Traveling Wave Solutions for Kolmogorov-Type Delayed Lattice Reaction-Diffusion Systems

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Abstract

This work investigates the existence and non-existence of traveling wave solutions for Kolmogorov-type delayed lattice reaction-diffusion systems. Employing the cross iterative technique coupled with the *explicit* construction of upper and lower solutions in the theory of quasi-monotone dynamical systems, we can find two threshold speeds c^* and c_* with $c^* \geq c_* > 0$. If the wave speed is greater than c^*, then we establish the existence of traveling wave solutions connecting two different equilibria. On the other hand, if the wave speed is smaller than c_*, we further prove the nonexistence result of traveling wave solutions. Finally, several ecological examples including one-species, two-species and three-species models with various functional responses and time delays are presented to illustrate the analytical results.

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1 Introduction

The purpose of this work is to investigate the existence and non-existence of traveling wave solutions for the following Kolmogorov-type delayed lattice reaction-diffusion systems:

$$\frac{d}{dt}u_{n,i}(t) = d_n\big(u_{n,i-1}(t) - 2u_{n,i}(t) + u_{n,i+1}(t)\big) + u_{n,i}(t)f_n\big((\boldsymbol{u}_i)_t(-\boldsymbol{\tau}_n)\big), \quad (1.1)$$

for $1 \le n \le N$, $t \ge 0$ and $i \in \mathbb{Z}$, where $d_n > 0$ are discrete diffusion coefficients; $f_n \in C^1(\mathbb{R}^N, \mathbb{R})$ and $\boldsymbol{\tau}_n := (\tau_{n,1}, \cdots, \tau_{n,N})$ is a nonnegative time-delayed vector. Note that the notation $(\boldsymbol{u}_i)_t(-\boldsymbol{\tau}_n)$ means

$$(\boldsymbol{u}_i)_t(-\boldsymbol{\tau}_n) = (u_{1,i}(t-\tau_{n,1}), \cdots, u_{N,i}(t-\tau_{n,N})).$$

Systems (1.1) describe the dynamics of coupled N layer equations distributed in one-dimensional lattice. On the nth-layer, the quantity $u_{n,i}$ at the site i is linearly coupled with the nearest sites $u_{n,i-1}$ and $u_{n,i+1}$ of the same layer n and nonlinearly coupled with all quantities at the same site of all layers through the nonlinearity $u_{n,i}(t)f_n\big((\boldsymbol{u}_i)_t(-\boldsymbol{\tau}_n)\big)$ with spatial homogeneous time delay $(\tau_{n,1}, \cdots, \tau_{n,N})$. We call such systems as discrete diffusive Kolmogorov-type systems. The geometrical configuration of systems (1.1) can be seen in Figure 1.

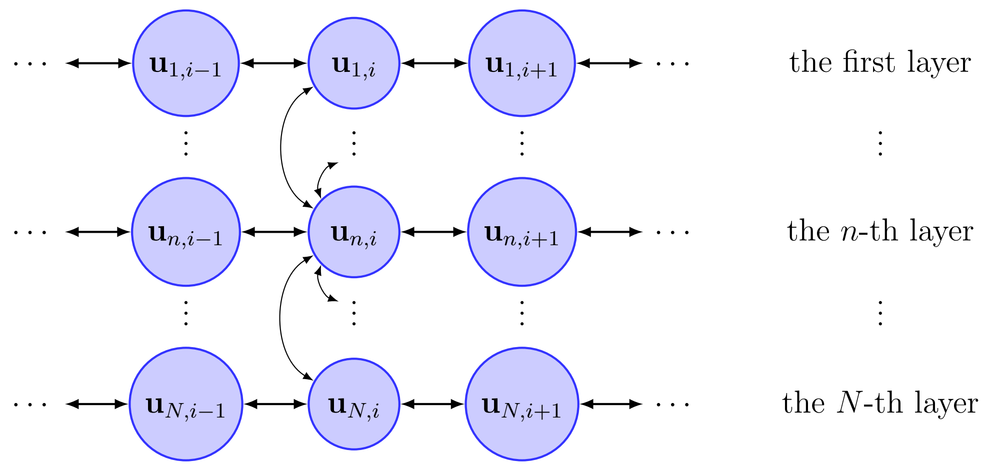

Figure 1: Geometrical configuration of systems (1.1).

Theoretically, we can regard systems (1.1) as the discrete version of the following time-delayed reaction-diffusion systems

$$\frac{\partial}{\partial t}u_n(x,t) = d_n\frac{\partial^2 u_n}{\partial x^2}(x,t) + u_n(x,t)f_n\big(u_1(x,t-\tau_{n,1}), \cdots, u_N(x,t-\tau_{n,N})\big). \quad (1.2)$$

In many situations, the discrete models can exhibit more complicated and colorful dynamics than the continuous models. Moreover, many lattice differential equations have also been proposed as models in various contexts. For examples, in ecological models, systems (1.1) can describe the dynamical interactions of N species distributed in the one-dimensional discrete habitat. Then $u_{n,i}$ denotes the population of the n-th species at the habitat i and the function f_n denotes the *per capita* growth rate. Additionally, from the viewpoint of neural network models, systems (1.1) represent that each cell $u_{n,i}$ not only diffusively (locally) interconnects with its neighbours on the same layer but also coupled with the cells of other layers. Many applications and mathematical results of neural network models were established in past years, see e.g. [3, 4, 9, 10, 17] and the references cited therein. Generally speaking, the investigation of dynamics for systems of lattice differential equations is more difficult than the scalar lattice differential equations, especially when the nonlinearities do not have monotonic properties. This gives us the main motivation to study dynamics of lattice differential systems with more general nonlinearities.

In the past decades, many researchers investigated the dynamics of lattice differential equations, see e.g. [2, 5–8, 14, 16–20, 22, 23, 27–29, 36, 37, 39, 40]. In particular, the existence of traveling wave solutions for lattice differential systems is an important and interesting subject which has attracted considerable attentions. For the related results of traveling wave solutions for lattice differential equations, we refer the readers to [2, 5–8, 14, 16–20, 22, 23, 27–29, 36, 39–41] and the references cited therein. Here we recall some techniques which were widely used in the mentioned literature. In the works [18, 19, 23, 24, 36, 41], the authors used the same framework to obtain the existence of traveling wave solutions for some discrete reaction-diffusion systems (or continuous reaction-diffusion systems). Their framework is sketched as follows. First, they transformed the existence problem of traveling wave solutions as a fixed point problem to a differential operator which depends mainly on the nonlinear reaction terms. Then, assuming some monotone conditions for the nonlinearities within suitable partial order, they proved that the operator is invariant on a nonempty compact set which can be defined by an admissible pair of upper-lower solution. Finally, they established the existence results by using the Schauder's fixed point theorem. According to the framework, two points are crucial in the investigation. One is to show the invariance of the differential operator under some quasimonotone condition, and another one is the construction of a pair of upper-lower solution. To achieve these two points, Wu and Zou [36, 41] first established this framework by assuming the nonlinearities satisfying the so-called quasimonotone (QM) condition. Then, Huang et al. [19] extended the quasimonotone condition to the so-called partial quasimonotone condition (PQM) and use the same framework to show the existence of traveling wave solutions for some epidemic models, which can not be obtained by Wu and Zou's results. Later than [19, 36, 41], Li et al. [23] further introduced the weak quasimonotone condition (WQM) for the nonlinearities and proved the existence of traveling wave solutions for some delayed diffusion-competition systems.

Recently, Hsu and Yang [18] considered the existence of traveling plane wave solutions of the following systems

$$\frac{d}{dt}u_{i,j}^n(t) = L_n[u_{i,j}^n](t) + u_{i,j}^n(t) f_n(\boldsymbol{u}_{i,j}(t), (\boldsymbol{u}_{i,j})_t^{\hat{n}}), \tag{1.3}$$

where all f_n are C^1 functions from $\mathbb{R}^{2N-1}$ to $\mathbb{R}$, $\boldsymbol{u}_{i,j}(t) := (u_{i,j}^1(t), \cdots, u_{i,j}^N(t))$,

$$L_n[u_{i,j}^n](t) := d_{n,1}u_{i+1,j}^n(t) + d_{n,2}u_{i,j+1}^n(t) + d_{n,3}u_{i-1,j}^n(t) + d_{n,4}u_{i,j-1}^n(t) - d_{n,0}u_{i,j}^n(t),$$
$$(\boldsymbol{u}_{i,j})_t^{\hat{n}} := (u_{i,j}^1(t-\tau_1), \cdots, u_{i,j}^{n-1}(t-\tau_{n-1}), u_{i,j}^{n+1}(t-\tau_{n+1}), \cdots, u_{i,j}^N(t-\tau_N)),$$

for $(i,j) \in \mathbb{Z}^2$ and $1 \le n \le N$. Here τ_i and $d_{i,j}$ are nonnegative real constants which represent the time delays and coupling coefficients respectively. The functions f_n are assumed to be of competitive Lotka-Volterra type. Employing the cross-iterative method coupled with the explicit construction of a pair of upper-lower solution, they obtained a speed c^* and show the existence of traveling plane wave solutions of (1.3) connecting two different equilibria when the wave speeds are greater than c^*.

Moreover, Lin et al. [24] also considered traveling wave solutions of the following reaction-diffusion systems

$$\frac{\partial u_n(x,t)}{\partial t} = d_n \frac{\partial^2 u_n(x,t)}{\partial x^2} + F_n(\boldsymbol{u}_t(x)), \text{ for all } t > 0 \text{ and } x \in \mathbb{R}, \tag{1.4}$$

where $d_n > 0$, $1 \le n \le N$, $\boldsymbol{u} = (u_1, \cdots, u_N) \in \mathbb{R}^N$, $\boldsymbol{F} = (F_1, \cdots, F_N) : C([-\tau, 0], \mathbb{R}^N) \to \mathbb{R}^N$ is continuous, and $\boldsymbol{u}_t(x) \in C([-\tau, 0], \mathbb{R}^N)$ for some $\tau > 0$ such that $\boldsymbol{u}_t(x)(\theta) = \boldsymbol{u}(x, t+\theta)$ for $\theta \in [-\tau, 0]$, $t \ge 0$ and $x \in \mathbb{R}$. By introducing the mixed quasimontone condition, the exponentially mixed quasimonotone condition and employing the Schauder's fixed point theorem, the authors reduced the existence problem of traveling wave solutions to that of finding a pair of admissible upper-lower solution of systems (1.4). Then they applied their results to obtain the existence of traveling wave solutions for some multi-species competition, cooperation and predator-prey delayed systems.

Motivated by the literature [18,19,23,24,36,41], in this work, we will consider the existence of positive traveling wave solutions of (1.1). For convenience, we first introduce the following notations.

Notation 1.1. *Let* $\mathbf{a} := (a_1, \cdots, a_N), \mathbf{b} := (b_1, \cdots, b_N) \in \mathbb{R}^N$ *and* $\Phi(s) := (\phi_1, \cdots, \phi_N(s))$, $\Psi(s) := (\psi_1(s), \cdots, \psi_N(s)) \in C(\mathbb{R}, \mathbb{R}^N)$.

(1) *The notation* $\mathbf{a} \preceq \mathbf{b}$ *means* $a_n \le b_n$ *for all* $1 \le n \le N$ *and the closed rectangle* $\{\mathbf{u} \in \mathbb{R}^N : \mathbf{a} \preceq \mathbf{u} \preceq \mathbf{b}\}$ *is denoted by* $[\mathbf{a}, \mathbf{b}]$. *Similarly, the notation* $\Phi \preceq \Psi$ *implies* $\Phi(s) \le \Psi(s)$ *for all* $s \in \mathbb{R}$.

(2) *Let* $C_b(\mathbb{R}, \mathbb{R}^N)$ *and* $C_{3\boldsymbol{K}}(\mathbb{R}, \mathbb{R}^N)$ *be the spaces defined by*

$$C_b(\mathbb{R}, \mathbb{R}^N) := \{\Phi(s) \mid \Phi(s) \in C(\mathbb{R}, \mathbb{R}^N) \textit{ is bounded and uniformly continuous}\},$$
$$C_{3\boldsymbol{K}}(\mathbb{R}, \mathbb{R}^N) := \{\Phi(s) \mid \Phi(s) \in C_b(\mathbb{R}, \mathbb{R}^N) \textit{ with } \mathbf{0} \preceq \Phi(s) \preceq 3\boldsymbol{K} \textit{ for all } s \in \mathbb{R}\},$$

where $3\boldsymbol{K} = (3k_1, \cdots, 3k_N)$*. Then* $C_b(\mathbb{R}, \mathbb{R}^N)$ *is a Banach space with the norm* $\|\Phi\| := \sup_{s\in\mathbb{R};1\le n\le N} |\phi_n(s)|$.

Throughout this paper, we assume the nonlinearities satisfy the following assumptions:

(H0) $f_n(\mathbf{0}) > 0$ and $f_n(\boldsymbol{K}) = 0$ for all $1 \le n \le N$, where $\mathbf{0} := (0, \cdots, 0)$ and $\boldsymbol{K} := (k_1, \cdots, k_N)$ is a positive constant vector with $k_j > 0$ for $j = 1, \cdots, N$.

(H1) For any $1 \le n \le N$, $f_n(x_1, \cdots, x_N)$ is strictly decreasing with respect to x_n on $[\mathbf{0}, 3\boldsymbol{K}]$ and monotone with respect to x_j for any $j \ne n$ on $[\mathbf{0}, 3\boldsymbol{K}]$,

(H2) For any $1 \le n \le N$, we have

$$\frac{\partial f_n(X)}{\partial x_n} k_n + \sum_{j\in I_n^+} \frac{\partial f_n(X)}{\partial x_j} k_j < \sum_{j\in I_n^-} \frac{\partial f_n(X)}{\partial x_j} k_j, \text{ for all } X \in [\mathbf{0}, 3\boldsymbol{K}],$$

where

$$\begin{aligned} I_n^+ &:= \big\{ j \ne n | \partial f_n(X)/\partial x_j \ge 0 \text{ for all } X \in [\mathbf{0}, 3\boldsymbol{K}] \big\}, \\ I_n^- &:= \big\{ j \ne n | \partial f_n(X)/\partial x_j < 0 \text{ for all } X \in [\mathbf{0}, 3\boldsymbol{K}] \big\}. \end{aligned}$$

(H3) $\max_{1\le j\le n}\{f_j(\mathbf{0})\} < \min_{1\le j\le n}\{2f_j(\mathbf{0})\}$.

