Results. Math. 62 (2012), 167–179

1422-6383/12/010167-13
published online May 5, 2011
DOI 10.1007/s00025-011-0137-6

Results in Mathematics

Incomplete Inverse Spectral and Nodal Problems for Differential Pencils

S. A. Buterin and C.-T. Shieh

Abstract. We prove uniqueness theorems for so-called half inverse spectral problem (and also for some its modification) for second order differential pencils on a finite interval with Robin boundary conditions. Using the obtained result we show that for unique determination of the pencil it is sufficient to specify the nodal points only on a part of the interval slightly exceeding its half.

Mathematics Subject Classification (2000). 34A55, 34B24, 47E05.

Keywords. Dfferential pencils, inverse spectral problems, inverse nodal problems, half inverse problem, oscillation theorem, nodal points.

1. Introduction

Consider the differential pencil $L = L(q_0(x), q_1(x), h, H)$ of the form

$$y'' + (\rho^2 - 2\rho q_1(x) - q_0(x))y = 0, \quad 0 < x < 1, \tag{1}$$

$$U(y) := y'(0) - hy(0) = 0, \quad V(y) := y'(1) + Hy(1) = 0, \tag{2}$$

where ρ is a spectral parameter, $q_j(x) \in W_1^j[0,1]$ are complex-valued functions, $h, H \in \mathbb{C}$. Differential equations with a nonlinear dependence on the spectral parameter frequently appear in mathematics as well as in applications (see [1–5]). We study inverse spectral and inverse nodal problems for differential pencil (1), (2) and establish a connection between them. Inverse spectral problems consist in recovering operators from given their spectral characteristics. Such problems play an important role in mathematics and have many applications in natural sciences and engineering (see [6–11, and the references therein]). Some aspects of inverse spectral problems for second order differential pencils were studied in [12–15] and other papers. In particular, in [15] it is proved that L is uniquely determined by specifying its Weyl function, which is equivalent to specification of the spectra of two boundary value problems for equation (1) with one common boundary condition. Inverse nodal problems, in turn, consist in constructing operators from given nodes (zeros) of their eigenfunctions (refer to [16–27]). From the physical point of view this corresponds to finding, e.g., the density of a string or a beam from the zero-amplitude positions of their eigenvibrations. In [25,26] inverse nodal problems were studied for equation (1) in the case of real coefficients, where it was proved in particular that the pencil L is determined uniquely by specifying a dense set of nodal points, provided that the mean value of $q_1(x)$ is known a priori and $q_1(x) \neq const.$ Otherwise we have a two-parametric set of pencils $L(q_0(x) + C_0, C_1, h, H)$ possessing the same nodal points for all constants C_0 and C_1. In [27] an inverse nodal problem was studied for second order differential pencils on a star-shaped graph.

In the first part of the paper we study so-called half inverse spectral problem for the pencil L, which consists in recovering the coefficients of (1), (2) from its spectrum $\{\rho_n\}$, provided that they are known a priori on one half of the segment $[0, 1]$, and prove the uniqueness of its solution. We note that in [28] a half inverse spectral problem for L was considered under an excessive assumption that $q_1(x)$ is known on the whole segment. An analogous problem for the Sturm–Liouville operator ($q_1(x) \equiv 0$) was investigated in [29–33] and other works. We also show that the lack of an arbitrary finite number of given eigenvalues can be compensated by specifying the functions $q_0(x)$, $q_1(x)$ on an interval $(0, a)$ for any $a > 1/2$. Further, using this fact and developing the idea of the works [22,24] we prove that for unique determination of L it is sufficient to specify the nodal points only on $(0, a), a > 1/2$, together with the mean values of the functions $q_0(x), q_1(x)$.

The paper is organized as follows. In the next section we prove uniqueness theorems for the incomplete inverse spectral problem. For the convenience of the reader in Sect. 3 some known results related to the inverse nodal problem for L are provided. In Sect. 4 we obtain additional properties of nodal points and prove a uniqueness theorem of the incomplete inverse nodal problem. The main results of the paper are contained in Theorems 1, 2, 6.

2. Incomplete Inverse Spectral Problems

Let the functions $\varphi(x, \rho), \psi(x, \rho), S(x, \rho)$ and $\Phi(x, \rho)$ be solutions of equation (1) under the conditions

$$\varphi(0, \rho) = \psi(1, \rho) = S'(0, \rho) = U(\Phi) = 1,$$
$$U(\varphi) = V(\psi) = S(0, \rho) = V(\Phi) = 0.$$

