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On a Free Boundary Problem for a Two-Species Weak Competition System

Jong-Shenq Guo · Chang-Hong Wu

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Abstract We study a Lotka–Volterra type weak competition model with a free boundary in a one-dimensional habitat. The main objective is to understand the asymptotic behavior of two competing species spreading via a free boundary. We also provide some sufficient conditions for spreading success and spreading failure, respectively. Finally, when spreading successfully, we provide an estimate to show that the spreading speed (if exists) cannot be faster than the minimal speed of traveling wavefront solutions for the competition model on the whole real line without a free boundary.

Keywords Lotka–Volterra model · Free boundary · Spreading–vanishing dichotomy · Spreading speed

Mathematics Subject Classification (2010) 35K51 · 35R35 · 92B05

1 Introduction

A variety of reaction-diffusion equations are used to describe some phenomena arising in population ecology. A typical model is the following Lotka–Volterra type competition system for two species in a one-dimensional habitat [34]:

$$u_t = d_1 u_{xx} + r_1 u(1 - b_1 u - a_1 v), \quad x, t \in \mathbb{R}, \tag{1.1}$$

$$v_t = d_2 v_{xx} + r_2 v(1 - b_2 v - a_2 u), \quad x, t \in \mathbb{R}, \tag{1.2}$$

where $u(x,t)$, $v(x,t)$ denote the population densities of two competing species at the position x and time t; d_1, d_2 are diffusion coefficients of species u, v; r_1, r_2 are net birth rates of species

J.-S. Guo · C.-H. Wu (✉)
Department of Mathematics, Tamkang University, Tamsui, New Taipei City 25137, Taiwan
e-mail: 896400012@ntnu.edu.tw

J.-S. Guo
e-mail: jsguo@mail.tku.edu.tw

u, v; $1/b_1$, $1/b_2$ are the carrying capacities of species u, v and a_1, a_2 are (inter-specific) competition coefficients of species u, v, respectively. All parameters are assumed to be positive. By setting

$$\hat{u} := b_1 u, \quad \hat{v} := b_2 v, \quad \hat{t} := r_1 t, \quad \hat{x} := \sqrt{\frac{r_1}{d_1}} x,$$
$$D := \frac{d_2}{d_1}, \quad k := \frac{a_1}{b_2}, \quad h := \frac{a_2}{b_1}, \quad r := \frac{r_2}{r_1},$$

and dropping the *hat* sign, (1.1)–(1.2) becomes the following nondimensional system:

$$u_t = u_{xx} + u(1 - u - kv), \quad x, t \in \mathbb{R}, \tag{1.3}$$
$$v_t = Dv_{xx} + rv(1 - v - hu), \quad x, t \in \mathbb{R}, \tag{1.4}$$

The global dynamics for the related kinetic system (in the absence of diffusion) to (1.3)–(1.4) is well-known. It has at least three constant equilibrium solutions $(u, v) = (0, 0)$, $(0, 1)$ and $(1, 0)$. Moreover, if either $h, k > 1$ or $0 < h, k < 1$, then there exists a unique positive constant equilibrium solution $(\frac{1-k}{1-hk}, \frac{1-h}{1-hk})$. For $0 < k < 1 < h$, $\lim_{t\to+\infty}(u, v)(t) = (1, 0)$; for $0 < h < 1 < k$, $\lim_{t\to+\infty}(u, v)(t) = (0, 1)$. For $0 < h, k < 1$, $\lim_{t\to+\infty}(u, v)(t) = (\frac{1-k}{1-hk}, \frac{1-h}{1-hk})$, this case is called the *weak competition* (*co-existence*) case. For $h, k > 1$, $(1, 0)$ and $(0, 1)$ are locally stable, almost every trajectory tends to $(1, 0)$ or $(1, 0)$ as $t \to +\infty$, and this case is called the case of *strong competition*.

To describe the invasion and spreading phenomenon for (1.3)–(1.4), there have been many interesting studies on the existence of positive traveling waves solutions connecting two different equilibria; see, for example [8,15,19,20,22,23,34,37] the references cited therein. Also, the study of asymptotic spreading speed plays an important role in invasion ecology since it can be used to predict the mean spreading rate of species. The concept of asymptotic spreading speed comes from Aronson and Weinberger [1–3] and then Lewis et al. [25,26,38] extended the result of asymptotic spreading speed to (1.3)–(1.4) and more general models. For related works, see [16,21,27,28] and the references cited therein. We also refer to [39] in which the author gave a review on traveling waves and asymptotic spreading speed.