From (H0), we know that $\mathbf{0}$ and $\boldsymbol{K}$ are equilibria of (1.1). A traveling wave solution of systems (1.1) is a solution of the form

$$u_{n,i}(t) = \phi_n(i + ct), \tag{1.5}$$

for all $i \in \mathbb{Z}$, $t \in \mathbb{R}$ and $1 \le n \le N$, where each $\phi_n \in C^1(\mathbb{R}, \mathbb{R})$ and $c \in \mathbb{R}$ is called the wave speed. Substituting (1.5) into systems (1.1) and using the moving coordinate $s = i + ct$, we can obtain the following profile equations:

$$c\phi_n'(s) = d_n \mathcal{L}(\phi_n)(s) + \phi_n(s) f_n(\Phi_s(-c\boldsymbol{\tau}_n)), \tag{1.6}$$

where

$$\begin{aligned} \Phi_s(-c\boldsymbol{\tau}_n) &:= (\phi_1(s - c\tau_{n,1}), \cdots, \phi_N(s - c\tau_{n,N})), \\ \mathcal{L}(\phi_n)(s) &:= \phi_n(s-1) - 2\phi_n(s) + \phi_n(s+1), \quad \text{for all } 1 \le n \le N. \end{aligned}$$

Our goal is to find solutions of (1.6) connecting $\mathbf{0}$ and $\boldsymbol{K}$, i.e.

$$\lim_{s\to-\infty} (\phi_1(s), \cdots, \phi_N(s)) = \mathbf{0} \text{ and } \lim_{s\to\infty} (\phi_1(s), \cdots, \phi_N(s)) = \boldsymbol{K}. \tag{1.7}$$

Based on assumptions (H0)$\sim$(H2), we can show that the nonlinearities of systems (1.1) satisfy the so-called exponentially mixed quasimonotone condition (see Definition 2.1 or [24]). Employing the cross iterative technique [19,23,24] combining with the explicit form of upper-lower solutions of (1.6), we can establish the existence of traveling waves satisfying (1.7). Note that there are several ecological systems satisfying the assumptions (H0) and (H1). Assumption (H2) can be rewritten by

$$-\frac{\partial f_n(X)}{\partial x_n}k_n > \sum_{j\in I_n^+}\frac{\partial f_n(X)}{\partial x_j}k_j + \sum_{j\in I_n^-} -\frac{\partial f_n(X)}{\partial x_j}k_j. \tag{1.8}$$

From the viewpoint of biology, (1.8) means that self-impact of one specie can dominate other species' impacts. The condition (H3) is a technical assumption which can help us to construct a pair of upper-lower solution of (1.6). Here we illustrate two examples which satisfy the assumptions. If

$$f_1(u_{1,i}, u_{2,i}) = r_1 - a_{11}u_{1,i} + a_{12}u_{2,i}, \tag{1.9}$$

$$f_2(u_{1,i}, u_{2,i}) = r_2 - a_{22}u_{2,i} + a_{21}u_{1,i}, \tag{1.10}$$

then systems (1.1) represent the two species delayed Lotka-Volterra ecological models, see [18, 24]. In addition, if

$$f_1(u_{1,i}, u_{2,i}, u_{3,i}) = b_1 - u_{1,i} - \alpha u_{2,i} - \frac{\eta u_{3,i}}{1+\omega_1 u_{1,i}}, \tag{1.11}$$

$$f_2(u_{1,i}, u_{2,i}, u_{3,i}) = b_2 - \beta u_{1,i} - u_{2,i} - \frac{\eta u_{3,i}}{1+\omega_2 u_{2,i}}, \tag{1.12}$$

$$f_3(u_{1,i}, u_{2,i}, u_{3,i}) = b_3 - \gamma u_{3,i} + \frac{d\eta u_{1,i}}{1+\omega_3 u_{1,i}} + \frac{d\eta u_{2,i}}{1+\omega_4 u_{2,i}}, \tag{1.13}$$

then systems (1.1) represent the model of one predator-two preys model with Holling-type II response. In Section 5, we will verify that (1.9)-(1.10) and (1.11)-(1.13) satisfy assumptions (H0)-(H3) for certain parameters.

Our main results are stated as follows.

Theorem 1.2. *Assume* (H0)∼(H3) *hold. Then there exist* c_* *and* c^* *with* $0 < c_* \le c^*$ *such that the following statements hold.*

(1) *For any* $c > c^*$, *there exists one* $\delta > 0$ (*depending on* c) *such that if* $\max\{\tau_{1,1}, \cdots, \tau_{N,N}\} \le \delta$, *then* (1.6) *has a positive solution satisfying* (1.7).

(2) *For any* $c < c_*$, (1.6) *has no positive solutions satisfying* (1.7).

Remark 1.3. (1) To establish a pair of upper-lower solution and prove the nonlinearities satisfying the exponentially mixed quasimonotone condition, we need small time delays condition in part (1) of Theorem 1.2.

(2) Although we apply the techniques similar to those of [24], there are some significant differences. In [24], the authors established a pair of upper-lower solution for three specific models to derive the existence of traveling wave solutions.

In contrast to their results, we construct a pair of upper-lower solutions of systems (1.1) *explicitly* for Kolmogorov-type nonlinearities to obtain the existence of traveling wave solutions.

(3) Our results can be applied to various ecological models, e.g. the cooperative systems [21], the competitive systems [18] and predator-prey systems, and so on. Moreover, we also prove that the nonexistence of traveling wave solutions of systems (1.1) satisfying (1.7) when the wave speeds are smaller than c_*.

(4) In [30], the authors considered the existence and non-existence of transition fronts for a spatially inhomogeneous (with localized inhomogeneity) KPP equations. The time delay terms considered in our work mimic such a spatially inhomogeneity as that of [30]. However, due to the general setting of the nonlinearities in our problem, it's difficult to prove that $c_* = c^*$ or explain what happens between c_* and c^*. The main difficulty is that our models are quasi-monotone systems. Note that the study of transition fronts between c_* and c^* is an interesting and challenging problem, we will continue the investigation.

The remainder of this paper is organized as follows. In Section 2, we first introduce some definitions and notations. Then we prove that the nonlinearities of (1.1) satisfy the exponentially mixed quasimonotone condition and establish the existence of a pair of upper-lower solution. Since the construction of a pair of upper-lower solution is quite complicated, we put the details in the final appendix. Next, we define the solution operator for equation (1.6) and examine their properties in Section 3. Applying the results obtained in Sections 2 and 3, we use the cross iteration scheme and Schauder's fixed point theorem in Section 4 to prove our main results. In the final section, we apply our results to several ecological models.

2 Preliminaries

We first introduce some definitions which will be used in the proof of the main theorem. Then we show the nonlinearities satisfy the exponentially mixed quasi-monotone condition, and construct a pair of upper-lower solution of (1.6).

Definition 2.1.

(1) *Let I be a subset of $\{1, 2, \cdots, N\}$. Then we denote the function $[\Phi|\Psi_I](s) := ([\Phi|\Psi_I]_1(s), \cdots, [\Phi|\Psi_I]_N(s))$ by*

$$[\Phi|\Psi_I]_i(s) = \begin{cases} \phi_i(s), & \text{if } i \notin I, \\ \psi_i(s), & \text{if } i \in I. \end{cases}$$

Similarly, if $\psi(s) \in C(\mathbb{R}, \mathbb{R})$, we also use the notation $[\Phi|\psi_{\{n\}}](s)$ to represent the function $(\phi_1(s), \cdots, \phi_{n-1}(s), \psi(s), \phi_{n+1}(s), \cdots, \phi_N(s))$.

(2) *Let* $\hat{\Phi}(s) := (\hat{\phi}_1(s), \cdots, \hat{\phi}_N(s))$ *and* $\check{\Phi}(s) := (\check{\phi}_1(s), \cdots, \check{\phi}_N(s))$ *be two functions in* $C_{3\boldsymbol{K}}(\mathbb{R}, \mathbb{R}^N)$ *which are continuously differentiable except for finitely many* s*. The function* $(\hat{\Phi}(s), \check{\Phi}(s))$ *is called a pair of upper-lower solution of* (1.6)*, respectively, if* $\check{\Phi} \preceq \hat{\Phi}$ *and the following inequalities hold except for finitely many* s*:*

$$c\check{\phi}_n'(s) \leq d_n \mathcal{L}(\check{\phi}_n)(s) + \check{\phi}_n(s) f_n\big([\check{\Phi}|\hat{\Phi}_{I_n^-}]_s(-c\boldsymbol{\tau}_n)\big); \tag{2.1}$$

$$c\hat{\phi}_n'(s) \geq d_n \mathcal{L}(\hat{\phi}_n)(s) + \hat{\phi}_n(s) f_n\big([\hat{\Phi}|\check{\Phi}_{I_n^-}]_s(-c\boldsymbol{\tau}_n)\big). \tag{2.2}$$

(3) *The functions* $f_1(\cdot), \cdots, f_N(\cdot)$ *in* (1.6) *are said to satisfy the* exponentially mixed quasimonotone condition *if each* $f_n(x_1, \cdots, x_N)$ *is monotone with respect to* x_j *for any* $j \neq n$ *and there exist positive numbers* $\tilde{\beta}_1, \cdots, \tilde{\beta}_N$ *such that*

$$\psi(t) f_n([\Phi|\psi_{\{n\}}]_t(-c\boldsymbol{\tau}_n)) - \phi_n(t) f_n(\Phi_t(-c\boldsymbol{\tau}_n)) + (c\tilde{\beta}_n - 2d_n)(\psi(t) - \phi_n(t)) \geq 0, \tag{2.3}$$

for any $\Phi(t) \in C_{3\boldsymbol{K}}(\mathbb{R}, \mathbb{R}^N)$ *and* $\psi(t) \in C(\mathbb{R}, \mathbb{R})$ *with* $0 \leq \phi_n(t) \leq \psi(t) \leq 3k_n$ *and* $e^{\tilde{\beta}_n t}(\psi(t) - \phi_n(t))$ *is nondecreasing in* t*.*

According to Definition 2.1, we have the following result.

Lemma 2.2. *Assume* (H1) *holds. There exists a* $\tilde{\tau} > 0$ *such that if* $\tau_{n,n} < \tilde{\tau}$ *for all* $1 \leq n \leq N$*, then* $f_1(\cdot), \cdots, f_N(\cdot)$ *satisfy the exponentially mixed quasimonotone condition.*

Proof. Assume $\Phi(t) = (\phi_1(t), \cdots, \phi_N(t)) \in C_{3\boldsymbol{K}}(\mathbb{R}, \mathbb{R}^N)$, $\psi(t) \in C(\mathbb{R}, \mathbb{R})$ with $0 \leq \phi_n(t) \leq \psi(t) \leq 3k_n$ and all $e^{\tilde{\beta}_n t}(\psi(t) - \phi_n(t))$ are nondecreasing in t. We only have to find some positive numbers $\tilde{\beta}_n$, $n = 1, \cdots, N$ and $\tilde{\tau}$ such that (2.3) holds when $\tau_{n,n} < \tilde{\tau}$. To this end, we denote

$$M_n := \max_{\boldsymbol{0} \preceq X \preceq 3\boldsymbol{K}} \{|f_n(X)|, |\partial f_n(X)/\partial x_n|\}, \text{ for } n = 1, \cdots N.$$

For each n, it is easy to see that there are positive numbers $\tilde{\beta}_n$ and $\tilde{\tau}_n$ such that

$$c\tilde{\beta}_n - 2d_n - M_n - 3e^{\tilde{\beta}_n c\tau_{n,n}} k_n M_n > 0, \text{ for any } \tau_{n,n} < \tilde{\tau}_n. \tag{2.4}$$

Then, by Mean Value Theorem, we have

$$\begin{aligned}
&(c\tilde{\beta}_n - 2d_n)(\psi(t) - \phi_n(t)) + \psi(t) f_n([\Phi|\psi_{\{n\}}]_t(-c\boldsymbol{\tau}_n)) - \phi_n(t) f_n(\Phi_t(-c\boldsymbol{\tau}_n)) \\
=&(c\tilde{\beta}_n - 2d_n)(\psi(t) - \phi_n(t)) + \psi(t)\big[f_n([\Phi|\psi_{\{n\}}]_t(-c\boldsymbol{\tau}_n)) - f_n(\Phi_t(-c\boldsymbol{\tau}_n))\big] + \\
&f_n(\Phi_t(-c\boldsymbol{\tau}_n))(\psi(t) - \phi_n(t)) \\
=&(c\tilde{\beta}_n - 2d_n)(\psi(t) - \phi_n(t)) + \psi(t) \frac{\partial f_n(\boldsymbol{\xi}_n)}{\partial x_n}\big[\psi(t - c\tau_{n,n}) - \phi_n(t - c\tau_{n,n})\big] + \\
&f_n(\Phi_t(-c\boldsymbol{\tau}_n))(\psi(t) - \phi_n(t)),
\end{aligned} \tag{2.5}$$

for some $\boldsymbol{\xi}_n \in [\mathbf{0}, 3\boldsymbol{K}]$ which depends on t. Since $e^{\tilde{\beta}_n t}(\psi(t) - \phi_n(t))$ is nondecreasing, it is clear that

$$\psi(t - c\tau_{n,n}) - \phi_n(t - c\tau_{n,n}) \leq e^{\tilde{\beta}_n c\tau_{n,n}}\big(\psi(t) - \phi_n(t)\big). \tag{2.6}$$

Then it follows from (2.5) and (2.6) that

$$\begin{aligned}&(c\tilde{\beta}_n - 2d_n)(\psi(t) - \phi_n(t)) + \psi(t) f_n([\Phi|\psi_{\{n\}}]_t(-c\boldsymbol{\tau}_n)) - \phi_n(t) f_n(\Phi_t(-c\boldsymbol{\tau}_n))\\ \geq&(c\tilde{\beta}_n - 2d_n - M_n - 3e^{\tilde{\beta}_n c\tau_{n,n}} k_n M_n)(\psi(t) - \phi_n(t)) > 0.\end{aligned}$$

Hence, the assertion follows by taking $\tilde{\tau} := \min\{\tilde{\tau}_1, \cdots, \tilde{\tau}_N\}$. □

Next, we construct a pair of upper-lower solution of (1.6). To this end, we first define the characteristic functions $\Delta_n(\lambda, c)$ for $n = 1, \cdots, N$ by

$$\Delta_n(\lambda, c) := -c\lambda + d_n(e^{-\lambda} + e^{\lambda} - 2) + f_n(\mathbf{0}). \tag{2.7}$$

Then the characteristic functions have the following properties.

Lemma 2.3. *For any $n = 1, \cdots N$, there exists a $c_n > 0$ such that the following statements hold.*

(1) *If $0 < c < c_n$ then $\Delta_n(\lambda, c)$ has no real roots.*

(2) *If $c > c_n$ then $\Delta_n(\lambda, c)$ has exactly two positive roots $\lambda_{n,1}$, $\lambda_{n,2}$ depending on c such that $\lambda_{n,1} < \lambda_{n,2}$ and*

$$\Delta_n(\lambda, c) \begin{cases} = 0, & \text{if } \lambda = \lambda_{n,1} \text{ or } \lambda_{n,2},\\ < 0, & \text{if } \lambda_{n,1} < \lambda < \lambda_{n,2},\\ > 0, & \text{if } \lambda < \lambda_{n,1} \text{ or } \lambda > \lambda_{n,2}.\end{cases} \tag{2.8}$$

Proof. (1) For any $n = 1, \cdots, N$, let's define $g_n(\lambda)$ by

$$g_n(\lambda) := d_n(e^{-\lambda} + e^{\lambda} - 2) + f_n(\mathbf{0}), \text{ for all } \lambda \in [0, \infty). \tag{2.9}$$

Then $\Delta_n(\lambda, c) = -c\lambda + g_n(\lambda)$. Hence the real roots of $\Delta_n(\lambda, c)$ are equivalent to the intersection points of the graphs of $c\lambda$ and $g_n(\lambda)$. Since $g_n(\lambda)$ is convex, one can easily verify that there is some $c_n > 0$ such that $\Delta_n(\lambda, c)$ has no root for $c < c_n$ and has exactly two positive roots $\lambda_{n,1}$ and $\lambda_{n,2}$ satisfying (2.8) for $c > c_n$.