The functions $\Phi(x,\rho)$ and $M(\rho) := \Phi(0,\rho)$ are called respectively the Weyl solution and the Weyl function of the pencil L. For each fixed $x \in [0,1]$ the functions $\varphi(x,\rho), \psi(x,\rho), S(x,\rho)$ together with their derivatives with respect to x are entire in ρ. The eigenvalues ρ_n of L with account of multiplicity coincide with the zeros of its characteristic function $\Delta(\rho) := \langle \psi(x,\rho), \varphi(x,\rho)\rangle$, where $\langle y,z\rangle := yz' - y'z$. Clearly $\Delta(\rho) = V(\varphi) = -U(\psi)$. The function $\Delta_0(\rho) := \psi(0,\rho)$ is a characteristic function of the boundary value problem for equation (1) with the boundary conditions

$$y(0) = V(y) = 0. \tag{3}$$

Let $\{\rho_n^0\}$ be the spectrum of (1), (3). We have

$$\Phi(x,\rho) = -\frac{\psi(x,\rho)}{\Delta(\rho)} = S(x,\rho) + M(\rho)\varphi(x,\rho), \quad M(\rho) = -\frac{\Delta_0(\rho)}{\Delta(\rho)}.$$

Clearly, $\{\rho_n\}\cap\{\rho_n^0\} = \emptyset$. Thus, $M(\rho)$ is a meromorphic function with the poles ρ_n and the zeros ρ_n^0. It is convenient to index eigenvalues ρ_n with account of multiplicity by $n = \pm 0, \pm 1, \pm 2, \ldots$. Then (see, e.g., [12]) $\{\rho_n\}$ have the form

$$\rho_n = \pi n + \omega_0 + \frac{\omega_1}{\pi n} + o\left(\frac{1}{n}\right), \quad |n| \to \infty, \tag{4}$$

where

$$\omega_0 = Q(1), \quad Q(x) = \int_0^x q_1(t)\,dt, \quad \omega_1 = h + H + \frac{1}{2}\int_0^\pi (q_0(t) + q_1^2(t))\,dt.$$

Denote by m_n the multiplicity of ρ_n. By virtue of (4) for sufficiently large $|n|$ we have $m_n = 1$.

Using the known method (see, e.g., [10]) one can prove the following auxiliary assertion.

Lemma 1. (i) *For $|\rho| \to \infty$ the following asymptotics holds*:

$$\psi(x,\rho) = \cos(\rho\ (1-x) - \omega_0 + Q(x)) + O\left(\frac{1}{\rho}\exp(|Im\rho|(1-x))\right), \tag{5}$$

$$\psi'(x,\rho) = \rho\sin(\rho(1-x) - \omega_0 + Q(x)) + O(\exp(|Im\rho|(1-x))) \tag{6}$$

uniformly with respect to $x \in [0,1]$.

(ii) *Fix $\delta > 0$. Then for sufficiently large $|\rho|$*

$$|\Delta(\rho)| \ge C_\delta |\rho| \exp(|Im\rho|), \quad \rho \in G_\delta, \tag{7}$$

where $G_\delta = \{\rho : |\rho - \pi n - \omega_0| \ge \delta,\ n \in \mathbb{Z}\}$.

Consider the following inverse problem.

Inverse Problem 1. Given the spectrum $\{\rho_n\}$ of $L = L(q_0(x), q_1(x), h, H)$, find L, provided that the value h and the functions $q_0(x)$, $q_1(x)$ on $(0, 1/2)$ are known a priori.

Here and below together with L we consider a pencil $\tilde{L}$ of the same form but with other coefficients $\tilde{q}_j(x)$, $\tilde{h}$, $\tilde{H}$. We agree that if a certain symbol α denotes an object related to L then the same symbol with the tilde $\tilde{\alpha}$ denotes the analogous object related to $\tilde{L}$, and $\hat{\alpha} = \alpha - \tilde{\alpha}$. The following uniqueness theorem holds.

Theorem 1. *Let* $h = \tilde{h}$, $q_j(x) \overset{\text{a.e.}}{=} \tilde{q}_j(x)$ *on* $(0, 1/2)$, $j = 0, 1$. *If* $\{\rho_n\} = \{\tilde{\rho}_n\}$, *then* $L = \tilde{L}$, *i.e.* $q_0(x) \overset{\text{a.e.}}{=} \tilde{q}_0(x), q_1(x) = \tilde{q}_1(x)$ *on* $[0, 1]$ *and* $H = \tilde{H}$. *Thus, the specification of the spectrum* $\{\rho_n\}$ *uniquely determines* L, *provided that* (1) *and* (2) *are known a priori on one half of the segment* $[0, 1]$.