Our main objective is to understand the long time behavior of two-competing species spreading via a free boundary. For this, we shall investigate the following problem **(FBP)**:

$$u_t = u_{xx} + u(1 - u - kv), \quad 0 < x < s(t), \quad t > 0, \tag{1.5}$$
$$v_t = Dv_{xx} + rv(1 - v - hu), \quad 0 < x < s(t), \quad t > 0, \tag{1.6}$$
$$u_x(0, t) = v_x(0, t) = 0, \quad u(s(t), t) = v(s(t), t) = 0, \quad t > 0, \tag{1.7}$$
$$s'(t) = -\mu[u_x(s(t), t) + \rho v_x(s(t), t)], \quad t > 0 \tag{1.8}$$
$$u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad 0 \le x \le s_0, \quad s(0) = s_0, \tag{1.9}$$

with the parameters $\mu, \rho > 0$ and the initial data (u_0, v_0, s_0) satisfying

$$\begin{cases} s_0 > 0, \quad u_0, v_0 \in C^2([0, s_0]), \quad u_0(x), v_0(x) > 0 \quad \text{for } x \in [0, s_0), \\ u_0(s_0) = v_0(s_0) = u_0'(0) = v_0'(0) = 0. \end{cases}$$

From a biological point of view, this model describes how the two competing species invade if they initially occupy the region $[0, s_0]$. It is assumed that the left boundary is fixed so that no flux across the left boundary $x = 0$, namely, we impose the zero Neumann boundary condition in (1.7) for $x = 0$. Also, we assume that both species have a tendency to emigrate from the right boundary to obtain their new habitat. Moreover, it is assumed that the expanding speed of the free boundary is proportional to the normalized population gradient at the free boundary, i.e., (1.8) holds, which is the well-known Stefan type condition. We call the free boundary $x = s(t)$ the *spreading front*. This setting for two competing species with a free boundary is motivated by the work of Du and Lin [11] who proposed a new approach to investigate how one species spreads and invades to a new environment (see also [9,12,14]). For more biological discussion, we refer to [4].

For the study of free boundary problems for some biological models, we refer to, for instance [4,6,7,9,10,12,14,17,18,29–32,35] and references cited therein.

In this paper, we only focus on the *weak competition* case:

(A1) $\quad 0 < h, k < 1.$

We now describe the main results of this paper as follows. Hereafter **(A1)** is always assumed. First, we have the following existence and uniqueness result for the solution.

Theorem 1 **(FBP)** *admits a unique global solution* $(u, v, s) \in C^{2,1}(\Omega) \times C^{2,1}(\Omega) \times C^1([0,\infty))$*, where* $\Omega := \{(x,t) : 0 \le x \le s(t),\ t > 0\}$*, such that* $0 < s'(t) \le \mu\Lambda$ *for all* $t \ge 0$ *with* $\Lambda > 0$ *depending only on* $D, r, \rho, u_0, v_0, s_0$*, and is independent of* μ*. More precisely, we have*

$$\Lambda := 2M_1 \max\{1, \|u_0\|_{L^\infty}\} + 2\rho M_2 \max\{1, \|v_0\|_{L^\infty}\}, \tag{1.10}$$

where

$$M_1 := \max\left\{\frac{4}{3}, \frac{-4}{3}\left(\min_{x\in[0,s_0]} u_0'(x)\right)\right\},$$
$$M_2 := \max\left\{\sqrt{\frac{r}{2D}}, \frac{4}{3}, \frac{-4}{3}\left(\min_{x\in[0,s_0]} v_0'(x)\right)\right\}.$$

In the sequel it is often to use the following three quantities:

$$s_\infty := \lim_{t\to+\infty} s(t) \quad \text{(the limit exists since } s'(t) > 0 \text{ for all } t > 0\text{)},$$
$$s_* := \min\left\{\frac{\pi}{2}, \frac{\pi}{2}\sqrt{\frac{D}{r}}\right\},$$
$$s^* := \begin{cases} \left(\dfrac{\pi}{2}\sqrt{\dfrac{D}{r}}\right)\dfrac{1}{\sqrt{1-h}} & \text{if } D < r; \\ \dfrac{\pi}{2}\dfrac{1}{\sqrt{1-k}} & \text{if } D > r; \\ \min\left\{\dfrac{\pi}{2}\dfrac{1}{\sqrt{1-k}}, \dfrac{\pi}{2}\dfrac{1}{\sqrt{1-h}}\right\} & \text{if } D = r. \end{cases}$$

Note that $s_* < s^*$.

In this paper, we say that the two species *vanish eventually* if $s_\infty < +\infty$ and

$$\lim_{t\to+\infty} \|u(\cdot,t)\|_{C([0,s(t)])} = \lim_{t\to+\infty} \|v(\cdot,t)\|_{C([0,s(t)])} = 0,$$

we say that the two species *spread successfully* if $s_\infty = +\infty$ and the two species persist in the sense that $\liminf_{t\to+\infty} u(x,t) > 0$ and $\liminf_{t\to+\infty} v(x,t) > 0$ uniformly in any compact subset of $[0, +\infty)$. In fact, we have the following simple criteria for the vanishing and spreading.