(2) Since $\Delta_n(\lambda, c)$ is convex with respect to λ, the result follows from the proof of part (1). □

Lemma 2.4. *There exists a $c_0 > \max\{c_1, \cdots, c_N\}$ such that*

$$\bigcap_{n=1}^{N} \big(\lambda_{n,1}, \min\{\lambda_{n,1} + \lambda_{1,1}, \cdots, \lambda_{n,1} + \lambda_{N,1}, \lambda_{n,2}\}\big) \tag{2.10}$$

is a non-empty set, where c_n are defined in Lemma 2.3.

Proof. For convenience, we relabel the indices of the nonlinearities such that

$$f_n(\mathbf{0}) \geq f_{n+1}(\mathbf{0}) \text{ and } d_{n+1} \leq d_n \text{ if } f_n(\mathbf{0}) = f_{n+1}(\mathbf{0}) \tag{2.11}$$

for any $1 \leq n \leq N-1$. By (H3) and (2.11), we have

$$f_1(\mathbf{0}) = \max_{1\leq j\leq n}\{f_j(\mathbf{0})\} < \min_{1\leq j\leq n}\{2f_j(\mathbf{0})\} = 2f_N(\mathbf{0}). \tag{2.12}$$

Moreover, it is easy to see that

$$\lim_{c\to\infty} \lambda_{n,1} = 0 \text{ and } \lim_{c\to\infty} \lambda_{n,2} = \infty. \tag{2.13}$$

We first claim that if c is large enough then

$$\min\{2\lambda_{1,1}, \cdots, 2\lambda_{N,1}\} > \max\{\lambda_{1,1}, \cdots, \lambda_{N,1}\}. \tag{2.14}$$

Since $h(\lambda) := d(e^{-\lambda} - 2 + e^{\lambda})$ is a convex function and $\lambda_{n,1}$ is the first intersection point of $h(\lambda)$ and the line $c\lambda$, there exists a $\hat{c}_1 > 0$ such that $\lambda_{n,1} \geq \lambda_{n+1,1}$ for any $1 \leq n \leq N-1$ and $c > \hat{c}_1$. Then, for $c > \hat{c}_1$, it follows that

$$\max\{\lambda_{n,1}, \cdots, \lambda_{N,1}\} = \lambda_{1,1} \quad \text{and} \quad \min\{\lambda_{n,1}, \cdots, \lambda_{N,1}\} = \lambda_{N,1}. \tag{2.15}$$

Moreover, by (2.12) and (2.13), we can find a $\hat{c}_2 > \hat{c}_1$ such that if $c > \hat{c}_2$ then

$$d_1\big(e^{-\lambda_{1,1}} - 2 + e^{\lambda_{1,1}}\big) - 2d_N\big(e^{-\lambda_{N,1}} - 2 + e^{\lambda_{N,1}}\big) < 2f_N(\mathbf{0}) - f_1(\mathbf{0}).$$

Thus we have

$$\lambda_{1,1} = \frac{d_1\big(e^{-\lambda_{1,1}} - 2 + e^{\lambda_{1,1}}\big) + f_1(\mathbf{0})}{c} < \frac{2d_N\big(e^{-\lambda_{N,1}} - 2 + e^{\lambda_{N,1}}\big) + 2f_N(\mathbf{0})}{c} = 2\lambda_{N,1} \tag{2.16}$$

for $c > \hat{c}_2$. Hence the claim follows from (2.15) and (2.16).

By (2.13), there also exists a $c_0 > \max\{\hat{c}_1, \hat{c}_2\}$ such that if $c > c_0$ then

$$\max\{\lambda_{n,1} + \lambda_{1,1}, \cdots, \lambda_{n,1} + \lambda_{N,1}\} < \lambda_{n,2}, \text{ for all } 1 \leq n \leq N. \tag{2.17}$$

Then (2.17) and (2.14) imply that

$$\begin{aligned}\min\{\lambda_{n,1} + \lambda_{1,1}, \cdots, \lambda_{n,1} + \lambda_{N,1}, \lambda_{n,2}\} &= \min\{\lambda_{n,1} + \lambda_{1,1}, \cdots, \lambda_{n,1} + \lambda_{N,1}\}\\ &\geq \min\{2\lambda_{1,1}, \cdots, 2\lambda_{N,1}\}\\ &> \max\{\lambda_{1,1}, \cdots, \lambda_{N,1}\}.\end{aligned}$$

Therefore, the set in (2.10) is non-empty. The proof is complete. □

According to Lemma 2.3, we denote the functions $\check{\Phi}(t) = (\check{\phi}_1(t), \cdots, \check{\phi}_N(t))$ and $\hat{\Phi}(t) = (\hat{\phi}_1(t), \cdots, \hat{\phi}_N(t))$ by

$$\check{\phi}_n(t) := \begin{cases} e^{\lambda_{n,1}t} - qe^{\eta\lambda_{n,1}t}, & \text{if } t < t_n, \\ k_n - (k_n - \frac{m_n}{\sigma})e^{-\gamma t}, & \text{if } t \geq t_n, \end{cases} \tag{2.18}$$

$$\hat{\phi}_n(t) := \begin{cases} e^{\lambda_{n,1}t} + qk_n e^{\kappa t}, & \text{if } t < \hat{t}_n, \\ k_n + k_n e^{-\gamma t}, & \text{if } t \geq \hat{t}_n, \end{cases} \tag{2.19}$$

where $q, \eta, m_n, \sigma, \kappa, \gamma, t_n$ and $\hat{t}_n$ are constants which will be decided in Appendix A. The graphs of each component of $\check{\Phi}(t)$ and $\hat{\Phi}(t)$ are illustrated in Figure **??**. If c is large enough then we can show that $(\check{\Phi}_n(t), \check{\Phi}(t))$ is a pair of upper-lower solution of (1.6).

Lemma 2.5. *There exist positive constants $\tilde{c}$, $\beta_1, \cdots, \beta_N, \tau^*$ with each $\beta_n > \tilde{\beta}_n$ such that if $c > \tilde{c}$ and $\tau_{n,n} < \tau^*$ for all $1 \leq n \leq N$ then $(\hat{\Phi}(t), \check{\Phi}(t))$ forms a pair of upper-lower solution of* (1.6) *and $e^{\beta_n t}(\check{\phi}_n(t) - \hat{\phi}_n(t))$ is nondecreasing in t.*

Proof. Since the proof requires many tedious computations, we put the details in Appendix A. □

Remark 2.6. *(1) The main motivation for constructing the upper-lower solution of* (1.6) *arises from the work [16]. The authors of [16] considered the existence of traveling wave solutions for some scalar quasimonotone lattice dynamical systems by using the monotone iteration method combining with a pair of upper-lower solution. Different to* (2.18) *and* (2.19), *they constructed the upper and lower solutions in the form*

$$U(s) = \begin{cases} x^+ & \text{if} \quad t \geq 0, \\ x^+ e^{\sigma t} & \text{if} \quad t \leq 0, \end{cases} \quad \text{and} \quad L(t) = \begin{cases} 0 & \text{if} \quad t \geq t_0, \\ \zeta(1 - he^{\epsilon t})e^{\sigma t} & \text{if} \quad t \leq s_0, \end{cases} \tag{2.20}$$

where 0 *and* x^+ *are equilibria,* $t_0 < 0$ *and* $\sigma, \zeta, h, \epsilon$ *are positive constants. Note that, for large t, $U(t)$ and $L(t)$ equal to the equilibria x^+ and* 0 *respectively. Unfortunately, since the nonlinearities of* (1.1) *are not quasimonotonic, we can not apply* (2.20) *directly to our problem. Therefore, by Lemmas 2.3 and 2.4, we modify the functions in* (2.20) *for large t into the form* (2.18) *and* (2.19). *Roughly speaking, we assume the upper and lower solutions have exponential decay rates for large $|t|$. It's not difficult to verify $(\check{\Phi}_n(t), \check{\Phi}(t))$ satisfy the inequalities* (2.1) *and* (2.2) *when $|t|$ is large enough. However, due to the non-quasimonotonicity of the systems, we have to adjust the corresponding parameters carefully to ensure $(\check{\Phi}_n(t), \check{\Phi}(t))$ satisfy the inequalities* (2.1) *and* (2.2) *when $t \in [t_n, \hat{t}_n]$ (see Appendix A.1-A.4). In fact, in our previous work [18], we also used such functions* (2.18) *and* (2.19) *to obtain the existence of traveling plane wave solutions of delayed lattice differential systems in competitive Lotka-Volterra type.*

(2) Since $\beta_n > \tilde{\beta}_n$ for all n, from Lemma 2.2, *it is easy to see that*

$$\psi(t)f_n([\Phi|\psi_{\{n\}}]_t(-c\boldsymbol{\tau}_n)) - \phi_n(t)f_n(\Phi_t(-c\boldsymbol{\tau}_n)) + (c\beta_n - 2d_n)(\psi(t) - \phi_n(t)) > 0,$$

for any $\Phi(t) \in C_{3\boldsymbol{K}}(\mathbb{R}, \mathbb{R}^N)$, $\psi(t) \in C(\mathbb{R}, \mathbb{R})$ *with* $0 \leq \phi_n(t) \leq \psi(t) \leq 3k_n$ *and* $e^{\beta_n t}(\psi(t) - \phi_n(t))$ *is nondecreasing in* t.

3 Properties of Solution Operator

To prove the existence of traveling wave solutions by Schauder's fixed point theorem, in this section we define the solution operator of the profile equations (1.6) and investigate its properties.

First, we denote the operators $H(\Phi)(s) = (H_1(\Phi)(s), \cdots, H_N(\Phi)(s))$ and $G(\Phi)(s) = (G_1(\Phi)(s), \cdots, G_N(\Phi)(s))$ on $C_{3\boldsymbol{K}}(\mathbb{R}, \mathbb{R}^N)$ by

$$H_n(\Phi)(s) := \frac{1}{c}\big\{d_n\mathcal{L}(\phi_n)(s) + \phi_n(s)f_n(\Phi_s(-c\boldsymbol{\tau}_n))\big\} + \beta_n\phi_n(s),$$
$$G_n(\Phi)(s) := e^{-\beta_n s}\int_{-\infty}^{s} e^{\beta_n z}H_n(\Phi)(z)dz,$$

where $s \in \mathbb{R}$, $1 \leq n \leq N$, $\Phi(s) = (\phi_1(s), \cdots, \phi_N(s)) \in C_{3\boldsymbol{K}}(\mathbb{R}, \mathbb{R}^N)$ and all β_n are the numbers defined in Lemma 2.5. It is clear that the profile equations (1.6) can be rewritten in the form

$$\phi_n'(s) + \beta_n\phi_n(s) - H_n(\Phi)(s) = 0$$

for each n, and any fixed point of the operator G is a solution of (1.6). Moreover, let $(\hat{\Phi}, (s)\check{\Phi}(s))$ be the pair of upper-lower solution (1.6) defined by Lemma 2.5, then it's easy to verify that

$$\hat{\phi}_n'(s) - H_n([\hat{\Phi}|\check{\Phi}_{I_n^-}])(s) + \beta_n\hat{\phi}_n(s) \geq 0, \tag{3.1}$$
$$\check{\phi}_n'(s) - H_n([\check{\Phi}|\hat{\Phi}_{I_n^-}])(s) + \beta_n\check{\phi}_n(s) \leq 0. \tag{3.2}$$

Some essential properties of the operator $G(\cdot)$ are established in the sequel.

Lemma 3.1. *Let* $c > 0$ *be fixed. Then the operator* $G(\cdot)$ *is continuous on* $C_{3\boldsymbol{K}}(\mathbb{R}, \mathbb{R}^N)$ *with respect to the sup-norm* $\|\cdot\|$.

Proof. First, we claim that $G(\Phi)(s) \in C_b(\mathbb{R}, \mathbb{R}^N)$ for all $\Phi(s) \in C_{3\boldsymbol{K}}(\mathbb{R}, \mathbb{R}^N)$. According to the definition of $G(\Phi)(s)$, it is obvious that $G(\Phi)(s)$ is bounded. Therefore, we only need to prove that $G(\Phi)(s)$ is uniformly continuous.

For any $1 \leq n \leq N$ and $s, h \in \mathbb{R}$, we have

$$\begin{aligned}&|G_n(\Phi)(s+h) - G_n(\Phi)(s)|\\ =&|e^{-\beta_n(s+h)}\int_{-\infty}^{s+h} e^{\beta_n z}H_n(\Phi)(z)dz - e^{-\beta_n s}\int_{-\infty}^{s} e^{\beta_n z}H_n(\Phi)(z)dz|.\end{aligned}$$

If $h < 0$ then

$$
\begin{aligned}
&|e^{-\beta_n(s+h)} \int_{-\infty}^{s+h} e^{\beta_n z} H_n(\Phi)(z)dz - e^{-\beta_n s} \int_{-\infty}^{s} e^{\beta_n z} H_n(\Phi)(z)dz| \\
\leq &(e^{-\beta_n h} - 1) \int_{-\infty}^{s+h} e^{\beta_n(z-s)} |H_n(\Phi)(z)|dz + \int_{s+h}^{s} e^{\beta_n(z-s)} |H_n(\Phi)(z)|dz. \qquad (3.3)
\end{aligned}
$$

Note that there is some $M > 0$ such that $|H_n(\Phi)(s)| < M$ for all $s \in \mathbb{R}$, $1 \leq n \leq N$ and $\Phi(s) \in C_{3\boldsymbol{K}}(\mathbb{R}, \mathbb{R}^N)$. Therefore, (3.3) implies that each $G_n(\Phi)(s)$ is uniformly continuous on $\mathbb{R}$. Similarly, if $h > 0$, we can also obtain the same result. Hence, all $G_n(\Phi)(s)$ are uniformly continuous.