Proof. Let $\Phi_0(x, \rho)$ be a solution of equation (1) under the boundary conditions

$$\Phi_0(0, \rho) = 1, \quad V(\Phi_0) = 0.$$

The functions $\Phi_0(x, \rho)$, $M_0(\rho) := U(\Phi_0(x, \rho))$ are called respectively the Weyl solution and the Weyl function of the boundary value problem (1), (3). Obviously,

$$\begin{aligned} \Phi_0(x, \rho) &= \frac{\psi(x, \rho)}{\psi(0, \rho)} = \varphi(x, \rho) + M_0(\rho) S(x, \rho), \\ M_0(\rho) &= \frac{U(\psi(x, \rho))}{\psi(0, \rho)} = -\frac{\Delta(\rho)}{\Delta_0(\rho)} = \frac{1}{M(\rho)}. \end{aligned} \tag{8}$$

Fix $a \in (0, 1)$ and consider the boundary value problem

$$y'' + (\rho^2 - 2\rho q_1(x) - q_0(x)) y = 0, \quad a < x < 1, \quad y(a) = V(y) = 0. \tag{9}$$

Let $\Phi_a(x, \rho), M_a(\rho)$ be respectively the Weyl solution and the Weyl function of (9), i.e. $\Phi_a(x, \rho)$ is a solution of the differential equation in (9) under the conditions $\Phi_a(a, \rho) = 1$, $V(\Phi_a) = 0$ and $M_a(\rho) = \Phi_a'(a, \rho)$. By virtue of the uniqueness theorem in [15] the specification of $M_a(\rho)$ uniquely determines $q_j(x), j = 0, 1$, on $(a, 1)$ and H. Thus, it is sufficient to prove that $\hat{M}_{\frac{1}{2}}(\rho) \equiv 0$. We have

$$\Phi_a(x, \rho) = \frac{\psi(x, \rho)}{\psi(a, \rho)}, \quad M_a(\rho) = \frac{\psi'(a, \rho)}{\psi(a, \rho)}, \quad \hat{M}_a(\rho) = \frac{d_a(\rho)}{\psi(a, \rho)\tilde{\psi}(a, \rho)}, \tag{10}$$

where $d_a(\rho) = \langle \tilde{\psi}(x, \rho), \psi(x, \rho) \rangle|_{x=a}$, and according to (8) we arrive at

$$\frac{d_a(\rho)}{\psi(0, \rho)\tilde{\psi}(0, \rho)} = \langle \tilde{\Phi}_0(x, \rho), \Phi_0(x, \rho) \rangle|_{x=a}.$$

Moreover, for all n and $\nu = \overline{0, m_n - 1}$ (8) gives

$$\frac{d^\nu}{d\rho^\nu} \frac{d_a(\rho)}{\psi(0, \rho)\tilde{\psi}(0, \rho)} \bigg|_{\rho=\rho_n} = \frac{d^\nu}{d\rho^\nu} \langle \tilde{\varphi}(x, \rho), \varphi(x, \rho) \rangle \bigg|_{x=a,\, \rho=\rho_n} = 0, \quad a \in [0, 1/2],$$

since under the conditions of the theorem we have $\varphi(x,\rho)=\tilde{\varphi}(x,\rho)$ for $x\in[0,1/2]$. Hence

$$d_a^{(\nu)}(\rho_n)=0,\quad \nu=\overline{0,m_n-1},\quad n=\pm 0,\pm 1,\dots,\quad a\in[0,1/2]. \tag{11}$$

Moreover, (5), (6) give

$$d_a(\rho)=\rho\sin(\hat{Q}(a)-\hat{\omega}_0)+O(\exp(2|Im\rho|(1-a))),\quad |\rho|\to\infty,$$

where $\hat{Q}(a)=0$ for $a\in[0,1/2]$ because $\hat{q}_1(x)=0$ on $[0,1/2]$. According to (4) the specification of the spectrum $\{\rho_n\}$ determines ω_0 up to a constant πk, where $k\in\mathbb{Z}$. Thus, under the conditions of the theorem we have $\hat{\omega}_0=\pi k$, and hence

$$d_a(\rho)=O(\exp(2|Im\rho|(1-a))),\quad |\rho|\to\infty,\quad a\in[0,1/2]. \tag{12}$$

Consider the function

$$\theta(\rho)=\frac{d_{\frac{1}{2}}(\rho)}{\Delta(\rho)},$$

which by virtue of (11) is entire in ρ. On the other hand, according to (7), (12) we have

$$\theta(\rho)=O\left(\frac{1}{\rho}\right),\quad |\rho|\to\infty,\quad \rho\in G_\delta.$$

Using the maximum modulus principle together with Liouville's theorem we arrive at $\theta(\rho)\equiv 0$. Consequently, $d_{\frac{1}{2}}(\rho)\equiv 0$ and according to (10) we obtain $M_{\frac{1}{2}}(\rho)=\tilde{M}_{\frac{1}{2}}(\rho)$. □

Below in Sect. 4 the following modification of Theorem 1 will be used.