Theorem 2 *Let* (u, v, s) *be a solution of* **(FBP)**. *Then the followings hold.*

(i) *If* $s_\infty \le s_*$, *then the two species vanish eventually.*
(ii) *If* $s_\infty > s^*$, *then the two species spread successfully.*

Theorem 2 does not provide any information for spreading–vanishing when $s_* < s_\infty \le s^*$. But if we add some more restrictions on the parameters for **(FBP)**, we can obtain a spreading–vanishing dichotomy, which was proposed initially by Du and Lin [11] for a single species case (see also [9,12,14]).

Before stating the following spreading–vanishing dichotomy result, we introduce the sets

$$A := \left\{0 < D < r,\ 0 < h \le 1 - \frac{D}{r},\ 0 < k < 1,\ \mu, \rho > 0\right\}, \tag{1.11}$$

$$B := \left\{0 < r < D,\ 0 < k \le 1 - \frac{r}{D},\ 0 < h < 1,\ \mu, \rho > 0\right\}. \tag{1.12}$$

Theorem 3 *Let* (u, v, s) *be a solution of* **(FBP)** *with* $(D, h, k, r, \mu, \rho) \in A \cup B$. *Then either* $s_\infty \le s_*$ *(and so the two species vanish eventually), or the two species spread successfully.*

Roughly speaking, Theorem 3 says that if D, r, μ and ρ are given and $D \ne r$, then a spreading–vanishing dichotomy can be assured either h or k is small enough.

Based on the previous results, we can provide some sufficient conditions for the spreading success and spreading failure via the initial data (u_0, v_0, s_0).

Corollary 1 *Let* (u, v, s) *be any solution of* **(FBP)**. *Then the followings hold.*

(i) *If* $s_0 \ge s^*$, *then the species* u *and* v *spread successfully.*
(ii) *Assume that* $(D, h, k, r, \mu, \rho) \in A \cup B$. *If* $s_0 \ge s_*$, *then the species* u *and* v *spread successfully.*
(iii) *If* $s_0 < s_*$ *and*

$$\max\{\ \|u_0\|_{L^\infty},\ \|v_0\|_{L^\infty}\ \} \le \cos\left(\frac{\pi}{2+\delta}\right) \frac{s_0^2 \alpha \delta (2+\delta)}{2\pi \mu (1+\rho)}, \tag{1.13}$$

then the species u *and* v *vanish eventually, where*

$$\delta := \frac{1}{2}\left[\frac{s_*}{s_0} - 1\right] > 0,$$

$$\alpha := \frac{1}{2} \min\left\{\left(\frac{\pi}{2}\right)^2 \frac{D}{(1+\delta)^2 s_0^2} - r,\ \left(\frac{\pi}{2}\right)^2 \frac{1}{(1+\delta)^2 s_0^2} - 1\right\} > 0.$$

In the case of spreading success, we have the following more precise asymptotic behavior.

Theorem 4 *Suppose that the two species spread successfully. Then*

$$(u, v)(x, t) \to \left(\frac{1-k}{1-hk}, \frac{1-h}{1-hk}\right) \text{ as } t \to +\infty, \tag{1.14}$$

uniformly in any compact subset of $[0, +\infty)$.

Our final result is to provide an upper bound for $\limsup_{t \to +\infty} [s(t)/t]$, which shows that the asymptotic spreading speed (if exists) for **(FBP)** with the weak competition cannot be faster than the minimal speed of traveling wavefront solutions to (1.3)–(1.4). Recall from Tang and Fife [37] that for $c \ge c_{\min} := \max\{2, 2\sqrt{rD}\}$ there exists a traveling wavefront solution of (1.3)–(1.4) with $u = U(x - ct)$ and $v = V(x - ct)$, connecting $(0, 0)$ with $(\frac{1-k}{1-hk}, \frac{1-h}{1-hk})$, while no such positive wavefronts exist for $c < c_{\min}$. Thus $c_{\min}$ is called the minimal speed of traveling wavefronts.

Theorem 5 *Let (u, v, s) be a solution of* **(FBP)** *with $s_\infty = +\infty$. Then*

$$\limsup_{t\to+\infty} \frac{s(t)}{t} \le c_{\min} = \max\{2, 2\sqrt{rD}\}.$$

The paper is organized as follows. In Sect. 2, we prove the solution of **(FBP)** exists globally (in time) and is unique (Theorem 1) using the contraction mapping theorem. Then, in Sect. 3, we derive several lemmas which are used to prove the main results. The main tool is the comparison principle. In Sect. 4, we study the long time behavior when the species spread successfully. A natural strategy is to find a pair of super and subsolutions with the same long time behavior, which is exactly what we desired, to squeeze the solution. However, it seems not easy to find such super and subsolutions for **(FBP)** at once. To overcome this difficulty we introduce a new idea. We first construct a non-trivial super/sub-solution to compare with the solution of **(FBP)** and introduce an iteration scheme so that we are able to construct better super and subsolutions step by step to derive the exact long time behavior of the solution. In Sect. 5, we give the proofs of Theorems 2–5. Finally, we give some discussion of our main results and some future directions in Sect. 6.