Next, we show that $G(\cdot)$ is continuous. Let $\Phi(s) = (\phi_1(s), \cdots, \phi_N(s))$, $\Psi(s) = (\psi_1(s), \cdots, \psi_N(s)) \in C_{3\boldsymbol{K}}(\mathbb{R}, \mathbb{R}^N)$. It is easy to see that

$$
\begin{aligned}
\|H(\Phi) - H(\Psi)\| &= \sup_{s\in\mathbb{R};1\leq n\leq N} |H_n(\Phi)(s) - H_n(\Psi)(s)| && (3.4) \\
&\leq \sup_{s\in\mathbb{R};1\leq n\leq N} \frac{|\phi_n(s) f_n\big(\Phi_s(-c\boldsymbol{\tau}_n)\big) - \psi_n(s) f_n\big(\Psi_s(-c\boldsymbol{\tau}_n)\big)|}{c} && (3.5) \\
&\quad + (\frac{4d_n}{c} + \beta_n)\|\Phi - \Psi\|. && (3.6)
\end{aligned}
$$

By the continuity of f_n and the compactness of $[0, 3\boldsymbol{K}]$, all f_n are uniformly continuous on $[0, 3\boldsymbol{K}]$. Therefore, by (3.4), the operator $H(\cdot)$ is continuous. Since

$$
\begin{aligned}
\|G(\Phi) - G(\Psi)\| &= \sup_{s\in\mathbb{R};1\leq n\leq N} |G_n(\Phi)(s) - G_n(\Psi)(s)| \\
&\leq \sup_{s\in\mathbb{R};1\leq n\leq N} e^{-\beta_n s} \int_{-\infty}^{s} e^{\beta_n z} |H_n(\Phi)(z) - H_n(\Psi)(z)|dz \\
&\leq \sup_{s\in\mathbb{R};1\leq n\leq N} \|H_n(\Phi) - H_n(\Psi)\| (e^{-\beta_n s} \int_{-\infty}^{s} e^{\beta_n z} dz) \\
&\leq (\max_{1\leq n\leq N} \frac{1}{\beta_n}) \|H(\Phi) - H(\Psi)\|,
\end{aligned}
$$

one can see that $G(\cdot)$ is also continuous. The proof is complete. □

Next, let $\Gamma(\check{\Phi}, \hat{\Phi})$ be the set of functions $\Phi(s) = (\phi_1(s), \cdots, \phi_N(s))$ satisfying the following two properties:

(1) $\Phi(s) \in C_{3\boldsymbol{K}}(\mathbb{R}, \mathbb{R}^N)$ with $\check{\Phi} \preceq \Phi \preceq \hat{\Phi}$;

(2) $e^{\beta_n s}[\hat{\phi}_n(s) - \phi_n(s)]$ and $e^{\beta_n s}[\phi_n(s) - \check{\phi}_n(s)]$ are both nondecreasing for $1 \leq n \leq N$.

Note that $(\check{\Phi}(s), \hat{\Phi}(s))$ is a pair of upper-lower solution of (1.6). Some properties of $G_n(\cdot)$ on $\Gamma(\check{\Phi}, \hat{\Phi})$ are further illustrated in the following lemmas.

Lemma 3.2. *Let $c > 0$ and $\Phi(s) \in \Gamma[\check{\Phi}, \hat{\Phi}]$ be fixed. Then we have*

$$\check{\phi}_n(s) \leq G_n[\check{\Phi}|\hat{\Phi}_{I_n^-}](s) \leq G_n(\Phi)(s) \leq G_n[\hat{\Phi}|\check{\Phi}_{I_n^-}](s) \leq \hat{\Phi}(s), \tag{3.7}$$

for any $s \in \mathbb{R}$ and $1 \leq n \leq N$.

Proof. We only prove $G_n(\Phi)(s) \leq G_n[\hat{\Phi}|\check{\Phi}_{I_n^-}](s) \leq \hat{\phi}_n(s)$ for any $s \in \mathbb{R}$ and $1 \leq n \leq N$. For the remainder part of (3.7), one can also prove it by the same arguments. By the definition of $G(\cdot)$ and inequality (3.1), we can obtain

$$\begin{aligned} G_n[\hat{\Phi}|\check{\Phi}_{I_n^-}](s) &= e^{-\beta_n s}\int_{-\infty}^{s} e^{\beta_n z} H_n[\hat{\Phi}|\check{\Phi}_{I_n^-}](z)\ dz \\ &\leq e^{-\beta_n s}\int_{-\infty}^{s} e^{\beta_n z}\big(\hat{\phi}_n'(z) + \beta_n \hat{\phi}_n(z)\big)\ dz = \hat{\phi}_n(s). \end{aligned}$$

To prove $G_n(\Phi)(s) \leq G_n[\hat{\Phi}|\check{\Phi}_{I_n^-}](s)$, it is sufficient to show that

$$H_n(\Phi)(s) \leq H_n[\hat{\Phi}|\check{\Phi}_{I_n^-}](s). \tag{3.8}$$

By direct computations, we have

$$\begin{aligned} & cH_n([\hat{\Phi}|\check{\Phi}_{I_n^-}])(s) - cH_n(\Phi)(s) \\ =& d_n\mathcal{L}(\hat{\phi}_n)(s) + \hat{\phi}_n(s) f_n\big([\hat{\Phi}|\check{\Phi}_{I_n^-}]_s(-c\boldsymbol{\tau}_n)\big) + c\beta_n\hat{\phi}_n(s) - d_n\mathcal{L}(\phi_n)(s) - \\ & \phi_n(s) f_n\big(\Phi_s(-c\boldsymbol{\tau}_n)\big) - c\beta_n\phi_n(s) \\ \geq& \hat{\phi}_n(s) f_n\big([\hat{\Phi}|\check{\Phi}_{I_n^-}]_s(-c\boldsymbol{\tau}_n)\big) - \phi_n(s) f_n\big([\hat{\Phi}|\check{\Phi}_{I_n^-}|(\phi_n)_{\{n\}}]_s(-c\boldsymbol{\tau}_n)\big) - \phi_n(s) f_n\big(\Phi_s(-c\boldsymbol{\tau}_n)\big) \\ & + \phi_n(s) f_n\big([\hat{\Phi}|\check{\Phi}_{I_n^-}|(\phi_n)_{\{n\}}]_s(-c\boldsymbol{\tau}_n)\big) + (c\beta_n - 2d_n)(\hat{\phi}_n(s) - \phi_n(s)), \end{aligned}$$

where

$$[\hat{\Phi}|\check{\Phi}_{I_n^-}|(\phi_n)_{\{n\}}]_j = \begin{cases} \phi_n, & \text{if } j = n, \\ \hat{\phi}_j, & \text{if } j \in I_n^+, \\ \check{\phi}_j, & \text{if } j \in I_n^-. \end{cases}$$

Since $\phi_n(s) \leq \hat{\phi}_n(s)$ for all $s \in \mathbb{R}$, it follows from Remark 2.6 that

$$\begin{aligned} 0 \leq& (c\beta_n - 2d_n)(\hat{\phi}_n(s) - \phi_n(s)) + \hat{\phi}_n(s) f_n\big([\hat{\Phi}|\check{\Phi}_{I_n^-}]_s(-c\boldsymbol{\tau}_n)\big) - \\ & \phi_n(s) f_n\big([\hat{\Phi}|\check{\Phi}_{I_n^-}|(\phi_n)_{\{n\}}]_s(-c\boldsymbol{\tau})\big). \end{aligned} \tag{3.9}$$

Moreover, by Mean Value Theorem and (H1), we have

$$\begin{aligned} & \phi_n(s) f_n\big([\hat{\Phi}|\check{\Phi}_{I_n^-}|(\phi_n)_{\{n\}}]_s(-c\boldsymbol{\tau})\big) - \phi_n(s) f_n\big(\Phi_s(-c\boldsymbol{\tau}_n)\big) \\ =& \phi_n(s) \sum_{j\in I_n^+} \frac{\partial f_n}{\partial x_j}(\boldsymbol{\xi})\big(\hat{\phi}_j(s - c\tau_{n,j}) - \phi_j(s - c\tau_{n,j})\big) + \\ & \phi_n(s) \sum_{j\in I_n^-} \frac{\partial f_n}{\partial x_j}(\boldsymbol{\xi})\big(\check{\phi}_j(s - c\tau_{n,j}) - \phi_j(s - c\tau_{n,j})\big) \geq 0. \end{aligned} \tag{3.10}$$

Then (3.9) and (3.10) imply that $H_n[\hat{\Phi}|\check{\Phi}_{I_n^-}](s) - H_n(\Phi)(s) \geq 0$ for all s. The proof is complete. □

Lemma 3.3. *Let $c > 0$ be fixed. Then the operator $G : \Gamma(\check{\Phi}, \hat{\Phi}) \to \Gamma(\check{\Phi}, \hat{\Phi})$ is compact.*

Proof. We first show that $G(\Gamma(\check{\Phi}, \hat{\Phi})) \subset \Gamma(\check{\Phi}, \hat{\Phi})$. Suppose $\Phi = (\phi_1, \cdots, \phi_N) \in \Gamma(\check{\Phi}, \hat{\Phi})$, then Lemma 3.2 implies that $\check{\Phi} \preceq G(\Phi) \preceq \hat{\Phi}$. By (3.1) and (3.8) and direct computations, we have

$$\begin{aligned}
&\frac{d}{ds} e^{\beta_n s}[\hat{\phi}_n(s) - G_n(\Phi)(s)] \\
=& e^{\beta_n s}\big[\beta_n \hat{\phi}_n(s) + \hat{\phi}_n'(s) - H_n(\Phi)(s)\big] \\
\geq& e^{\beta_n s}\big[\beta_n \hat{\phi}_n(s) + \hat{\phi}_n'(s) - H_n([\hat{\Phi}|\check{\Phi}_{I_n^-}])(s) + H_n([\hat{\Phi}|\check{\Phi}_{I_n^-}])(s) - H_n(\Phi)(s)\big] \\
\geq& 0
\end{aligned}$$

for all except finite s. Hence all $e^{\beta_n s}[\hat{\phi}_n(s) - G_n(\Phi)(s)]$ are nondecreasing for any $s \in \mathbb{R}$. Similarly, all $e^{\beta_n s}[G_n(\Phi)(s) - \check{\phi}_n(s)]$ are also nondecreasing. Hence $G(\Gamma(\check{\Phi}, \hat{\Phi})) \subset \Gamma(\check{\Phi}, \hat{\Phi})$.

The proof of compactness of operator G is similar to the proof in [23], so we omit it here. □

4 Existence and Nonexistence for Traveling Wave Solutions

Before to prove Theorem 1.2, we first provide two important lemmas which play important roles in estimating the lower bound of wave speeds.

Lemma 4.1. (See [6, Theorem 4].) *Let $c > 0$ and $B(\cdot)$ be a continuous function such that $\lim_{x \to \pm\infty} B(x) = B(\pm\infty) < \infty$. Suppose $z(\cdot)$ is a measurable function satisfying*

$$cz(x) = e^{\int_x^{x+1} z(s)ds} + e^{\int_x^{x-1} z(s)ds} + B(x) \quad \textit{for all } x \in \mathbb{R},$$

then z is uniformly continuous and bounded. Moreover, the limits $\lim_{x \to \pm\infty} z(x) = \omega^{\pm}$ exist and which are real roots of the equation

$$c\omega = e^{\omega} + e^{-\omega} + B(\pm\infty).$$

Lemma 4.2. *Suppose* (1.6) *has a positive solution satisfying* (1.7)*, then $c > 0$.*

Proof. Let $\Phi = (\phi_1, \cdots, \phi_N)$ be a positive solution of equations (1.6) satisfying (1.7), then $\lim_{t \to -\infty} \Phi(t) = \mathbf{0}$. Note that $f_n(\mathbf{0}) > 0$ and

$$f_n(\Phi_t(-c\boldsymbol{\tau}_n)) = f_n(\phi_1(t - c\tau_{n,1}), \cdots, \phi_N(t - c\tau_{n,N})).$$

The continuity of f_n implies that

$$\lim_{t\to-\infty} f_n(\Phi_t(-c\boldsymbol{\tau}_n)) = \lim_{t\to-\infty} f_n(\phi_1(t-c\tau_{n,1}),\cdots,\phi_N(t-c\tau_{n,N}))$$
$$= f_n(\lim_{t\to-\infty}\phi_1(t-c\tau_{n,1}),\cdots,\lim_{t\to-\infty}\phi_N(t-c\tau_{n,N})) = f_n(\mathbf{0}).$$

Then there exists a $\hat{L}\in\mathbb{R}$ such that $f_n(\Phi_t(-c\boldsymbol{\tau}_n)) > f_n(\mathbf{0})/2$ for $t\in(-\infty,\hat{L})$ and $1\le n\le N$. It's easy to check that the integrals

$$\int_{-\infty}^{L}\phi_n'(t)dt,\quad \int_{-\infty}^{L}\phi_n(t-1)-\phi_n(t)dt \quad\text{and}\quad \int_{-\infty}^{L}\phi_n(t+1)-\phi_n(t)dt \tag{4.1}$$

exist. Since Φ satisfies (1.6), by (4.1), the integral

$$\int_{-\infty}^{L}\phi_n(t)f_n(\Phi_t(-c\boldsymbol{\tau}_n))dt \tag{4.2}$$

also exists, which implies that $\int_{-\infty}^{\xi}\phi_n(t)dt$ converges for all $\xi\in(-\infty,\hat{L})$. Now we define $R_n(\xi) := \int_{-\infty}^{\xi}\phi_n(t)dt$ for all $\xi\in(-\infty,\hat{L})$. Since each $\phi_n>0$, then $R_n(\xi)$ is increasing, $R_n'(\xi)=\phi_n(\xi)$ and $\lim_{\xi\to-\infty}R_n(\xi)=0$. By direct computations, we have

$$c\phi_n(\xi) \ge d_n\int_{\xi}^{\xi+1}\phi_n(t)dt - d_n\int_{\xi-1}^{\xi}\phi_n(t)dt + \frac{f_n(\mathbf{0})}{2}\int_{-\infty}^{\xi}\phi_n(t)dt. \tag{4.3}$$

Then, integrating (4.3) over $(-\infty,\hat{L})$, we obtain that $c>0$. The proof is complete. $\square$

Proof of Theorem 1.2.

(1). Let $c^* := \tilde{c}$ and $\delta := \min\{\tau^*,\tilde{\tau}\}$, where $\tilde{c}$ is defined by (A.1), $\tilde{\tau}$ is defined in Lemma 2.2 and τ^* are defined in Lemma 2.5. We assume that $c>c^*$ and $\tau_{n,n}<\delta$ for all n. Then the statements in Lemma 2.5 hold. By Lemmas 3.1, 3.3 and the Schauder's fixed point theorem, (1.6) has a positive solution $\Phi(t) = (\phi_1(t),\cdots,\phi_N(t))$. On the other hand, by Lemma 2.5 again, the upper- and lower solutions satisfy (1.7). Therefore, it is easy to see that $\Phi(t)$ satisfies the conditions (1.7).

(2). Let $c_* := \min\{c_n \mid 1\le n\le N\}$, where each c_n is defined in Lemma 2.3. For $c<c_*$, we claim that (1.6) has no positive solution (with c as the parameter) satisfying the conditions of (1.7). Suppose the claim is false, i.e. there exists a solution of (1.6) satisfying (1.7) with $c<c_*$. Without loss of generality, we may assume that $c_*=c_1$. According to Lemma 4.2, we have $c>0$. Let $\rho(t) := \phi_1'(t)/\phi_1(t)$. By the equation (1.6) (with $n=1$), we know that

$$c\rho(t) = d_1(e^{\int_t^{t+1}\rho(s)ds} + e^{\int_t^{t-1}\rho(s)ds} - 2) + f_1(\Phi_t(-c\boldsymbol{\tau}_1)) = 0.$$

The result of Lemma 4.1 implies the equation

$$-c\lambda + d_1(e^{-\lambda}+e^{\lambda}-2) + f_1(\mathbf{0}) = 0$$

has one real root. Thus the definition of c_1 implies that $c\ge c_1=c_*$, which gives a contradiction. Hence the claim follows and the proof is complete.