Theorem 2. *Fix arbitrary $\varepsilon>0$ and $N\in\mathbb{N}$. Let $h=\tilde{h}$, $q_j(x)\overset{\text{a.e.}}{=}\tilde{q}_j(x)$ on $(0,1/2+\varepsilon)$, $j=0,1$. If $\{\rho_n\}_{|n|\ge N}=\{\tilde{\rho}_n\}_{|n|\ge N}$, then $L=\tilde{L}$, i.e. $q_0(x)\overset{\text{a.e.}}{=}\tilde{q}_0(x), q_1(x)=\tilde{q}_1(x)$ on $[0,1]$ and $H=\tilde{H}$. Thus, the specification of the spectrum with exception of any finite subset uniquely determines L, provided that the number h and the functions $q_0(x)$, $q_1(x)$ on $(0,1/2+\varepsilon)$ are known a priori.*

Proof. Using the same arguments as in the proof of Theorem 1 we obtain

$$d_{\frac{1}{2}+\varepsilon}^{(\nu)}(\rho_n)=0,\quad \nu=\overline{0,m_n-1},\quad |n|\ge N, \tag{13}$$

$$d_{\frac{1}{2}+\varepsilon}(\rho)=O(\exp(|Im\rho|(1-2\varepsilon))),\quad |\rho|\to\infty. \tag{14}$$

Consider the function

$$\theta_1(\rho)=\frac{d_{\frac{1}{2}+\varepsilon}(\rho)}{\Delta(\rho)}\prod_{n=0}^{N-1}(\rho-\rho_{-n})(\rho-\rho_{+n}),$$

which by virtue of (13) is entire in ρ. According to (7), (14) we have

$$\theta_1(\rho)=O(\rho^{2N-1}\exp(-2\varepsilon|Im\rho|)),\quad |\rho|\to\infty,\quad \rho\in G_\delta.$$

Phragmen–Lindelöf's and Liouville's theorems infer $\theta_1(\rho) \equiv 0$. Therefore $M_{\frac{1}{2}+\varepsilon}(\rho) = \tilde{M}_{\frac{1}{2}+\varepsilon}(\rho)$ and consequently $q_0(x) \overset{\text{a.e.}}{=} \tilde{q}_0(x)$, $q_1(x) = \tilde{q}_1(x)$ on $[1/2+\varepsilon, 1]$ and $H = \tilde{H}$. □

3. Inverse Nodal Problem

In this section we provide auxiliary results on an inverse nodal problem for the pencil L (see [25,26]). Here and in the next section we assume that its coefficients $q_0(x), q_1(x), h, H$ are real, then for sufficiently large $|n|$ the eigenfunctions $\varphi(x,\rho_n)$ are real too.

Theorem 3. *For sufficiently large $|n|$ the eigenfunction $\varphi(x,\rho_n)$ has exactly $|n|$ zeros x_n^j in the interval $(0,1)$:*

$$0 < x_n^1 < x_n^2 < \cdots < x_n^n < 1 \quad \textit{for } n > 0$$

and

$$0 < x_n^0 < x_n^{-1} < \cdots < x_n^{n+1} < 1 \quad \textit{for } n < 0.$$

Consider the following inverse problem.

Inverse Problem 2. Given the set of nodal points X, find $q_0(x)$, $q_1(x)$, h, H.

Here and below the notion "set of nodal points" is understood with account of their indices. In other words, $\{x_n^j\}_{(n,j)\in I} = \{\tilde{x}_n^j\}_{(n,j)\in \tilde{I}}$ if and only if $I = \tilde{I}$ and $x_n^j = \tilde{x}_n^j$ for all $(n,j) \in I$.

Shifting the spectral parameter $\rho \to \rho + C$ we obtain the pencil

$$L_{\{C\}} = L(q_0(x) + 2Cq_1(x) - C^2, q_1(x) - C, h, H),$$

which possesses the same eigenfunctions as L does. Without loss of generality we assume that

$$\omega_0 = 0. \tag{15}$$

Denote

$$p(x) = q_0(x) - \int_0^1 q_0(t)\,dt,$$

$$\alpha_n(x) = \frac{1}{2}\left(\int_0^x (p(t) + q_1^2(t))\cos(2\pi nt - 2Q(t))\,dt - \int_0^x q_1'(t)\sin(2\pi nt - 2Q(t))\,dt\right),$$

$$\omega_1(x) = h + H + \frac{1}{2}\int_0^x (q_0(t) + q_1^2(t))\,dt,$$

$$\omega_2(x) = hq_1(0) + Hq_1(1) + \frac{1}{2}\int_0^x (q_0(t) + q_1^2(t))q_1(t)\,dt.$$

For solving Inverse Problem 2 the following asymptotics of the nodal points is used.