2 Existence and Uniqueness

In this section, we will prove Theorem 1. The proof can be done by modifying the arguments of [6,11] (see also [29]). We provide the details of proof here for the reader's convenience.

Lemma 2.1 *The problem* **(FBP)** *has a unique local solution $(u, v, s) \in C^{2+\alpha,1+\frac{\alpha}{2}}(\Omega_T) \times C^{2+\alpha,1+\frac{\alpha}{2}}(\Omega_T) \times C^{1+\frac{\alpha}{2}}([0, T])$ for any $\alpha \in (0, 1)$ for some $T > 0$ small enough, where*

$$\Omega_T := \{(x, t) : \ 0 \le x \le s(t), \ t \in (0, T]\}.$$

Proof Let $\zeta \in C^3([0, \infty))$ such that

$$\zeta(y) = 1 \text{ if } |y - s_0| \le \frac{s_0}{4}, \quad \zeta(y) = 0 \ \text{ if } |y - s_0| > \frac{s_0}{2}, \quad |\zeta'(y)| \le \frac{6}{s_0} \quad \text{for all } y.$$

Following [6], we introduce a transformation to straighten the free boundary:

$$(x, t) \to (y, t), \quad x = y + \zeta(y)(s(t) - s_0), \quad 0 \le y < +\infty.$$

Note that as long as $|s(t) - s_0| \le s_0/8$, $(x, t) \to (y, t)$ is a diffeomorphism from $[0, \infty)$ to $[0, \infty)$. Moreover,

$$0 \le x \le s(t) \iff 0 \le y \le s_0,$$
$$x = s(t) \iff y = s_0.$$

It is easy to get that

$$\frac{\partial y}{\partial x} = \frac{1}{1 + \zeta'(y)(s(t) - s_0)} := \sqrt{P(y, s(t))},$$
$$\frac{\partial^2 y}{\partial x^2} = \frac{-\zeta''(y)(s(t) - s_0)}{[1 + \zeta'(y)(s(t) - s_0)]^3} := Q(y, s(t)),$$
$$\frac{\partial y}{\partial t} = \frac{-s'(t)\zeta(y)}{1 + \zeta'(y)(s(t) - s_0)} := -s'(t)R(y, s(t)).$$

We now define $U(y,t) := u(x,t)$ and $V(y,t) := v(x,t)$ and set

$$F(U,V) := U(1-U-kV), \quad G(U,V) := rV(1-V-hU),$$

then the problem **(FBP)** becomes

$$U_t = PU_{yy} + (Q + s'(t)R)U_y + F(U,V), \quad 0 < y < s_0, \quad t > 0, \tag{2.1}$$
$$V_t = PDV_{yy} + (DQ + s'(t)R)V_y + G(U,V), \quad 0 < y < s_0, \quad t > 0, \tag{2.2}$$
$$U_y(0,t) = V_y(0,t) = 0, \quad U(s_0,t) = V(s_0,t) = 0, \quad t > 0, \tag{2.3}$$
$$s'(t) = -\mu[U_y(s_0,t) + \rho V_y(s_0,t)], \quad t > 0 \tag{2.4}$$
$$s(0) = s_0, \quad U(y,0) = U_0(y), \quad V(y,0) = V_0(y), \quad y \in [0,s_0], \tag{2.5}$$

where $U_0(y) = u_0(x)$ and $V_0(y) = v_0(x)$.

As in [11], we shall prove the local existence by using the contraction mapping theorem. To do so, choose T such that

$$0 < T \le \frac{s_0}{8(1+\tilde{s})}, \tag{2.6}$$

where $\tilde{s} := -\mu(U_0'(s_0) + \rho V_0'(s_0)) \ge 0$.

Introduce function spaces:

$$X_{1T} := \{U \in C(D) : \ U(y,0) = U_0(y), \ \|U - U_0\|_{C(D)} \le 1\},$$
$$X_{2T} := \{V \in C(D) : \ V(y,0) = V_0(y), \ \|V - V_0\|_{C(D)} \le 1\},$$
$$X_{3T} := \{s \in C^1([0,T]) : \ s(0) = 0, \ s'(0) = \tilde{s}, \ \|s' - \tilde{s}\|_{C([0,T])} \le 1\},$$

where $D := \{(x,t) : \ 0 \le y \le s_0, \ 0 \le t \le T\}$. Then, $X_T := X_{1T} \times X_{2T} \times X_{3T}$ is a complete metric space with the metric:

$$d((U_1,V_1,s_1),(U_2,V_2,s_2)) := \|U_1 - U_2\|_{C(D)} + \|V_1 - V_2\|_{C(D)} + \|s_1' - s_2'\|_{C([0,T])}.$$

For each $(U,V,s) \in X_T$, by (2.6),

$$|s(t) - s_0| \le \int_0^T |s'(r)|dr \le T(1+\tilde{s}) \le \frac{s_0}{8},$$

so that the mapping $(x,t) \to (y,t)$ is diffeomorphism.