5 Applications

In this section, we will apply our main theorem to show the existence of traveling wave solutions for various types of lattice reaction-diffusion systems. For convenience, we use the notation $D[u_i](t) := u_{i-1}(t) - 2u_i(t) + u_{i+1}(t)$ in the sequel of this section.

5.1 A Delayed Food-Limited Population Model

The discrete food-limited population model with delay can be described by

$$u_i'(t) = dD[u_i](t) + \frac{u_i(t)[K - u_i(t-\tau)]}{K + \gamma u_i(t-\tau)}, \tag{5.1}$$

where $t \in \mathbb{R}$, $i \in \mathbb{Z}$, $d, K > 0$ and $\tau, \gamma \geq 0$.

The simplest version of system (5.1) with $d = \gamma = \tau = 0$ is the well-known logistic system with the environmental carrying capacity K. When γ is positive and $d = \tau = 0$, the equation

$$u'(t) = \frac{u(t)[K - u(t)]}{K + \gamma u(t)} \tag{5.2}$$

was first proposed by Smith [33] as a food-limited mathematical model for population of daphnia magna. Moreover, equation (5.2) can also describe the effect of environmental toxicants on aquatic populations, see [15]. In the past, there are many mathematical results for equation (5.2). For examples, the global attractivity of trivial solution was studied in [11, 25, 26]; the oscillation of positive solutions, and the global attractivity of the positive equilibrium K with or without delay were investigated in [13, 34]. Furthermore, the delayed food-limited population model incorporating spatial dispersal was studied in [12, 31, 32].

Recently, the existence of traveling wave solutions for (5.1) was investigated by Huang, Lu, and Zou [20]. The authors proved the existence of traveling wave solutions connecting the trivial solution and the positive equilibrium K. However, by Theorem 1.2, we not only obtain the existence result of traveling wave solutions as in [20], but also derive the non-existence of traveling wave solutions.

More precisely, let $\phi(t)$ be a traveling wave solution of (5.1) connecting 0 and K, the profile equation yields

$$\phi'(t) = d(\phi(t-1) - 2\phi(t) + \phi(t+1)) + \phi(t)\frac{K - \phi(t - c\tau)}{K + \gamma\phi(t - c\tau)}, \quad t \in \mathbb{R}. \tag{5.3}$$

Then we look for solutions of (5.3) satisfying the following conditions:

$$\lim_{t\to-\infty} \phi(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} \phi(t) = K. \tag{5.4}$$

It is easy to check that the assumptions (H0)$\sim$(H3) hold. Hence we obtain the results stated in Theorem 1.2 for the food-limited population model.

5.2 Two Species Delayed Lotka-Volterra Ecological Models

The two species delayed Lotka-Volterra ecological model can be described by the following system:

$$\begin{cases} u_i'(t) = d_1 D[u_i](t) + \ u_i(t)\big(r_1 - a_{11}u_i(t-\tau_1) + a_{12}v_i(t-\tau_2)\big), \\ v_i'(t) = d_2 D[v_i](t) + v_i(t)\big(r_2 - a_{22}v_i(t-\hat{\tau}_1) + a_{21}u_i(t-\hat{\tau}_2)\big), \end{cases} \tag{5.5}$$

where $i \in \mathbb{Z}$, $t \in \mathbb{R}$, $d_n, r_n, a_{nn} > 0$, $\hat{\tau}_n, \tau_n \geq 0$ and $a_{12}, a_{21} \neq 0$.

System (5.5) arises from the study of the interaction between two species with discrete diffusion or migration when the habitat is of one-dimensional and divided into niches. According to the signs of parameters a_{12} and a_{21}, one can distinguish (5.5) into three types: *cooperative*, *competitive* and *predator-prey*. Recently, traveling wave solutions for competitive type has been studied in [14,18].

Here we apply Theorem 1.2 to (5.5) for $N = 2$. Without loss of generality, we may assume

$$r_2 \leq r_1 < 2r_2, \tag{5.6}$$

$$a_{11}a_{22} - a_{12}a_{21} > 0,\ r_1a_{22} + r_2a_{12} > 0 \text{ and } r_2a_{11} + r_1a_{21} > 0. \tag{5.7}$$

By direct computations, it is clear that

$$(k_1, k_2) = \Big(\frac{r_1a_{22} + r_2a_{12}}{a_{11}a_{22} - a_{12}a_{21}},\ \frac{r_2a_{11} + r_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}}\Big)$$

is the positive equilibrium of (5.5). It's easy to see that assumptions (H0), (H1) and (H3) hold. Therefore, we only have to verify assumption (H2). Note that inequalities in (H2) can be represented as the following form:

$$a_{11}k_1 > |a_{12}|k_2 \quad \text{and} \quad a_{22}k_2 > |a_{21}|k_1. \tag{5.8}$$

Then we consider the following four cases:

∘ Assume $a_{12} > 0$ and $a_{21} > 0$.

By the formula of (k_1, k_2) and (5.7), the condition (5.8) hold.

∘ Assume $a_{12} > 0$ and $a_{21} < 0$.

By the formula of (k_1, k_2), the condition (5.8) is equivalent to

$$-2a_{21}a_{22}r_1 < (a_{11}a_{22} + a_{12}a_{21})r_2. \tag{5.9}$$

∘ Assume $a_{12} < 0$ and $a_{21} > 0$.

By the formula of (k_1, k_2), the condition (5.8) is equivalent to

$$-2a_{11}a_{12}r_2 < (a_{12}a_{21} + a_{11}a_{22})r_1. \tag{5.10}$$

∘ Assume $a_{12} < 0$ and $a_{21} < 0$.

By the formula of (k_1, k_2), the condition (5.8) is equivalent to (5.9) and (5.10). Thus, we have the following result immediately.

Theorem 5.1. *Assume* (5.6) *and* (5.7) *hold. Then the statements of* Theorem 1.2 *hold for system* (5.5) *if one of the following conditions holds:*

(1) $a_{12} > 0$ *and* $a_{21} > 0$*;*

(2) $a_{12} > 0$, $a_{21} < 0$ *and condition* (5.9) *holds;*

(3) $a_{12} < 0$, $a_{21} > 0$ *and condition* (5.10) *holds;*

(4) $a_{12} < 0$, $a_{21} < 0$ *and conditions* (5.9) *and* (5.10) *hold.*

5.3 A Discrete Predator-Prey Model with Modified Leslie-Gower and Holling-Type II Schemes

The discrete predator-prey model with modified Leslie-Gower and Holling-Type II schemes can be described by

$$\begin{cases} \dfrac{du_i(t)}{dt} = d_1 D[u_i](t) + u_i(t)\big[a_1 - bu_i(t-\tau_1) - \dfrac{\alpha_1 v_i(t-\tau_2)}{u_i(t-\tau_1)+\gamma_1}\big], \\ \dfrac{dv_i(t)}{dt} = d_2 D[v_i](t) + v_i(t)\big[a_2 - \dfrac{\alpha_2 v_i(t-\hat{\tau}_1)}{u_i(t-\hat{\tau}_2)+\gamma_2}\big], \end{cases} \tag{5.11}$$

where $i \in \mathbb{Z}$, $t \in \mathbb{R}$, $\tau_1, \tau_2, \hat{\tau}_1, \hat{\tau}_2 \geq 0$ and all parameters $d_1, d_2, a_1, a_2, \alpha_1, \alpha_2, \gamma_1, \gamma_2, b$ are positive. System (5.11) can be regarded as the discrete version of the system

$$\begin{cases} u_t(x,t) = d_1 u_{xx}(x,t) + u(x,t)\big[a_1 - bu(x,t-\tau_1) - \dfrac{\alpha_1 v(x,t-\tau_2)}{u(x,t-\tau_1)+\gamma_1}\big], \\ v_t(x,t) = d_2 v_{xx}(x,t) + v(x,t)\big[a_2 - \dfrac{\alpha_2 v(x,t-\hat{\tau}_1)}{u(x,t-\hat{\tau}_2)+\gamma_2}\big]. \end{cases} \tag{5.12}$$

If $d_1 = d_2 = 0$, then the ODE system

$$\begin{cases} \dfrac{du}{dt} = u(a_1 - bu - \dfrac{\alpha_1 v}{u+\gamma_1}), \\ \dfrac{dv}{dt} = v(a_2 - \dfrac{\alpha_2 v}{u+\gamma_2}). \end{cases} \tag{5.13}$$

was first introduced by [1]. The authors of [1] studied the boundedness of solutions, existence of an attracting set and global stability of the coexisting interior equilibrium. Later, Zhou [38] considered the steady-state solutions of (5.12) for bounded domain Ω with a smooth boundary.

To apply our main result to system (5.11), we assume

$$d_1 = d_2 = d, a_1 = a_2 = a, \gamma_1 = \gamma_2 = 1 \tag{5.14}$$

and denote

$$(k_1, k_2) := \big(\frac{a\alpha_2 - a\alpha_1}{b\alpha_2}, \frac{a^2\alpha_2 - a^2\alpha_1 + ab\alpha_2}{b\alpha_2^2}\big).$$

One can easily verify that assumptions (H0)$\sim$(H3) hold if and only if

$$\alpha_2 > \alpha_1,\ \alpha_1 k_2 < b,\ \alpha_1 k_2(k_1+1) < a(1-\frac{\alpha_1}{\alpha_2}) \text{ and } k_1 < 1. \tag{5.15}$$

In fact, the condition (5.15) holds when α_2 and b are large and α_1 is small. Thus, we have the following result immediately.

Theorem 5.2. *Assume* (5.14) *holds. If α_1 is small and α_2, b are large enough such that* (5.15) *holds, then the statements of* Theorem 1.2 *hold for system* (5.11).

5.4 One Predator-Two Preys Model with Holling-Type II Response and Diffusion

Finally, we provide a one predator-two preys model with Holling-type II response and diffusion to illustrate our main result. The model is described as follows

$$\begin{cases} \dfrac{dx_i(t)}{dt} = d_1 D[x_i](t) + x_i(t)\big[b_1 - x_i(t-\tau_1) - \alpha y_i(t-\tau_2) - \dfrac{\eta z_i(t-\tau_3)}{1+\omega_1 x_i(t-\tau_1)}\big], \\ \dfrac{dy_i(t)}{dt} = d_2 D[y_i](t) + y_i(t)\big[b_2 - \beta x_i(t-\hat\tau_1) - y_i(t-\hat\tau_2) - \dfrac{\eta z_i(t-\hat\tau_3)}{1+\omega_2 y_i(t-\hat\tau_2)}\big], \\ \dfrac{dz_i(t)}{dt} = d_3 D[z_i](t) + z_i(t)\big[b_3 + \dfrac{d\eta x_i(t-\bar\tau_1)}{1+\omega_3 x_i(t-\bar\tau_1)} + \dfrac{d\eta y_i(t-\bar\tau_2)}{1+\omega_4 y_i(t-\bar\tau_2)} - \gamma z_i(t-\bar\tau_3)\big], \end{cases} \tag{5.16}$$

where $i \in \mathbb{Z}$, $t \in \mathbb{R}$ and all parameters $d_i, d, b_i, \alpha, \beta, \gamma, \omega_i$ are all nonnegative.

If $b_3 < 0$, $\omega_1 = \omega_3$, $\omega_2 = \omega_4$ and $d_1 = \cdots = d_3 = \gamma = 0$, the dynamics of the following ODE system

$$\begin{cases} \dfrac{dx(t)}{dt} = x(t)\big[b_1 - x(t) - \alpha y(t) - \dfrac{\eta z(t)}{1+\omega_1 x(t)}\big], \\ \dfrac{dy(t)}{dt} = y(t)\big[b_2 - \beta x(t) - y(t) - \dfrac{\eta z(t)}{1+\omega_2 y(t)}\big], \\ \dfrac{dz(t)}{dt} = z(t)\big[b_3 + \dfrac{d\eta x(t)}{1+\omega_1 x(t)} + \dfrac{d\eta y(t)}{1+\omega_2 y(t)}\big], \end{cases} \tag{5.17}$$

have been considered in [35]. To apply Theorem 1.2 to (5.16), we assume b_1, b_2 and b_3 are positive numbers,

$$\max_{i=1,2,3}\{b_i\} < 2\min_{i=1,2,3}\{b_i\},\ \alpha = \beta = \omega_3 = \omega_4 = 0,\ d\eta^2 = 1 \text{ and } \sqrt{\gamma} > \max\{\omega_1, \omega_2\}. \tag{5.18}$$

Then system (5.16) can be rewritten as the form

$$
\begin{cases}
\dfrac{dx_i(t)}{dt} = d_1 D[x_i](t) + x_i(t)\big[b_1 - x_i(t-\tau_1) - \dfrac{\eta z_i(t-\tau_3)}{1+\omega_1 x_i(t-\tau_1)}\big], \\
\dfrac{dy_i(t)}{dt} = d_2 D[y_i](t) + y_i(t)\big[b_2 - y_i(t-\hat{\tau}_2) - \dfrac{\eta z_i(t-\hat{\tau}_3)}{1+\omega_2 y_i(t-\hat{\tau}_2)}\big], \\
\dfrac{dz_i(t)}{dt} = d_3 D[z_i](t) + z_i(t)\big[b_3 + x_i(t-\bar{\tau}_1)/\eta + y_i(t-\bar{\tau}_2)/\eta - \gamma z_i(t-\bar{\tau}_3)\big],
\end{cases} \tag{5.19}
$$

To find the positive equilibrium of (5.19) is equivalent to find the positive solution of the following system

$$
\begin{cases}
\gamma b_1 - \eta b_3 + (\gamma\omega_1 b_1 - \gamma - 1)x - \gamma\omega_1 x^2 = y, \\
\gamma b_2 - \eta b_3 + (\gamma\omega_2 b_2 - \gamma - 1)y - \gamma\omega_2 y^2 = x, \\
b_3 + x/\eta + y/\eta = \gamma z.
\end{cases} \tag{5.20}
$$

Under the assumptions

$$
b_1\omega_1\gamma - \gamma - 1, \quad b_2\omega_2\gamma - \gamma - 1 > 0, \tag{5.21}
$$

$$
\Big[b_2\omega_2\gamma - \gamma - 1 + \sqrt{(b_2\omega_2\gamma - \gamma - 1)^2 + 4\gamma\omega_2(\gamma b_2 - \eta b_3)}\Big]/2\gamma\omega_2 < \gamma b_1 - \eta b_3, \tag{5.22}
$$

$$
\Big[b_1\omega_1\gamma - \gamma - 1 + \sqrt{(b_1\omega_1\gamma - \gamma - 1)^2 + 4\gamma\omega_1(\gamma b_1 - \eta b_3)}\Big]/2\gamma\omega_1 < \gamma b_2 - \eta b_3, \tag{5.23}
$$

it is easy to check that (5.20) has a unique positive solution. Note that (5.21), (5.22) and (5.23) hold for large γ, ω_1 and ω_2, since

$$
\lim_{\gamma\to\infty} \gamma b_1 - \eta b_3 = \lim_{\gamma\to\infty} \gamma b_2 - \eta b_3 = \infty, \tag{5.24}
$$

$$
\lim_{\gamma,\omega_1\to\infty} b_1\omega_1\gamma - \gamma - 1 = \lim_{\gamma,\omega_2\to\infty} b_2\omega_1\gamma - \gamma - 1 = \infty, \tag{5.25}
$$

$$
\lim_{\gamma,\omega_2\to\infty} \frac{b_2\omega_2\gamma - \gamma - 1 + \sqrt{(b_2\omega_2\gamma - \gamma - 1)^2 + 4\gamma\omega_2(\gamma b_2 - \eta b_3)}}{2\gamma\omega_2} = b_2, \tag{5.26}
$$

$$
\lim_{\gamma,\omega_1\to\infty} \frac{b_1\omega_1\gamma - \gamma - 1 + \sqrt{(b_1\omega_1\gamma - \gamma - 1)^2 + 4\gamma\omega_1(\gamma b_1 - \eta b_3)}}{2\gamma\omega_1} = b_1. \tag{5.27}
$$

In other words, if γ, ω_1 and ω_2 are large enough, then (H0) holds. Furthermore, one can verify that (H1) and (H2) also hold for large γ, ω_1 and ω_2. In conclusion, if γ, ω_1 and ω_2 are large enough and (5.18) is true, then (H0)$\sim$(H3) hold. Thus, we have the following result.