Theorem 4. *The following representation holds*:

$$\begin{aligned} x_n^j &= \frac{1}{n}\left(j - \frac{1}{2}\right) + \frac{1}{\pi n}Q(x_n^j) \\ &+ \frac{1}{(\pi n)^2}(\omega_1(x_n^j) - \omega_1(1)x_n^j - H + \alpha_n(x_n^j) - \alpha_n(1)x_n^j) \\ &+ \frac{1}{(\pi n)^3}(\omega_2(x_n^j) - \omega_2(1)x_n^j - Hq_1(1)) + o\left(\frac{1}{n^3}\right), \quad |n| \to \infty, \end{aligned}$$

uniformly with respect to j.

Corollary 1. *The set* X *of all nodal points is dense in* $[0, 1]$.

The following assertion is also an immediate corollary of Theorem 4.

Lemma 2. *Fix* $x \in [0, 1]$. *Choose* $\{j_n\}$ *such that* $x_n^{j_n} \to x$ *as* $|n| \to \infty$. *Then there exist the following finite limits and the corresponding equalities hold:*

$$Q(x) = \lim_{|n|\to\infty} \beta_n, \quad \beta_n = \pi(nx_n^{j_n} - j_n) + \frac{\pi}{2}, \tag{16}$$

$$f(x) = \lim_{|n|\to\infty} \gamma_n, \quad \gamma_n = \pi n(\beta_n - Q(x_n^{j_n})), \tag{17}$$

$$g(x) = \pi \lim_{|n|\to\infty} n(\gamma_n - f(x_n^{j_n}) - \alpha_n(x_n^{j_n}) + \alpha_n(1)x_n^{j_n}), \tag{18}$$

where

$$f(x) = h(1 - x) - Hx + \frac{1}{2}\left(\int_0^x (q_0(t) + q_1^2(t))\,dt - x\int_0^1 (q_0(t) + q_1^2(t))\,dt\right),$$

$$g(x) = h(1 - x)q_1(0) - Hxq_1(1) + \frac{g_1(x)}{2}, \tag{19}$$

$$g_1(x) = \int_0^x (q_0(t) + q_1^2(t))q_1(t)\,dt - x\int_0^1 (q_0(t) + q_1^2(t))q_1(t)\,dt.$$

Let us for definiteness exclude the Sturm–Liouville operator $(q_1(x) \equiv 0)$ from the consideration, i.e. with account of (15) we assume that

$$q_1(x) \neq const. \tag{20}$$

Then according to the following uniqueness theorem the pencil $L_{\{C\}}$ is a unique modification of L that leaves the nodal points unchanged.

Theorem 5. *The specification of any dense subset $X_0 \subset X$ uniquely determines the functions $q_0(x)$, $q_1(x)$ and the numbers h, H, which can be found by the following algorithm.*

Algorithm 1. *Let a dense subset X_0 of the nodal points be given. Then*

(i) *for each $x \in [0,1]$ choose a sequence $\{x_n^{j_n}\} \subset X_0$ such that $x_n^{j_n} \to x$ as $|n| \to \infty$;*

(ii) *find the function $Q(x)$ via (16) and calculate*

$$q_1(x) = Q'(x);$$

(iii) *calculate $f(x)$ by formula (17) and obtain*

$$h = f(0), \quad H = -f(1),$$

$$p(x) = 2(h+H) + 2f'(x) - q_1^2(x) + \int_0^1 q_1^2(t)\,dt;$$

(iv) *fix an arbitrary $x \in [0,1]$ such that $Q(x) \neq 0$, find $g_1(x)$ via (18), (19) and put*

$$A = \frac{1}{Q(x)}\left(g(x) - \int_0^x (p(t) + q_1^2(t))q_1(t)\,dt + x\int_0^1 (p(t) + q_1^2(t))q_1(t)\,dt\right);$$

(v) *finally, calculate the function $q_0(x)$ by the formula*

$$q_0(x) = p(x) + A.$$

Corollary 2. *Let (15) be not assumed while (20) be kept. Then the specification of X_0 uniquely determines the functions $q_1(x) - \omega_0$, $q_0(x) + 2\omega_0 q_1(x) - \omega_0^2$, and the numbers h, H.*

Remark 1. As in [26] Algorithm 1 can be modified so that $q_0(x)$, $q_1(x)$ will be approximated directly, i.e. not via their primitive functions.

Remark 2. It is easy to see that not assuming (15), (20) one also can recover L from a dense set of nodes X_0, provided that mean values of the functions $q_0(x)$, $q_1(x)$ are known a priori. Moreover, the set $X_1 := X_0 \cap (a,b)$ determines the coefficients of L on $[a,b]$.

4. Incomplete Inverse Nodal Problem

In this section we prove that for unique determination of L it is sufficient to specify the nodal points only on the part of the interval together with the mean values of $q_0(x)$, $q_1(x)$. More precisely, the following theorem holds.