For each $(U,V,s) \in X_T$, we consider the initial-boundary problem (P_U):

$$\bar{U}_t = P\bar{U}_{yy} + (Q + s'(t)R)\bar{U}_y + F(U,V), \quad 0 < y < s_0, \quad t > 0,$$
$$\bar{U}_y(0,t) = \bar{U}(s_0,t) = 0, \quad t > 0,$$
$$\bar{U}(y,0) = U_0(y), \quad 0 \le y \le s_0.$$

By using the L^p theory and the Sobolev embedding theorem, the system (P_U) has a unique solution $\bar{U}$ with

$$\|\bar{U}\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(D)} \le K_1, \tag{2.7}$$

for any $\alpha \in (0,1)$, where K_1 depends on s_0, α, $\|U_0\|_{C^2([0,s_0])}$ and $\|V_0\|_{C^2([0,s_0])}$. Similarly, for each $(U,V,s) \in X_T$, there exists a unique solution $\bar{V}$ satisfying

$$\bar{V}_t = DP\bar{V}_{yy} + (DQ + s'(t)R)\bar{V}_y + G(U,V), \quad 0 < y < s_0, \quad t > 0,$$
$$\bar{V}_y(0,t) = \bar{V}(s_0,t) = 0, \quad t > 0,$$
$$\bar{V}(y,0) = V_0(y), \quad 0 \le y \le s_0.$$

with

$$\|\bar{V}\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(D)} \le K_2, \tag{2.8}$$

for any $\alpha \in (0,1)$, where K_2 depends on $s_0, \alpha, \|U_0\|_{C^2([0,s_0])}$ and $\|V_0\|_{C^2([0,s_0])}$.

Also, set

$$\bar{s}(t) := s_0 - \mu \int_0^t [\bar{U}_y(s_0,\tau) + \rho \bar{V}_y(s_0,\tau)]d\tau. \tag{2.9}$$

Then, $\bar{s}'(t) = -\mu[\bar{U}_y(s_0,t) + \rho\bar{V}_y(s_0,t)] \in C^{\frac{\alpha}{2}}([0,T])$ with

$$\|\bar{s}'\|_{C^{\frac{\alpha}{2}}([0,T])} \le K_3, \tag{2.10}$$

where K_3 depends on $\mu, \rho, s_0, \alpha, \|U_0\|_{C^2([0,s_0])}$ and $\|V_0\|_{C^2([0,s_0])}$.

Now, introduce the mapping W such that $W(U,V,s) := (\bar{U},\bar{V},\bar{s})$. We shall prove that W has a unique fixed point, which implies that (2.1)–(2.5) admits a unique solution. By (2.7)–(2.10), and if

$$0 < T \le \min\left\{K_1^{\frac{-2}{1+\alpha}}, K_2^{\frac{-2}{1+\alpha}}, K_3^{\frac{-2}{\alpha}}\right\},$$

we have

$$\begin{aligned}
\|\bar{U} - U_0\|_{C(D)} &\le \|\bar{U}\|_{C^{\frac{1+\alpha}{2},0}} T^{\frac{1+\alpha}{2}} \le K_1 T^{\frac{1+\alpha}{2}} \le 1,\\
\|\bar{V} - V_0\|_{C(D)} &\le \|\bar{V}\|_{C^{\frac{1+\alpha}{2},0}} T^{\frac{1+\alpha}{2}} \le K_2 T^{\frac{1+\alpha}{2}} \le 1,\\
\|\bar{s}' - \tilde{s}\|_{C([0,T])} &\le \|\bar{s}'\|_{C^{\frac{\alpha}{2}}} T^{\frac{\alpha}{2}} \le K_3 T^{\frac{\alpha}{2}} \le 1,
\end{aligned}$$

which imply that W maps X_T into itself.

Finally, we show that W is a contraction mapping for sufficiently small T. Let $(\bar{U}_i, \bar{V}_i, \bar{s}_i) \in X_T$ for $i = 1,2$ and set $\mathcal{U} := \bar{U}_1 - \bar{U}_2$, $\mathcal{V} := \bar{V}_1 - \bar{V}_2$, we see that $\mathcal{U}$ satisfies

$$\begin{aligned}
&\mathcal{U}_t = P(y,s_2)\mathcal{U}_{yy} + [Q(y,s_2) + s_2' R(y,s_2)]\mathcal{U}_y + \mathcal{F},\\
&\mathcal{U}_y(0,t) = 0,\ \mathcal{U}(s_0,t) = 0,\ t > 0,\\
&\mathcal{U}(y,0) = 0,\ 0 \le y \le s_0.
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{F} = {}& [P(y,s_1) - P(y,s_2)]\bar{U}_{1,yy} + [Q(y,s_1) - Q(y,s_2) + s_1' R(y,s_1) - s_2' R(y,s_2)]\bar{U}_{1,y}\\
&+ (U_1 - U_2)[1 - (U_1 + U_2) - kV_2] - kU_1(V_1 - V_2) \in L^p(D).
\end{aligned}$$