Theorem 5.3. *Assume* (5.18) *holds. If* γ, ω_1 *and* ω_2 *are large enough, then the statement of* Theorem 1.2 *holds for system* (5.16).

A Appendix

In this appendix, we will prove the result of Lemma 2.5. Let's recall that $\lambda_{n,1}$, $\lambda_{n,2}$, c_n, c_0 and $\tilde{\beta}_1, \cdots, \tilde{\beta}_N$ are the numbers defined in Lemma 2.3, Lemma 2.4 and Lemma 2.2 respectively. Then we define the number $\tilde{c}$ by

$$\tilde{c} := \max\{c_1, \cdots, c_N, c_0\}. \tag{A.1}$$

For any fixed $c > \tilde{c}$, we further introduce two numbers η and κ which satisfy

$$1 < \eta < \min\left\{\frac{\lambda_{n,2}}{\lambda_{n,1}}, \frac{\lambda_{n,1}+\lambda_{m,1}}{\lambda_{n,1}} \mid 1 \le n, m \le N\right\} \le 2; \tag{A.2}$$

$$\kappa \in \bigcap_{n=1}^{N} \big(\lambda_{n,1}, \min_{1\le j\le N}\{\lambda_{n,1}+\lambda_{j,1}, \lambda_{n,2}\}\big). \tag{A.3}$$

Note that κ is well-defined by Lemma 2.4. Let $\check{\Phi}(s) = (\check{\phi}_1(s), \cdots, \check{\phi}_N(s))$ and $\hat{\Phi}(s) = (\hat{\phi}_1(s), \cdots, \hat{\phi}_N(s))$ be the functions defined by (2.18) and (2.19) respectively. If $t_n, \hat{t}_n \in \mathbb{R}$ are fixed, then it is easy to see that $(\check{\Phi}(s), \hat{\Phi}(s))$ satisfies (1.7) for any $\gamma > 0$. To show that $(\hat{\Phi}(s), \check{\Phi}(s))$ is a pair of upper-lower solution of (1.6) and $e^{\beta_n t}(\check{\phi}_n(t) - \hat{\phi}_n(t))$ is nondecreasing in t, we first provide the following lemma.

Lemma A.1. *For any fixed n, there exist positive numbers $\zeta_{n,1}, \cdots, \zeta_{n,N}, L_n$ such that*

$$\frac{\partial f_n(X)}{\partial x_n} k_n \zeta_{n,n} + \sum_{j\in I_n^+} \frac{\partial f_n(X)}{\partial x_j} k_j \zeta_{n,j} + \sum_{j\in I_n^-} \frac{-\partial f_n(X)}{\partial x_j} k_j \zeta_{n,j} < -L_n \tag{A.4}$$

for all $\mathbf{0} \preceq X \preceq 3\boldsymbol{K}$, where the positive constants $\zeta_{n,n} \in (0,1)$, $\zeta_{n,j} \in (1,\infty)$ for $j \ne n$,

$$I_n^+ := \{j \ne n | \partial f_n/\partial x_j \ge 0\} \quad \text{and} \quad I_n^- := \{j \ne n | \partial f_n/\partial x_j < 0\}.$$

Proof. Since $[\mathbf{0}, 3\boldsymbol{K}]$ is compact and $\partial f_n/\partial x_j$ is continuous for any $1 \le n, j \le N$, (H1) and (H2) imply that

$$0 \le \max_{\mathbf{0}\preceq X\preceq 3\boldsymbol{K}} \big\{\big(\sum_{j\in I_n^+} \frac{\partial f_n(X)}{\partial x_j} k_j - \sum_{j\in I_n^-} \frac{\partial f_n(X)}{\partial x_j} k_j\big)/\big(-\frac{\partial f_n(X)}{\partial x_n} k_n\big)\big\} < 1.$$

Then there exists a rational number $p_2/p_1 > 1$ with $p_1, p_2 > 0$ such that

$$0 \le \frac{p_2}{p_1} \cdot \max_{\mathbf{0}\preceq X\preceq 3\boldsymbol{K}} \big\{\big(\sum_{j\in I_n^+} \frac{\partial f_n(X)}{\partial x_j} k_j - \sum_{j\in I_n^-} \frac{\partial f_n(X)}{\partial x_j} k_j\big)/\big(-\frac{\partial f_n(X)}{\partial x_n} k_n\big)\big\} < 1.$$

Moreover, for any $r \in (p_1, p_2)$, we have

$$\frac{\partial f_n(X)}{\partial x_n} k_n \frac{p_1}{r} + \sum_{j\in I_n^+} \frac{\partial f_n(X)}{\partial x_j} k_j \frac{p_2}{r} + \sum_{j\in I_n^-} \frac{-\partial f_n(X)}{\partial x_j} k_j \frac{p_2}{r} < 0.$$

Let

$$\zeta_{n,j} = \begin{cases} p_1/r, & \text{if } j = n, \\ p_2/r, & \text{if } j \neq n, \end{cases}$$

by the continuity of each $\partial f_n/\partial x_j$ and the compactness of $[\mathbf{0}, 3\boldsymbol{K}]$ again, the assertion of the lemma follows. □

According to the above lemma, we will choose the parameters q, m_n, β_n, σ, γ, t_n, $\hat{t}_n$, τ^* suitably in the sequel.

A.1 The choice of q and m_n

For any $q > 0$ and $1 \leq n \leq N$, we define two functions as follows

$$\check{h}_{n,q}(t) := e^{\lambda_{n,1} t} - q e^{\eta \lambda_{n,1} t} \text{ and } \hat{h}_{n,q}(t) := e^{\lambda_{n,1} t} + q k_n e^{\kappa t}.$$

By direct computations, one can verify that $\check{h}_{n,q}(\check{t}^*_{n,q}) = 0$ and $\check{h}_{n,q}(t)$ has a unique global maximum $m_{n,q}$ at $t = \check{t}_{n,q}$, where

$$m_{n,q} := (1 - \frac{1}{\eta})(\frac{1}{q\eta})^{\frac{1}{\eta-1}}, \ \check{t}_{n,q} := \frac{1}{\lambda_{n,1}(\eta - 1)} \ln(\frac{1}{q\eta}), \ \check{t}^*_{n,q} := \frac{1}{\lambda_{n,1}(\eta - 1)} \ln \frac{1}{q}.$$

Moreover, one can also verify that there are real numbers $\hat{t}_{n,q}$ and $\hat{t}^*_{n,q}$ such that

$$\hat{h}_{n,q}(\hat{t}_{n,q}) = k_n \quad \text{and} \quad \hat{h}_{n,q}(\hat{t}^*_{n,q}) = 3k_n.$$

It is clear that

$$\lim_{q\to\infty} \check{t}_{n,q} = \lim_{q\to\infty} \check{t}^*_{n,q} = \lim_{q\to\infty} \hat{t}_{n,q} = \lim_{q\to\infty} \hat{t}^*_{n,q} = -\infty. \tag{A.5}$$

By (A.5), we have the following lemma.

Lemma A.2. *There exists a positive constant $q^* > 1$ such that if $q > q^*$ then the following inequalities hold:*

(1) $\hat{t}^*_{n,q} + c, \check{t}^*_{n,q} + c \leq 0$;

(2) $q\Delta_n(\eta\lambda_{n,1}, c)e^{\eta\lambda_{n,1}t} + Me^{\lambda_{n,1}t}\big[e^{\lambda_{n,1}t} + \sum_{j\in I_n^-}(e^{\lambda_{j,1}t} + qk_j e^{\kappa t})\big] \leq 0$, *for* $t < \check{t}^*_{n,q}$;

(3) $q\Delta_n(\kappa, c)k_n e^{\kappa t} + M(e^{\lambda_{n,1}t} + qk_n e^{\kappa t})\sum_{j\in I_n^+} e^{\lambda_{j,1}t} \leq 0$, *for* $t < \hat{t}^*_{n,q}$,

where $1 \leq n \leq N$ and M is a constant defined by

$$M := \max_{1\leq n,j\leq N} \max_{\mathbf{0}\preceq X\preceq 3\boldsymbol{K}} |\partial f_n(X)/\partial x_j|.$$

Proof. (1) By (A.5), it is easy to see that the result holds if q is large enough.

(2) By (A.2) and part (1) of Lemma 2.3, we have $q\Delta_n(\eta\lambda_{n,1}, c) < 0$. Then, it follows from (A.2) and (A.3) that

$$\eta\lambda_{n,1} < \min\{2\lambda_{n,1}, \lambda_{n,1} + \lambda_{1,1}, \cdots, \lambda_{n,1} + \lambda_{N,1}, \lambda_{n,1} + \kappa\}.$$

Therefore, there exists a $T < 0$ such that if $t < T$ then

$$q\Delta_n(\eta\lambda_{n,1}, c)e^{\eta\lambda_{n,1}t} + Me^{\lambda_{n,1}t}\big[e^{\lambda_{n,1}t} + \sum_{j\in I_n^-}(e^{\lambda_{j,1}t} + qk_je^{\kappa t})\big] \leq 0.$$

By (A.5), if q is large enough, we have $\check{t}_{n,q}^* < T$ for all $1 \leq n \leq N$. The proof is complete.

(3) The assertion of this part follows by the same arguments as the proof of part (2). Hence we skip the details. □

Hereinafter, we always assume q and m_n be the numbers satisfying the following condition:

(N1) $q > q^*$ and $m_n := m_{n,q}$.

A.2 The choice of β_n and σ

Lemma A.3. *For any fixed $q > q^*$, there are positive numbers $\beta_1^*, \cdots, \beta_N^*$ and $\sigma^* > 0$ such that*

$$\frac{\beta_n + \lambda_{n,1}}{k_n\beta_n}e^{\lambda_{n,1}\check{t}_{n,q}} + \frac{(\beta_n + \kappa)q}{\beta_n}e^{\kappa\check{t}_{n,q}} > 1 - \frac{(k_n - m_n/\sigma)(\beta_n - \gamma)}{k_n\beta_n}$$

for $\beta_n > \beta_n^*$, $\sigma > \sigma^*$ and $\gamma \in (0, 1)$.

Proof. Since q is fixed, we have

$$\lim_{\sigma,\beta_n\to\infty} \frac{(k_n - m_n/\sigma)(\beta_n - \gamma)}{k_n\beta_n} = 1, \text{ for any } \gamma \in (0, 1).$$

Note that the above convergence is uniform with respect to $\gamma \in (0, 1)$. Therefore, one can easily obtain the assertion. The proof is complete. □

According to Lemma A.1 and Lemma A.3, we also let $\beta_1, \cdots, \beta_N$ and σ be fixed numbers which satisfy the following condition:

(N2) $\beta_n > \max\{\beta_n^*, \tilde{\beta}_n\}$ and

$$\sigma > \max\big\{1, \sigma^*, \frac{m_1}{k_1}, \cdots, \frac{m_N}{k_N}, \frac{m_1}{k_1(1-\zeta_{1,1})}, \cdots, \frac{m_N}{k_N(1-\zeta_{N,N})}\big\}.$$

A.3 The choice of γ, t_n and $\hat{t}_n$

To choose the parameters γ, t_n and $\hat{t}_n$, we consider the following two functions

$$\check{g}_{n,\gamma}(t) := k_n - (k_n - \frac{m_n}{\sigma})e^{-\gamma t} \quad \text{and} \quad \hat{g}_{n,\gamma}(t) := k_n + k_n e^{-\gamma t}.$$

Some properties of $\check{g}_{n,\gamma}(t)$ and $\hat{g}_{n,\gamma}(t)$ can be characterized in the following lemma.

Lemma A.4. *There is a positive number $\gamma^* < 1$ such that the following inequalities hold for $\gamma \in (0, \gamma^*)$ and $1 \le n \le N$.*

(1) $m_n/(2\sigma) < \check{g}_{n,\gamma}(t) < m_n/\sigma$, *for all* $t \in [\check{t}_{n,q}, \check{t}^*_{n,q}]$;

(2) $k_n < \hat{g}_{n,\gamma}(t) < 3k_n$, *for all* $t \in [\hat{t}_{n,q}, \hat{t}^*_{n,q}]$;

(3) $\lambda_{n,1}e^{\lambda_{n,1}t} + qk_n\kappa e^{\kappa t} - \gamma(k_n - m_n/\sigma)e^{-\gamma t}$ *for any* $t \in [\check{t}_{n,q}, \check{t}^*_{n,q}] \cup [\hat{t}^*_{n,q}, \hat{t}_{n,q}]$;

(4) $-(k_n - \frac{m_n}{\sigma})(c\gamma + d_n e^{\gamma} - 2d_n + d_n e^{-\gamma}) + \check{\phi}_n(\check{t}_n)L_n/2 \ge 0$ *for all n;*

(5) $c\gamma + d_n(e^{\gamma} - 2 + e^{-\gamma}) - k_n L_n/2 \le 0$ *for all n.*

Proof. The assertions of parts (4) and (5) are easy to see, therefore we only prove the results of parts (1)-(3).

(1) Since γ is positive, by part (1) of Lemma A.2, it is clear that

$$k_n - (k_n - m_n/\sigma)e^{-\gamma \check{t}_{n,q}} \le \check{g}_n(\gamma, t) \le k_n - (k_n - m_n/\sigma)e^{-\gamma \check{t}^*_{n,q}} \le m_n/\sigma$$

for all $t \in [\check{t}_{n,q}, \check{t}^*_{n,q}]$, which implies that $\check{g}_n(\gamma, t) < m_n/\sigma$ for any $t \in [\check{t}_{n,q}, \check{t}^*_{n,q}]$. Since the interval $[\check{t}_{n,q}, \check{t}^*_{n,q}]$ is compact and $\check{g}_n(0, t) = m_n/\sigma > m_n/(2\sigma)$ on this interval, one can verify that

$$\check{g}_{n,\gamma}(t) > m_n/(2\sigma), \text{ for small } \gamma \in (0, \gamma_1) \text{ and any } t \in [\check{t}_{n,q}, \check{t}^*_{n,q}].$$

Hence proof of this part is complete.