Theorem 6. *Fix arbitrary $a > 1/2$. Let*

$$\int_0^1 \hat{q}_j(x)\,dx = 0, \quad j = 0, 1.$$

If $X \cap (0, a) = \tilde{X} \cap (0, a)$ then $L = \tilde{L}$. Thus, the specification of the nodes on any interval $(0, a)$, $a > 1/2$, together with the mean values of $q_0(x)$, $q_1(x)$ uniquely determines the functions $q_0(x)$, $q_1(x)$ and the coefficients h, H of the boundary conditions.

Remark 3. Obviously, the interval $(0, a)$ in Theorem 6 can be replaced with $(1 - a, 1)$.

First we prove some auxiliary assertions. Denoting $Q(x, \rho) := \rho^2 - 2\rho q_1(x) - q_0(x)$ we have $\frac{\partial}{\partial\rho} Q(x, \rho) = 2(\rho - q_1(x))$. Thus, for each fixed $x \in [0, 1]$ the function $Q(x, \rho)$ increases for $\rho > q_1(x)$ and decreases for $\rho < q_1(x)$. Put $m := \min q_1(x)$, $M := \max q_1(x)$, $x \in [0, 1]$. Let $z_j(x)$, $j = 1, 2$, be solutions of the equations

$$z_j'' + Q(x, \mu_j)z_j = 0, \quad 0 < x < 1,$$

where $\mu_2 > \mu_1 \geq M$ or $\mu_2 < \mu_1 \leq m$. Then according to Sturm's comparison theorem (see, e.g., Lemma 1.2.1 in [10]) between any two zeros of the function $z_1(x)$ there is at least one zero of the function $z_2(x)$.

We consider the function $\varphi(x, \rho)$ for real ρ. Repeating word by word the proof of the corresponding assertion for the case $q_1(x) = 0$ (see, e.g., Lemma 1.2.2. in [10]) one can prove that the zeros of $\varphi(x, \rho)$ with respect to x are continuous functions of ρ, i.e. the following lemma holds.

Lemma 3. *Let $\varphi(x^*, \rho^*) = 0$. For sufficiently small $\varepsilon > 0$ there exists $\delta > 0$ such that if $|\rho - \rho^*| < \delta$, then the function $\varphi(x, \rho)$ has exactly one zero in the interval $|x - x^*| < \varepsilon$.*

Lemma 4. *Fix μ_1, μ_2 such that $\mu_2 > \mu_1 \geq M$ or $\mu_2 < \mu_1 \leq m$. Let $\varphi(x_1, \mu_1) = 0$ while $\varphi(x, \mu_1) > 0$ for $x \in [0, x_1)$. Then there exists $x_2 \in (0, x_1)$ such that $\varphi(x_2, \mu_2) = 0$.*

Proof. Subtracting the relation

$$\varphi(x, \mu_1)(\varphi''(x, \mu_2) + Q(x, \mu_2)\varphi(x, \mu_2)) = 0$$

termwise from

$$\varphi(x, \mu_2)(\varphi''(x, \mu_1) + Q(x, \mu_1)\varphi(x, \mu_1)) = 0$$

we arrive at

$$\frac{d}{dx}\langle \varphi(x, \mu_2), \varphi(x, \mu_1)\rangle = (Q(x, \mu_2) - Q(x, \mu_1))\varphi(x, \mu_1)\varphi(x, \mu_2).$$

Integrating the latter relation with respect to x on the interval $(0, x_1)$, we get

$$\int_0^{x_1} (Q(x,\mu_2) - Q(x,\mu_1))\varphi(x,\mu_1)\varphi(x,\mu_2)\,dx = \varphi(x_1,\mu_2)\varphi'(x_1,\mu_1). \quad (21)$$

Let us assume to the contrary that $\varphi(x,\mu_2) > 0$ for $x \in (0, x_1)$, then, in particular, $\varphi(x_1,\mu_2)\varphi'(x_1,\mu_1) \le 0$. On the other hand, since $Q(x,\mu_2) > Q(x,\mu_1)$, the integral in the left-hand side of (21) is strictly positive. This contradiction proves the lemma. □

Corollary 3. *Summarizing Lemmas 3, 4 together with the above reasoning we get that if real ρ moves away from the interval (m, M), then the zeros of $\varphi(x,\rho)$ on the interval $(0, 1]$ continuously move to the left. New zeros can appear only through the point $x = 1$.*

Consider the boundary value problem $L_1 = L_1(q_0(x), q_1(x), h)$ for equation (1) with the boundary conditions $U(y) = y(1) = 0$. The eigenvalues $\rho_{n,1}$, $n = \pm 0, \pm 1, \pm 2, \ldots$, of L_1 have the form

$$\rho_{n,1} = \pi\left(n + \frac{\operatorname{sign} n}{2}\right) + \omega_0 + O\left(\frac{1}{n}\right), \quad |n| \to \infty. \quad (22)$$

Clearly, $\rho_{n,1}$ are real for large $|n|$. Moreover, by virtue of (4), (22) for sufficiently large $|n|$ eigenvalues of the pencils L, L_1 are alternating in the following way: $\rho_n < \rho_{n,1} < \rho_{n+1}$ for $n > 0$ and $\rho_{n-1} < \rho_{n,1} < \rho_n$ for $n < 0$. Therefore, according to Theorem 3 and Corollary 3 for large $|n|$ the eigenfunction $\varphi(x, \rho_{n,1})$ of the pencil L_1 has precisely $|n|$ zeros in the interval $(0, 1)$.