Again, using the L^p theory and the Sobolev embedding theorem,

$$\|\mathcal{U}\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(D)} \le K_4\left(\|U_1 - U_2\|_{C(D)} + \|V_1 - V_2\|_{C(D)} + \|s_1 - s_2\|_{C^1([0,T])}\right), \tag{2.11}$$

for some $K_4 > 0$ which depends only on P, Q, R and K_i, $i = 1,2,3$. Similarly, we have also

$$\|\mathcal{V}\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(D)} \le K_5\left(\|U_1 - U_2\|_{C(D)} + \|V_1 - V_2\|_{C(D)} + \|s_1 - s_2\|_{C^1([0,T])}\right), \tag{2.12}$$

for some $K_5 > 0$ which depends only on P, Q, R and K_i, $i = 1, 2, 3$. By (2.9),

$$\|\bar{s}_1' - \bar{s}_2'\|_{C^{\frac{\alpha}{2}}([0,T])} \le K_6 \left(\|U_1 - U_2\|_{C(D)} + \|V_1 - V_2\|_{C(D)} + \|s_1 - s_2\|_{C^1([0,T])} \right), \tag{2.13}$$

for some $K_6 > 0$ which depends only on μ, ρ and K_i, $i = 4, 5$.

On the other hand, we have

$$\begin{aligned} &\|\mathcal{U}\|_{C(D)} + \|\mathcal{V}\|_{C(D)} + \|\bar{s}_1' - \bar{s}_2'\|_{C([0,T])} \\ &\le T^{\frac{1+\alpha}{2}} \|\mathcal{U}\|_{C^{1+\alpha, \frac{1+\alpha}{2}}(D)} + T^{\frac{1+\alpha}{2}} \|\mathcal{V}\|_{C^{1+\alpha, \frac{1+\alpha}{2}}(D)} + T^{\frac{\alpha}{2}} \|\bar{s}_1' - \bar{s}_2'\|_{C^{\frac{\alpha}{2}}([0,T])}, \end{aligned} \tag{2.14}$$

Together with (2.11)–(2.14), and if $T \in (0, 1]$, then

$$\begin{aligned} &\|\mathcal{U}\|_{C(D)} + \|\mathcal{V}\|_{C(D)} + \|\bar{s}_1' - \bar{s}_2'\|_{C([0,T])} \\ &\le K_7 T^{\frac{\alpha}{2}} (\|U_1 - U_2\|_{C(D)} + \|V_1 - V_2\|_{C(D)} + \|s_1' - s_2'\|_{C([0,T])}), \end{aligned}$$

where $K_7 := \max\{K_4, K_5, K_6\}$. By choosing

$$T := \frac{1}{2} \min \left\{ 1, \frac{s_0}{8(1 + \tilde{s})}, K_1^{\frac{-2}{1+\alpha}}, K_2^{\frac{-2}{1+\alpha}}, K_3^{\frac{-2}{\alpha}}, K_7^{\frac{-2}{\alpha}} \right\}$$

we can apply the contraction mapping theorem, then W has a unique fixed point in X_T. This completes the proof of Lemma 2.1. □

To prove the existence of solution for all $t > 0$, we need the following lemma.

Lemma 2.2 *Let* (u, v, s) *be a solution of* **(FBP)** *for* $t \in [0, T]$ *for some* $T > 0$. *Then*

$$0 < u(x, t) \le \max\{1, \|u_0\|_{L^\infty}\} \quad \text{for } x \in [0, s(t)),\ t \in [0, T], \tag{2.15}$$

$$0 < v(x, t) \le \max\{1, \|v_0\|_{L^\infty}\} \quad \text{for } x \in [0, s(t)),\ t \in [0, T], \tag{2.16}$$

$$0 < s'(t) \le \mu\Lambda \quad \text{for } t \in (0, T], \tag{2.17}$$

where $\Lambda > 0$ *depending only on* D, r, ρ, $\|u_0\|_{L^\infty}, \|v_0\|_{L^\infty}, \min_{x\in[0,s_0]} u_0'(x)$ *and* $\min_{x\in[0,s_0]} v_0'(x)$.

Proof The strong maximal principle yields that $u > 0$ and $v > 0$ for $x \in [0, s(t))$, $t \in [0, T]$. Thus, we see from (1.7) that $u_x(s(t), t) < 0$ and $v_x(s(t), t) < 0$ for $t \in (0, T]$. By (1.8), $s'(t) > 0$ for $t \in (0, T]$.