(2) The proof is similar to that of part (1) and we skip it here.

(3) Note that for any fixed q, the interval $[\check{t}_{n,q}, \check{t}^*_{n,q}] \cup [\hat{t}^*_{n,q}, \hat{t}_{n,q}]$ is bounded. Then it is easy to see that the function $\lambda_{n,1}e^{\lambda_{n,1}t} + qk_n\kappa e^{\kappa t}$ is positive and the function $(k_n - m_n/\sigma)e^{-\gamma t}$ is bounded on this interval. Therefore, if γ is small enough, then assertion follows obviously. The proof is complete. □

By part (1) of Lemma A.4, for any $\gamma \in (0, \gamma^*)$, we have

$$\begin{aligned} \check{g}_{n,\gamma}(\check{t}_{n,q}) - \check{h}_{n,q}(\check{t}_{n,q}) &= \check{g}_{n,\gamma}(\check{t}_{n,q}) - m_n < \check{g}_{n,\gamma}(\check{t}_{n,q}) - m_n/\sigma < 0, \\ \check{g}_{n,\gamma}(\check{t}^*_{n,q}) - \check{h}_{n,q}(\check{t}^*_{n,q}) &= \check{g}_{n,\gamma}(\check{t}^*_{n,q}) > 0. \end{aligned}$$

Thus, by Intermediate Value Theorem, one can see that graphs of functions $\check{g}_{n,\gamma}(t)$ and $\check{h}_{n,q}(t)$ intersect in the interval $(\check{t}_{n,q}, \check{t}^*_{n,q})$. Similarly, by part (2) of Lemma A.4, we also can obtain that graphs of functions $\hat{g}_{n,\gamma}(t)$ and $\hat{h}_{n,q}(t)$ intersect in the interval $(\hat{t}_{n,q}, \hat{t}^*_{n,q})$. Therefore, in the sequel, we assume the parameters γ, t_n and $\hat{t}_n$ satisfying the following assumption:

(**N3**) $\gamma \in (0, \gamma^*)$, $t_n \in (\check{t}_{n,q}, \check{t}^*_{n,q})$, $\hat{t}_n \in (\hat{t}_{n,q}, \hat{t}^*_{n,q})$,

$$\check{g}_{n,\gamma}(t_n) = \check{h}_{n,q}(t_n) \text{ and } \hat{g}_{n,\gamma}(\hat{t}_n) = \hat{h}_{n,q}(\hat{t}_n).$$

A.4 The choice of τ^*

According to the assumptions (**N1**)–(**N3**), we can represent the functions $\check{\Phi}(t)$ and $\hat{\Phi}(t)$ (see (2.18), (2.19) and Figure **??**) by the following equivalent forms:

$$\check{\phi}_n(t) = \max\{e^{\lambda_{n,1}t} - qe^{\eta\lambda_{n,1}t},\ k_n - (k_n - \frac{m_n}{\sigma})e^{-\gamma t}\}, \tag{A.6}$$

$$\hat{\phi}_n(t) = \min\{e^{\lambda_{n,1}t} + qk_n e^{\kappa t},\ k_n + k_n e^{-\gamma t}\}. \tag{A.7}$$

By the continuity of functions $\check{\Phi}(t)$ and $\hat{\Phi}(t)$, one can easily see that there exists a $\tau^* > 0$ such that the following result holds.

Lemma A.5. *There exists a $\tau^* \in (0, 1)$ such that if $\tau_{n,n} < \tau^*$ for $1 \leq n \leq N$, then the following assertions hold:*

(1) $e^{-\kappa c\tau_{n,n}} \geq \zeta_{n,n}$ *for all* $1 \leq n \leq N$;

(2) $|\check{\phi}_n(t - c\tau_{n,n}) - \check{\phi}_n(t_n)| < L_n/(2M)$ *for all* $t \in [t_n, t_n + c\tau_{n,n}]$;

(3) $|\hat{\phi}_n(t - c\tau_{n,n}) - \hat{\phi}_n(\hat{t}_n)| < L_n/(2M)$ *for all* $t \in [\hat{t}_n, \hat{t}_n + c\tau_{n,n}]$.

By the assumptions of (**N1**)–(**N3**), if $0 \leq \tau < \tau^*$, we are now ready to show the results of Lemma 2.5.

A.5 Proof of Lemma 2.5.

Let's prove the results of Lemma 2.5 by checking the following statements:

(1) $\hat{\phi}_n(t) \geq \check{\phi}_n(t)$ for all $t \in \mathbb{R}$;

(2) $e^{\beta t}(\hat{\phi}_n(t) - \check{\phi}_n(t))$ is nondecreasing in t;

(3) $c\check{\phi}'_n(t) \leq d_n\mathcal{L}(\check{\phi}_n)(t) + \check{\phi}_n(t)f_n\big([\check{\Phi}|\hat{\Phi}_{I_n^-}]_t(-c\boldsymbol{\tau}_n)\big)$;

(4) $c\hat{\phi}'_n(t) \geq d_n\mathcal{L}(\hat{\phi}_n)(t) + \hat{\phi}_n(t)f_n\big([\hat{\Phi}|\check{\Phi}_{I_n^-}]_t(-c\boldsymbol{\tau}_n)\big)$.

To check the inequality of (1), we consider the following four cases:

(1-1) $t \leq \min\{t_n, \hat{t}_n\}$; (1-2) $t \geq \max\{t_n, \hat{t}_n\}$;

(1-3) $t \in (\hat{t}_n, t_n)$ and $\hat{t}_n < t_n$; (1-4) $t \in (t_n, \hat{t}_n)$ and $t_n < \hat{t}_n$.

All the cases (1-1)–(1-4) can be checked by elementary computations. To avoid tedious computations, here we only consider the case (1-4). According to Figure **??**, we only have to show that

$$F_n(t) := e^{\lambda_{n,1}t} + qk_n e^{\kappa t} + (k_n - \frac{m_n}{\sigma})e^{-\gamma t} - k_n > 0$$

for $t \in (t_n, \hat{t}_n)$ and $1 \le n \le N$. By (**N3**), it is clear that

$$\begin{aligned} F_n(t_n) =& e^{\lambda_{n,1}t_n} + qk_n e^{\kappa t_n} - \big[k_n - (k_n - \frac{m_n}{\sigma})e^{-\gamma t_n}\big] \\ =& e^{\lambda_{n,1}t_n} + qk_n e^{\kappa t_n} - \big[e^{\lambda_{n,1}t_n} - qe^{\eta\lambda_{n,1}\check{t}_n}\big] > 0. \end{aligned} \tag{A.8}$$

Moreover, by part (3) of Lemma A.4 and (**N3**), we have

$$F_n'(t) = \lambda_{n,1}e^{\lambda_{n,1}t} + q\kappa k_n e^{\kappa t} - \gamma(k_n - \frac{m_n}{\sigma})e^{-\gamma t} > 0. \tag{A.9}$$

Then it follows from (A.8) and (A.9) that $F(t) > 0$ for all $t \in (t_n, \hat{t}_n)$. Hence the assertion of (1) follows.

Next, we check the statement of (2) by considering the following four cases:

(2-1) $t \le \min\{t_n, \hat{t}_n\}$; (2-2) $t \ge \max\{t_n, \hat{t}_n\}$;

(2-3) $t \in (\hat{t}_n, t_n)$ and $\hat{t}_n < t_n$; (2-4) $t \in (t_n, \hat{t}_n)$ and $t_n < \hat{t}_n$.

To avoid tedious computations, we also only consider the case (2-4). For convenience, we set

$$F_n(t) := e^{\beta_n t}(\hat{\phi}_n(t) - \check{\phi}_n(t)), \text{ for all } t \in (t_n, \hat{t}_n).$$

By (**N3**) and the result of part (1) of Lemma A.2, we have $\hat{t}_n < 0$ for all $1 \le n \le N$. Since $\hat{t}_n < 0$, Lemma A.3 implies that

$$\begin{aligned} F_n'(t) =& e^{\beta_n t}\big[(\beta_n + \lambda_{n,1})e^{\lambda_{n,1}t} + (\beta_n + \kappa)qk_n e^{\kappa t} - k_n\beta_n + (k_n - \frac{m_n}{\sigma})(\beta_n - \gamma)e^{-\gamma t}\big] \\ \ge& e^{\beta_n t}\big[(\beta_n + \lambda_{n,1})e^{\lambda_{n,1}t} + (\beta_n + \kappa)qk_n e^{\kappa t} - k_n\beta_n + (k_n - \frac{m_n}{\sigma})(\beta_n - \gamma)\big] > 0, \end{aligned}$$

for $t \in (t_n, \hat{t}_n)$. Therefore $e^{\beta t}(\hat{\phi}_n(t) - \check{\phi}_n(t))$ is nondecreasing in t.

To cheek the inequality of case (3), we consider the following two cases :

(3-1) $t < t_n$ and (3-2) $t \ge t_n$.

For case (3-1), by (2.18) and (2.19), we know that

$$\begin{aligned} & d_n\mathcal{L}(\check{\phi}_n)(t) + \check{\phi}_n(t)f_n\big([\check{\Phi}|\hat{\Phi}_{I_n^-}]_t(-c\boldsymbol{\tau}_n)\big) - c\check{\phi}_n'(t) \\ =& d_n(e^{\lambda_{n,1}(t-1)} - qe^{\eta\lambda_{n,1}(t-1)}) - 2d_n(e^{\lambda_{n,1}t} - qe^{\eta\lambda_{n,1}t}) + d_n(e^{\lambda_{n,1}(t+1)} - qe^{\eta\lambda_{n,1}(t+1)}) + \\ & \check{\phi}_n(t)f_n\big([\check{\Phi}|\hat{\Phi}_{I_n^-}]_t(-c\boldsymbol{\tau}_n)\big) - c\lambda_{n,1}(e^{\lambda_{n,1}t} - q\eta e^{\eta\lambda_{n,1}t}) \\ =& \check{\phi}_n(t)f_n\big([\check{\Phi}|\hat{\Phi}_{I_n^-}]_t(-c\boldsymbol{\tau}_n) - f_n(\mathbf{0})\big) - q\Delta_n(\eta\lambda_{n,1}, c)e^{\eta\lambda_{n,1}t}. \end{aligned} \tag{A.10}$$

Then, applying the Mean Value Theorem, we have

$$\begin{aligned}
&\check{\phi}_n(t)\big(f_n([\check{\Phi}|\hat{\Phi}_{I_n^-}]_t(-c\boldsymbol{\tau}_n)) - f_n(\mathbf{0})\big)\\
=&\check{\phi}_n(t)\Big[\frac{\partial f_n(\boldsymbol{\xi}_n)}{\partial x_n}\check{\phi}_n(t-c\tau_{n,n}) + \sum_{j\in I_n^+}\frac{\partial f_n(\boldsymbol{\xi}_j)}{\partial x_j}\check{\phi}_j(t-c\tau_{n,j}) + \sum_{j\in I_n^-}\frac{\partial f_n(\boldsymbol{\xi}_j)}{\partial x_j}\hat{\phi}_j(t-c\tau_{n,j})\Big],\\
\geq&\check{\phi}_n(t)\Big[\frac{\partial f_n(\boldsymbol{\xi}_n)}{\partial x_n}\check{\phi}_n(t-c\tau_{n,n}) + \sum_{j\in I_n^-}\frac{\partial f_n(\boldsymbol{\xi}_j)}{\partial x_j}\hat{\phi}_j(t-c\tau_{n,j})\Big],\\
\geq&-Me^{\lambda_{n,1}t}\big[e^{\lambda_{n,1}t} + \sum_{j\in I_n^-}(e^{\lambda_{j,1}t} + qk_je^{\kappa t})\big],
\end{aligned} \tag{A.11}$$

where M is the number defined in Lemma A.2 and $\boldsymbol{\xi}_1,\cdots,\boldsymbol{\xi}_N \in \mathbb{R}^N$ are vectors which depend on t. Note that $t_n < \check{t}_{n,q}$. By the equations (A.10), (A.11) and the result of part (2) of Lemma A.2, we obtain

$$-c\check{\phi}_n'(t) + d_n\mathcal{L}(\check{\phi}_n)(t) + \check{\phi}_n(t)f_n\big([\check{\Phi}|\hat{\Phi}_{I_n^-}]_t(-c\boldsymbol{\tau}_n)\big) \geq 0,$$

i.e. inequality (2.1) holds for $t < t_n$.

For case (3-2), by (2.18) and (2.19) again, we obtain that

$$\begin{aligned}
&d_n\mathcal{L}(\check{\phi}_n)(t) + \check{\phi}_n(t)f_n\big([\check{\Phi}|\hat{\Phi}_{I_n^-}]_t(-c\boldsymbol{\tau}_n)\big) - c\check{\phi}_n'(t)\\
\geq&d_n\big(k_n - (k_n - \frac{m_n}{\sigma})e^{-\gamma(t-1)}\big) - 2d_n\big(k_n - (k_n - \frac{m_n}{\sigma})e^{-\gamma t}\big) +\\
&d_n\big(k_n - (k_n - \frac{m_n}{\sigma})e^{-\gamma(t+1)}\big) + \check{\phi}_n(t)f_n\big([\check{\Phi}|\hat{\Phi}_{I_n^-}]_t(-c\boldsymbol{\tau}_n)\big) - c\gamma(k_n - \frac{m_n}{\sigma})e^{-\gamma t}\\
=&-(k_n - \frac{m_n}{\sigma})e^{-\gamma t}(c\gamma + d_ne^{\gamma} - 2d_n + d_ne^{-\gamma}) + \check{\phi}_n(t)f_n\big([\check{\Phi}|\hat{\Phi}_{I_n^-}]_t(-c\boldsymbol{\tau}_n)\big).
\end{aligned} \tag{A.12}$$

Since $f_n(\boldsymbol{K}) = 0$ for all $1 \leq n \leq N$, by (A.6), (A.7) and Mean Value Theorem again, we have

$$\begin{aligned}
&f_n\big([\check{\Phi}|\hat{\Phi}_{I_n^-}]_t(-c\boldsymbol{\tau}_n)\big)\\
=&\frac{\partial f_n(\boldsymbol{\xi}_n)}{\partial x_n}(\check{\phi}_n(t-c\tau_{n,n}) - k_n) + \sum_{j\in I_n^+}\frac{\partial f_n(\boldsymbol{\xi}_j)}{\partial x_j}(\check{\phi}_j(t-c\tau_{n,j}) - k_j) +\\
&\qquad\sum_{j\in I_n^-}\frac{\partial f_n(\boldsymbol{\xi}_j)}{\partial x_j}(\hat{\phi}_j(t-c\tau_{n,j}) - k_j)\\
\geq&\frac{\partial f_n(\boldsymbol{\xi}_n)}{\partial x_n}(\check{\phi}_n(t-c\tau_{n,n}) - k_n) + \sum_{j\in I_n^+}\frac{-\partial f_n(\boldsymbol{\xi}_j)}{\partial x_j}(k_j - \frac{m_j}{\sigma})e^{-\gamma(t-c\tau_{n,j})} +\\
&\qquad\sum_{j\in I_n^-}\frac{\partial f_n(\boldsymbol{\xi}_j)}{\partial x_j}k_je^{-\gamma(t-c\tau_{n,j})}.
\end{aligned} \tag{A.13}$$