According to Theorem 3 one can choose sufficiently large N_1 such that for all $n \ge N_1$ there are exactly two eigenfunctions of the pencil L namely $\varphi(x,\rho_n)$ and $\varphi(x,\rho_{-n})$ that possess precisely n zeros in the interval $(0, 1)$ and $\rho_n > 0, \rho_{-n} < 0$. The same assertion with the same N_1 is obviously valid for the pencil L_b of the form

$$y'' + (\rho^2 - 2\rho q_1(x) - q_0(x))y = 0, \quad 0 < x < b \le 1, \quad U(y) = y(b) = 0,$$

whose eigenvalues $\rho_{n,b}$, $n = \pm 0, \pm 1, \pm 2, \ldots$, have the asymptotics

$$\rho_{n,b} = \frac{\pi}{b}\left(n + \frac{\operatorname{sign} n}{2}\right) + \frac{1}{b}\int_0^b q_1(x)\,dx + O\left(\frac{1}{n}\right), \quad |n| \to \infty,$$

i.e. the following lemma holds.

Lemma 5. *For all $b \in (0, 1]$ and $n \ge N_1$ there are exactly two eigenfunctions of the pencil L_b namely $\varphi(x, \rho_{n,b})$ and $\varphi(x, \rho_{-n,b})$ that have precisely n zeros in the interval $(0, b)$ and $\rho_{n,b} > 0, \rho_{-n,b} < 0$.*

Proof of Theorem 6. Choose $N \ge N_1$ so that $x_n^{N_1+1} < a$ and $x_{-n}^{-N_1} < a$ for $n \ge N$. Fix $n \ge N$ and put $b := x_n^{N_1+1}$. Consider the pencil L_b. According to Lemma 5 we obtain $\{\rho_n, \tilde\rho_n\} \subset \{\rho_{N_1,b}, \rho_{-N_1,b}\}$. Since $\rho_n\tilde\rho_n > 0$ and

$\rho_{N_1,b}\rho_{-N_1,b} < 0$, we get $\rho_n = \tilde{\rho}_n$. Analogously putting $b := x_{-n}^{-N_1}$ we arrive at $\rho_{-n} = \tilde{\rho}_{-n}$. Thus, we have $\rho_n = \tilde{\rho}_n$ for $|n| \geq N$ and it remains to apply Remark 2 and Theorem 2. □

Acknowledgements

This research was supported in part by Russian Foundation for Basic Research and Taiwan National Science Council (projects 10-01-00099 and 10-01-92001-NSC-a) and by Mikhail Lomonosov Program of the Ministry of Education and Science of Russia and DAAD (project 5007).

References

[1] Tamarkin, J.D.: On some problems of the theory of ordinary linear differential equations, Petrograd, 1917

[2] Keldysh, M.V.: On eigenvalues and eigenfunctions of some classes of non-selfadjoint equations. Dokl. Akad. Nauk. SSSR **77**, 11–14 (1951)

[3] Kostyuchenko, A.G., Shkalikov, A.A.: Selfadjoint quadratic operator pencils and elliptic problems. Funk. Anal. i Prilozhen. **17**(2), 38–61 (1983); English transl.: Funct. Anal. Appl. **17**, 109–128 (1983)

[4] Markus, A.S.: Introduction to the spectral theory of polynomial operator pencils, Shtinitsa, Kishinev (1986); English transl., AMS, Providence (1988)

[5] Yurko, V.A.: An inverse problem for systems of differential equations with nonlinear dependence on the spectral parameter. Diff. Uravneniya **33**(3), 390–395 (1997) (Russian); English transl., Diff. Equ. **33**(3), 388–394 (1997)

[6] Marchenko, V.A.: Sturm–Liouville Operators and their Applications. Naukova Dumka, Kiev (1977); English transl., Birkhäuser, Basel (1986)

[7] Levitan, B.M.: Inverse Sturm–Liouville Problems. Nauka, Moscow (1984); English transl., VNU Science Press, Utrecht (1987)

[8] McLaughlin, J.R.: Analytical methods for recovering coefficients in differential equations from spectral data. SIAM Rev. **28**, 53–72 (1986)