Let $\bar{u} = \bar{u}(t)$ be the solution of $u' = u(1 - u)$ with $\bar{u}(0) = \|u_0\|_{L^\infty}$. The comparison principle implies that $u(x, t) \le \bar{u}(t) \le \max\{1, \|u_0\|_{L^\infty}\}$ for all $x \in [0, s(t)]$, $t \in [0, T]$. Similarly, we have $v(x, t) \le \max\{1, \|v_0\|_{L^\infty}\}$ for $x \in [0, s(t)]$, $t \in [0, T]$.

To prove (2.17), we shall compare u and v with some auxiliary functions (cf. [11]). Note that the solution (u, v) can be extended from $(x, t) \in [0, s(t)] \times [0, T]$ into $(x, t) \in [-s(t), s(t)] \times [0, T]$ by letting $u(x, t) := u(-x, t)$ and $v(x, t) := v(-x, t)$ for $x \in [-s(t), 0]$ and $t \in [0, T]$. Define

$$w(x, t) := R \left[2M_1(s(t) - x) - M_1^2 (s(t) - x)^2 \right],$$

where

$$M_1 := \max \left\{ \frac{4}{3}, \frac{-4}{3} \left(\min_{x\in[0,s_0]} u_0'(x) \right) \right\}, \quad R := \max\{1, \|u_0\|_{L^\infty}\}. \tag{2.18}$$

Also, we set

$$\eta(t) := \max\left\{-s(t),\ s(t) - \frac{1}{M_1}\right\}, \quad \eta(0) := \eta_0.$$

Hereafter we shall apply the comparison principle over

$$\Omega_{M_1} := \{(x,t);\ \eta(t) \le x \le s(t),\ 0 \le t \le T\}.$$

Firstly, we derive that $w(x,0) \ge u_0(x)$ for $x \in [\eta_0, s_0]$. Indeed, since $w(\cdot, 0)$ is concave, we obtain that

$$w_x(x,0) \le w_x(s_0 - (2M_1)^{-1}, 0) = -RM_1 \le -\frac{3}{4}RM_1 \tag{2.19}$$

for $x \in [s_0 - (2M_1)^{-1}, s_0]$. By using (2.18) and the fact that $R \ge 1$,

$$RM_1 \ge \frac{-4}{3} \min_{x\in[0,s_0]} u_0'(x). \tag{2.20}$$

Combing (2.19) and (2.20), if necessary we may define $u_0'(x) \equiv 0$ for $x \in [s_0 - (2M_1)^{-1}, -s_0)$, then we obtain

$$w_x(x,0) \le \min_{x\in[0,s_0]} u_0'(x) \le u_0'(x)$$

for $x \in [s_0 - (2M_1)^{-1}, s_0]$, Integrating over $[x, s_0]$ and using $w(s_0, 0) = 0 = u_0(s_0)$, we have

$$w(x,0) \ge u_0(x) \quad \text{for } x \in \left[s_0 - (2M_1)^{-1}, s_0\right]. \tag{2.21}$$

Thus, we get that $w(x,0) \ge u_0(x)$ for $x \in [\eta_0, s_0]$ if $\eta_0 \ge s_0 - (2M_1)^{-1}$.

It suffices to consider the case that $\eta_0 < s_0 - (2M_1)^{-1}$. Again, using the concavity of $w(\cdot,0)$ and $w_x(s_0 - M_1^{-1}, 0) = 0$, then for all $x \in [\eta_0, s_0 - (2M_1)^{-1}]$,

$$w(x,0) \ge w\left(s_0 - (2M_1)^{-1}, 0\right) = \frac{3}{4}RM_1 \ge \|u_0\|_{L^\infty} \ge u_0(x).$$

Hence, together with (2.21) we have proved that $w(x,0) \ge u_0(x)$ for $x \in [\eta_0, s_0]$.

On the other hand, one can easily compute that

$$w_t - w_{xx} \ge 2M_1^2 R \ge u(1-u-kv) = u_t - u_{xx}$$

in Ω_M, due to $M \ge 1/\sqrt{2}$. Also, note that $w(s(t),t) = 0 = u(s(t),t)$ for all $t \in [0,T]$. Moreover, by (2.15), we have

$$w(s(t) - M_1^{-1}, t) = R \ge u(s(t) - M_1^{-1}, t) \quad \text{for all } t \in [0,T].$$

Together with the fact that $u(-s(t),t) = 0$ for $t \in [0,T]$, it follows that $w(\eta(t),t) \ge u(\eta(t),t)$ for $t \in [0,T]$. Then the comparison principle yields that $w \ge u$ in Ω_M.