We further consider two sub-cases:

$$\text{(3-2-1) } t_n < t \le t_n + c\tau_{n,n} \quad \text{and} \quad \text{(3-2-2) } t > t_n + c\tau_{n,n}.$$

For case (3-2-1), by the part (2) of Lemma A.5, we have

$$|\check{\phi}_n(t - c\tau_{n,n}) - \check{\phi}_n(t_n)| < L_n/(2M). \tag{A.14}$$

Then, by (A.14), part (1) of Lemma A.2, Lemma A.1, the definition of σ and equation (A.13), one can see that

$$\begin{aligned}
&f_n\big([\check{\Phi}|\hat{\Phi}_{I_n^-}]_t(-c\boldsymbol{\tau}_n)\big)\\
\ge& e^{-\gamma t}\Big[\sum_{j\in I_n^-}\frac{\partial f_n(\boldsymbol{\xi}_j)}{\partial x_j}k_j e^{\gamma c\tau_{n,j}} - \sum_{j\in I_n^+}\frac{\partial f_n(\boldsymbol{\xi}_j)}{\partial x_j}(k_j - \frac{m_j}{\sigma})e^{\gamma c\tau_{n,j}}\Big] -\\
&e^{-\gamma t}\Big[\frac{\partial f_n(\boldsymbol{\xi}_n)}{\partial x_n}(k_n - \frac{m_n}{\sigma})e^{-\gamma(t_n - t)}\Big] + \frac{\partial f_n(\boldsymbol{\xi}_n)}{\partial x_n}\frac{L_n}{2M}\\
\ge& e^{-\gamma t}\Big[\sum_{j\in I_n^+}\frac{-\partial f_n(\boldsymbol{\xi}_j)}{\partial x_j}k_j e^{\gamma c\tau_{n,j}} + \sum_{j\in I_n^-}\frac{\partial f_n(\boldsymbol{\xi}_j)}{\partial x_j}k_j e^{\gamma c\tau_{n,j}}\Big] -\\
&e^{-\gamma t}\Big[\frac{-\partial f_n(\boldsymbol{\xi}_n)}{\partial x_n}(k_n - \frac{m_n}{\sigma})\Big] - L_n/2 \ge e^{-\gamma t}L_n/2.
\end{aligned} \tag{A.15}$$

Combining the equations (A.12), (A.15), part (4) of Lemma A.4 and the inequality

$$\check{\phi}_n(t) \ge \check{\phi}_n(t_n) = k_n - (k_n - \frac{m_n}{\sigma})e^{-\gamma t_n} \quad \text{for} \quad t \ge t_n, \tag{A.16}$$

we obtain

$$\begin{aligned}
&- c\check{\phi}_n'(t) + d_n\mathcal{L}(\check{\phi}_n)(t) + \check{\phi}_n(t)f_n\big([\check{\Phi}|\hat{\Phi}_{I_n^-}]_t(-c\boldsymbol{\tau}_n)\big)\\
\ge& - e^{-\gamma t}\big[(k_n - \frac{m_n}{\sigma})(c\gamma + d_n e^{\gamma} - 2d_n + d_n e^{-\gamma}) - \check{\phi}_n(\check{t}_n)L_n/2\big] \ge 0.
\end{aligned}$$

For the case (3-2-2), by (A.13), it is easy to check that

$$f_n\big([\check{\Phi}|\hat{\Phi}_{I_n^-}]_t(-c\boldsymbol{\tau}_n)\big) \ge e^{-\gamma t}L_n. \tag{A.17}$$

Similarly, combining the inequalities (A.12), (A.16), (A.17) and result of part (4) of Lemma A.4, we can also show that

$$-c\check{\phi}_n'(t) + d_n\mathcal{L}(\check{\phi}_n)(t) + \check{\phi}_n(t)f_n\big([\check{\Phi}|\hat{\Phi}_{I_n^-}]_t(-c\boldsymbol{\tau}_n)\big) \ge 0$$

for $t > t_n + c\tau_{n,n}$.

Finally, let's check the inequality of (4). Similarly, we also consider the following two cases:

$$\text{(4-1) } t < \hat{t}_n \quad \text{and} \quad \text{(4-2) } t \ge \hat{t}_n.$$

For case (4-1), by (2.7) and direct computations, we have

$$\begin{aligned}&-c\hat{\phi}_n'(t)+d_n\mathcal{L}(\hat{\phi}_n)(t)+\hat{\phi}_n(t)f_n([\hat{\Phi}|\check{\Phi}_{I_n^-}]_t(-c\boldsymbol{\tau}_n))\\ \le& d_n\big[e^{\lambda_{n,1}(t-1)}+k_nqe^{\kappa(t-1)}+e^{\lambda_{n,1}(t+1)}+k_nqe^{\kappa(t+1)}-2(e^{\lambda_{n,1}t}+k_nqe^{\kappa t})\big]+\\ &\hat{\phi}_n(t)f_n([\hat{\Phi}|\check{\Phi}_{I_n^-}]_t(-c\boldsymbol{\tau}_n))-c\big(\lambda_{n,1}e^{\lambda_{n,1}t}+k_nq\kappa e^{\kappa t}\big)\\ =&qk_n\Delta_n(\kappa,c)e^{\kappa t}+\hat{\phi}_n(t)(f_n([\hat{\Phi}|\check{\Phi}_{I_n^-}]_t(-c\boldsymbol{\tau}_n))-f_n(\mathbf{0})).\end{aligned} \tag{A.18}$$

Applying the Mean Value Theorem and using (A.7), we can derive

$$\begin{aligned}f_n([\hat{\Phi}|\check{\Phi}_{I_n^-}]_t(-c\boldsymbol{\tau}_n))-f_n(\mathbf{0})\le&\frac{\partial f_n(\boldsymbol{\xi}_n)}{\partial x_n}\hat{\phi}_n(t-c\tau_{n,n})+\sum_{j\in I_n^+}\frac{\partial f_n(\boldsymbol{\xi}_j)}{\partial x_j}\hat{\phi}_j(t-c\tau_{n,j})\\ \le&\frac{\partial f_n(\boldsymbol{\xi}_n)}{\partial x_n}e^{\lambda_{n,1}(t-c\tau_{n,n})}+\sum_{j\in I_n^+}\frac{\partial f_n(\boldsymbol{\xi}_j)}{\partial x_j}e^{\lambda_{j,1}(t-c\tau_{n,j})}+\\ &\Big[\frac{\partial f_n(\boldsymbol{\xi}_n)}{\partial x_n}k_ne^{-\kappa c\tau_{n,n}}+\sum_{j\in I_n^+}\frac{\partial f_n(\boldsymbol{\xi}_j)}{\partial x_j}k_je^{-\kappa c\tau_{n,j}}\Big]qe^{\kappa t}.\end{aligned} \tag{A.19}$$

Moreover, by part (1) of Lemma A.5, we have

$$\frac{\partial f_n(\boldsymbol{\xi}_n)}{\partial x_n}k_ne^{-\kappa c\tau_{n,n}}+\sum_{j\in I_n^+}\frac{\partial f_n(\boldsymbol{\xi}_j)}{\partial x_j}k_je^{-\kappa c\tau_{n,j}}<\frac{\partial f_n(\boldsymbol{\xi}_n)}{\partial x_n}k_ne^{-\kappa c\tau_{n,n}}+\sum_{j\in I_n^+}\frac{\partial f_n(\boldsymbol{\xi}_j)}{\partial x_j}k_j<-L_n. \tag{A.20}$$

Then, by (H1), (A.19) and (A.20), we know that

$$f_n([\hat{\Phi}|\check{\Phi}_{I_n^-}]_t(-c\boldsymbol{\tau}_n))-f_n(\mathbf{0})\le M\sum_{j\in I_n^+}e^{\lambda_{j,1}t}. \tag{A.21}$$

Hence, by this inequalities (A.18), (A.21) and result of part (3) of Lemma A.2, we obtain

$$\begin{aligned}&-c\hat{\phi}_n'(t)+d_n\mathcal{L}(\hat{\phi}_n)(t)+\hat{\phi}_n(t)f_n([\hat{\Phi}|\check{\Phi}_{I_n^-}]_t(-c\boldsymbol{\tau}_n))\\ \le& q\Delta_n(\kappa,c)k_ne^{\kappa t}+M(e^{\lambda_{n,1}t}+qk_ne^{\kappa t})\sum_{j\in I_n^+}e^{\lambda_{j,1}t}\le 0.\end{aligned}$$

Therefore, the inequality (2.2) holds for $t<\hat{t}_n$.

Next, we consider the case (4-2). By (2.18) and (2.19) again, we have

$$\begin{aligned}&-c\hat{\phi}_n'(t)+d_n\mathcal{L}(\hat{\phi}_n)(t)+\hat{\phi}_n(t)f_n([\hat{\Phi}|\check{\Phi}_{I_n^-}]_t(-c\boldsymbol{\tau}_n))\\ \le& e^{-\gamma t}k_n[c\gamma+d_n(e^{\gamma}-2+e^{-\gamma})]+\hat{\phi}_n(t)f_n([\hat{\Phi}|\check{\Phi}_{I_n^-}]_t(-c\boldsymbol{\tau}_n)).\end{aligned} \tag{A.22}$$

Since $f_n(\boldsymbol{K}) = 0$ for all $1 \le n \le N$, by Mean Value Theorem again, we can also obtain the following inequality

$$\begin{aligned}
&f_n([\hat{\Phi}|\check{\Phi}_{I_n^-}]_t(-c\boldsymbol{\tau}_n)) \\
=&\frac{\partial f_n(\boldsymbol{\xi}_n)}{\partial x_n}\big(\hat{\phi}_n(t-c\tau_{n,n})-k_n\big)+\sum_{j\in I_n^+}\frac{\partial f_n(\boldsymbol{\xi}_j)}{\partial x_j}\big(\hat{\phi}_j(t-c\tau_{n,j})-k_n\big)+ \\
&\qquad\qquad\qquad\qquad\qquad\qquad \sum_{j\in I_n^-}\frac{\partial f_n(\boldsymbol{\xi}_j)}{\partial x_j}\big(\check{\phi}_j(t-c\tau_{n,j})-k_n\big) \\
\le&\frac{\partial f_n(\boldsymbol{\xi}_n)}{\partial x_n}\big(\hat{\phi}_n(t-c\tau_{n,n})-k_n\big)+e^{-\gamma t}\sum_{j\in I_n^+}\frac{\partial f_n(\boldsymbol{\xi}_j)}{\partial x_j}k_je^{\gamma c\tau_{n,j}}+ \\
&\qquad\qquad\qquad\qquad\qquad\qquad e^{-\gamma t}\sum_{j\in I_n^-}\frac{-\partial f_n(\boldsymbol{\xi}_j)}{\partial x_j}k_je^{\gamma c\tau_{n,j}}. \qquad \text{(A.23)}
\end{aligned}$$

Similarly, we consider the following two sub-cases:

$$\text{(4-2-1) } \hat{t}_n \le t \le \hat{t}_n + c\tau_{n,n} \quad \text{and} \quad \text{(4-2-2) } t > \hat{t}_n + c\tau_{n,n}.$$

For case (4-2-1), by part (3) of Lemma A.5, we have

$$|\hat{\phi}_n(t-c\tau_{n,n}) - \hat{\phi}_n(\hat{t}_n)| < L_n/(2M).$$

Furthermore, the result of part (1) of Lemma A.2 and equation (A.23) imply that

$$\begin{aligned}
&f_n([\hat{\Phi}|\check{\Phi}_{I_n^-}]_t(-c\boldsymbol{\tau}_n)) \\
\le&e^{-\gamma t}\Big[\sum_{j\in I_n^+}\frac{\partial f_n(\boldsymbol{\xi}_{n,j})}{\partial x_j}k_je^{\gamma c\tau_{n,j}}-\sum_{j\in I_n^-}\frac{\partial f_n(\boldsymbol{\xi}_{n,j})}{\partial x_j}k_je^{\gamma c\tau_{n,j}}+\frac{\partial f_n(\boldsymbol{\xi}_n)}{\partial x_n}k_ne^{-\gamma(\hat{t}_n-t)}\Big]+ \\
&-\frac{\partial f_n(\boldsymbol{\xi}_n)}{\partial x_n}\frac{L_n}{2M} \\
\le&e^{-\gamma t}\Big[\frac{\partial f_n(\boldsymbol{\xi}_n)}{\partial x_n}k_n+\sum_{j\in I_n^+}\frac{\partial f_n(\boldsymbol{\xi}_{n,j})}{\partial x_j}k_je^{\gamma c\tau_{n,j}}+\sum_{j\in I_n^-}\frac{-\partial f_n(\boldsymbol{\xi}_{n,j})}{\partial x_j}k_je^{\gamma c\tau_{n,j}}\Big]+\frac{L_n}{2} \\
\le&e^{-\gamma t}(-L_n+L_n/2) = -e^{-\gamma t}L_n/2. \qquad \text{(A.24)}
\end{aligned}$$

Therefore, combining the equations (A.22) and (A.24), the part (5) of Lemma A.4 and the fact

$$\hat{\phi}_n(t) \ge k_n, \text{ for all } t \ge \hat{t}_n, \qquad \text{(A.25)}$$

we have

$$\begin{aligned}
&-c\hat{\phi}_n'(t)+d_n\boldsymbol{\mathcal{L}}(\hat{\phi}_n)(t)+\hat{\phi}_n(t)f_n([\hat{\Phi}|\check{\Phi}_{I_n^-}]_t(-c\boldsymbol{\tau}_n)) \\
\le&e^{-\gamma t}k_n\big(c\gamma+d_n(e^{\gamma}-2+e^{-\gamma})-k_nL_n/2\big)\le 0.
\end{aligned}$$

For case (4-2-2), by (A.23), we see that

$$\begin{aligned}
&f_n\big([\hat{\Phi}|\check{\Phi}_{I_n^-}]_t(-c\boldsymbol{\tau}_n)\big)\\
\le& e^{-\gamma t}\Big[\sum_{j\in I_n^+\cup\{n\}}\frac{\partial f_n(\boldsymbol{\xi}_j)}{\partial x_j}k_j e^{\gamma c\tau_{n,j}}+\sum_{j\in I_n^-}\frac{-\partial f_n(\boldsymbol{\xi}_j)}{\partial x_j}k_j e^{\gamma c\tau_{n,j}}\Big]\le -L_n e^{-\gamma t}. \qquad (A.26)
\end{aligned}$$

Therefore, by (A.22), (A.25), (A.26) and the part (5) of Lemma A.4, we have

$$\begin{aligned}
&-c\hat{\phi}_n'(t)+d_n\mathcal{L}(\hat{\phi}_n)(t)+\hat{\phi}_n(t)f_n([\hat{\Phi}|\check{\Phi}_{I_n^-}]_t(-c\boldsymbol{\tau}_n))\\
\le& e^{-\gamma t}k_n\big(c\gamma+d_n(e^{\gamma}-2+e^{-\gamma})-k_nL_n\big)\le 0.
\end{aligned}$$

The proof is complete.

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