[9] Chadan, K., Colton, D., Päivärinta, L., Rundell, W.: An Introduction to Inverse Scattering and Inverse Spectral Problems. SIAM Monographs on Mathematical Modeling and Computation, SIAM, Philadelphia (1997)

[10] Freiling, G., Yurko, V.A.: Inverse Sturm–Liouville Problems and Their Applications. NOVA Science Publishers, New York (2001)

[11] Yurko, V.A.: Method of Spectral Mappings in the Inverse Problem Theory. Inverse and Ill-posed Problems Series. VSP, Utrecht (2002)

[12] Gasymov, M.G., Gusejnov, G.Sh.: Determination of a diffusion operator from spectral data. Akad. Nauk Azerb. SSR Dokl. **37**(2), 19–23 (1981)

[13] Yamamoto, M.: Inverse eigenvalue problem for a vibration of a string with viscous drag. J. Math. Anal. Appl. **152**(1), 20–34 (1990)

[14] Yurko, V.A.: An inverse problem for pencils of differential operators. Matem. Sbornik **191**(10), 137–160 (2000) (Russian); English transl., Sbornik: Mathematics **191**(10), 1561–1586

[15] Buterin, S.A., Yurko, V.A.: Inverse Spectral Problem for Pencils of Differential Operators on a Finite Interval. vol. 4, pp. 8–12. Vestnik Bashkirsk University (Russian) (2006)

[16] McLaughlin, J.R.: Inverse spectral theory using nodal points as data—a uniqueness result. J. Diff. Equ. **73**(2), 354–362 (1988)

[17] Hald, O.H., McLaughlin, J.R.: Solutions of inverse nodal problems. Inverse Problems **5**, 307–347 (1989)

[18] Shen, C.-L., Tsai, T.-M.: On a uniform approximation of the density function of a string equation using eigenvalues and nodal points and some related inverse nodal problems. Inverse Problems **11**, 1113–1123 (1995)

[19] Browne, P.J., Sleeman, B.D.: Inverse nodal problems for Sturm–Liouville equations with eigenparameter dependent boundary conditions. Inverse Problems **12**, 377–381 (1996)

[20] Yang, X.-F.: A solution of the inverse nodal problem. Inverse Problems **13**, 203–213 (1997)

[21] Law, C.K., Yang, C.-F.: Reconstracting the potential function and its derivatives using nodal data. Inverse Problems **14**, 299–312 (1998)

[22] Yang, X.-F.: A new inverse nodal problem. J. Diff. Equ. **169**, 633–653 (2001)

[23] Cheng, Y.H., Law, C.K.: The inverse nodal problem for Hill's equation. Inverse Problems **22**, 891–901 (2006)

[24] Shieh, C.-T., Yurko, V.A.: Inverse nodal and inverse spectral problems for discontinuous boundary value problems. J. Math. Anal. Appl. **374**, 266–272 (2008)

[25] Buterin, S.A.: Inverse nodal problem for differential pencils of second order. Spect. Evol. Problems (Russian) **18**, 46–51 (2008)

[26] Buterin, S.A., Shieh, C.-T.: Inverse nodal problem for differential pencils. Appl. Math. Lett. **22**, 1240–1247 (2009)

[27] Yang, C.-F., Yang, X.-P.: Inverse nodal problems for differential pencils on a star-shaped graph. Inverse Problems **26**, 085008 (2010)

[28] Koyunbakan, H., Panakhov, E.S.: Half-inverse problem for diffusion operators on the finite interval. J. Math. Anal. Appl. **326**(2), 1024–1030 (2007)

[29] Hochschtadt, H., Liebermann, B.: An inverse Sturm–Liouville problem with mixed given data. SIAM J. Appl. Math. **34**, 676–680 (1978)

[30] Gesztesy, F., Simon, B.: Inverse spectral analysis with partial information on the potential. II. The case of discrete spectrum. Trans. Am. Math. Soc. **352**(6), 2765–2787 (2000)

[31] Sakhnovich, L.: Half-inverse problem on the finite interval. Inverse Problems **17**, 527–532 (2001)

[32] Hryniv, R.O., Mykytyuk, Y.V.: Half-inverse spectral problems for Sturm–Liouville operators with singular potentials. Inverse Problems **20**, 1423–1444 (2004)

[33] Martinyuk, O., Pivovarchik, V.: On the Hochstadt–Lieberman theorem. Inverse Problems **26**, 035011 (2010)

S. A. Buterin
Department of Mathematics
Saratov State University
Astrakhanskaya 83
Saratov 410012
Russia
e-mail: buterinsa@info.sgu.ru

C.-T. Shieh
Department of Mathematics
Tamkang University
151 Ying-chuan Road Tamsui
Taipei 25137
Taiwan, ROC
e-mail: ctshieh@math.tku.edu.tw

Received: April 11, 2011.
Accepted: April 21, 2011.