, Tp)], \frac{d(p, Sp)[1 + d(p, Tp) + d(p, Sp)]}{2}\right\}$$

$$= 0,$$

and (6) is satisfied trivially.

As yet the only direct application of Theorem 1 has been to the paper of Beg and Azam[5]. The presence of Theorem 1 should serve to eliminate future theorems of this type.

We now state the single-valued version of Theorem 1. For single-valued maps, (1) takes the form $x_0 \in K$, $y_n = (1-\beta_n)x_n + Tx_n$, $x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n$.

***Theorem* 2** — *Let X be a Banach space, K a closed convex subset of X, S and T selfmaps of K. Let $\{x_n\}$ be an Ishikawa method satisfying (1), (i) and (ii). Suppose there exist nonnegative numbers $\alpha, \beta, \gamma, \delta$, $\beta < 1$, such that, for all n sufficiently* *large, it is possible to write*

$$\| Tx_n - Sy_n \| \leq \alpha \| x_n - Sy_n \| + \beta \| x_n - Tx_n \| \quad \text{... (11)}$$

and

$$\| Tx_n - Sp \| \leq \alpha \| x_n - p \| + \gamma \| x_n - Tx_n \| + \delta \| p - Tx_n \| + \beta \max \{\| p - Sp \|, \| x_n - Sp \|\}. \quad \text{... (12)}$$

Then p is a fixed point of S.

If also

$$\| Tp - Sp \| \leq \beta [\| p - Tp \| + \| p - Sp \|], \quad \text{... (13)}$$

then p is a common fixed point of S and T.

Since (3) — (5) become (11) — (13) for single-valued maps, there is nothing to prove.

Corollary 4 (Guay and Singh[6], Theorem 1.1) — Let X be a normed linear space, K a closed convex subset of X, S and T selfmaps of K satisfying

$$\| Sx - Ty \| \leq k \max \{\| x - y \|, \| x - Sx \|, \| y - Ty \|, \| x - Ty \| + \| y - Sx \|\} \quad \text{... (14)}$$

for all x, y in K, $0 \leq k < 1$. If $\{\alpha_n\}$ satisfies (i) and (ii), $\{\beta_n\}$ satisfies (iv), and $\{x_n\}$ converges to a point p, then p is a common fixed point of S and T.

PROOF : From (14),

$$\begin{aligned} \| Sy_n - Tx_n \| &\leq k \max \{\| y_n - x_n \|, \| y_n - Sy_n \|, \| x_n - Tx_n \|, \| x_n - Sy_n \| + \| y_n - Tx_n \|\} \\ &\leq k \{\beta_n \| x_n - Tx_n \|, \| y_n - x_n \| + \| x_n - Sy_n \|, \| x_n - Tx_n \|, \| x_n - Sy_n \| + (1 - \beta_n) \| x_n - Tx_n \|\} \\ &\leq \max \{k\beta_n, k, k(1 - \beta_n)\} \| x_n - Tx_n \| + k \| x_n - Sy_n \|, \end{aligned}$$

and (11) is satisfied.

From (14),

$$\begin{aligned} \| Sp - Tx_n \| &\leq k \max \{\| p - x_n \|, \| p - Sp \|, \| x_n - Tx_n \|, \| x_n - Sp \| + \| p - Tx_n \|\} \\ &\leq k \| p - x_n \| + k \| x_n - Tx_n \| + k \| p - Tx_n \| + k \max \{\| p - Sp \|, \| x_n - Sp \|\}, \end{aligned}$$

and (12) is satisfied.

Finally, from (14),

$$\| Sp - Tp \| \leq k \max \{0, \| p - Sp \|, \| p - Tp \|, \| p - Sp \| + \| p - Tp \|\},$$

and (13) is satisfied.

Theorem 2.2 of Rashwan and Saddeek[7] is a special case of Corollary 4.

We take this opportunity to point out that Theorem 1 of Tiwary and Debnath[8] is a special case of the theorem of Rhoades[2].

Mann iteration can be obtained from Ishikawa iteration by setting $\beta_n \equiv 0$. Therefore, each of the results of this paper has an obvious analogous statement for Mann iteration.

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