Since $w(s(t),t) = 0 = u(s(t),t)$, we then obtain that

$$u_x(s(t),t) \ge w_x(s(t),t) = -2M_1 R. \tag{2.22}$$

Similarly, we can prove that

$$v_x(s(t),t) \ge -2M_2 \max\{1, \|v_0\|_{L_\infty}\}, \tag{2.23}$$

where

$$M_2 := \max\left\{\sqrt{\frac{r}{2D}},\ \frac{4}{3},\ \frac{-4}{3}\left(\min_{x\in[0,s_0]} v_0'(x)\right)\right\}.$$

Combing (2.22) and (2.23) and (1.8), we have proved that $s'(t) \le \mu\Lambda$, where

$$\Lambda := 2M_1 \max\{1, \|u_0\|_{L_\infty}\} + 2\rho M_2 \max\{1, \|v_0\|_{L_\infty}\}.$$

This completes the proof of Lemma 2.2. □

Combing Lemma 2.1 and Lemma 2.2, we can prove Theorem 1 as follows.

Proof of Theorem 1 By Lemma 2.1, we can define $T_{\max} > 0$ as the maximal existence time of the solution. We assume that $T_{\max} < \infty$ for contradiction. By Lemma 2.2, there exists a positive constant K which dose not depend on $T_{\max}$ such that $0 \le u(x,t), v(x,t), s'(t) \le K$ for all $x \in [0, s(t)]$ and $t \in [0, T_{\max})$. In particular,

$$s_0 \le s(t) \le s_0 + Kt \quad \text{for all } t \in [0, T_{\max}).$$

Fix $\epsilon \in (0, T_{\max})$ and $A > T_{\max}$, it follows from the standard regularity theory that there exists $K' > 0$ depending only on ϵ, A, K such that

$$\|u(\cdot,t)\|_{C^2([0,s(t)])},\ \|v(\cdot,t)\|_{C^2([0,s(t)])} \le K' \quad \forall\, t \in [\epsilon, T_{\max}).$$

Following the proof of Lemma 2.1, there exists a $\tau > 0$ which depends only on K and K' such that the solution of (**FBP**) with any initial time $t \in [\epsilon, T_{\max})$ can be uniquely extended to the interval $[t, t+\tau)$. This contradicts with the definition of $T_{\max}$ because the solution with the initial time $T_{\max} - \tau/2$ can be uniquely extended to the time $T_{\max} + \tau/2$. Thus, $T_{\max} = \infty$. This completes the proof of Theorem 1. □

3 Preliminaries

The basic technique we use in this section is the comparison principle over some suitable parabolic regions. The first lemma will be used later frequently. It can be thought as a special case of Proposition 3.3 in [5]. Consider the problem (P_0):

$$\begin{aligned} &u_t = Du_{xx} + ru(1-bu), \quad x \in (0,l), \quad t > 0,\\ &u_x(0,t) = 0,\ u(l,t) = 0, \quad \text{for } t > 0, \end{aligned}$$

for given $b, r, D > 0$.

Lemma 3.1 *Let* $l^* := \frac{\pi}{2}\sqrt{\frac{D}{r}}$. *Then we have:* (i) *all positive solutions of* (P_0) *tend to zero in* $C([0,l])$ *as* $t \to \infty$, *if* $l \le l^*$, (ii) *there exists a unique positive stationary solution* ϕ *of* (P_0) *such that all positive solutions of* (P_0) *approach* ϕ *in* $C([0,l])$ *as* $t \to \infty$, *if* $l > l^*$.

Furthermore, we need the following lemma which is a special case of the Corollary 1 of Murray and Sperb [33].

Lemma 3.2 *Given* $\kappa \in \mathbb{R}$. *The the smallest eigenvalue* $\lambda_1(\kappa)$ *of*

$$w'' + \kappa w' + \lambda w = 0,\ x \in [-l, l] \ \text{ with } w(\pm l) = 0$$

is given by

$$\lambda_1(\kappa) = \left(\frac{\pi}{2l}\right)^2 + \left(\frac{\kappa}{2}\right)^2.$$

For the smallest eigenvalue $\lambda_1(\kappa)$, it is easy to find an eigenfunction

$$w(x) = e^{-\frac{\kappa}{2}x} \cos\left(\frac{\pi x}{2l}\right). \tag{3.1}$$

Note that $w'(x) < 0$ for all $x \in [0, l]$. The monotonicity of w on $[0, l]$ plays an important role later on.

Lemma 3.3 *Let* (u, v, s) *be a solution of* **(FBP)**. *If* $s_\infty < +\infty$, *then*

$$s'(t) \to 0 \quad \text{as } t \to +\infty$$

Proof By the standard transformation

$$y := \frac{x}{s(t)}, \quad \hat{u}(y, t) := u(x, t) \quad \text{and} \quad \hat{v}(y, t) := v(x, t).$$

The free boundary problem **(FBP)** can be transformed into a fixed boundary problem. Then by the standard L^p theory and the embedding theorem, we see that $\hat{u}$ and $\hat{v}$ have a uniform $C^{1+\alpha,(1+\alpha)/2}$ bound over $\{(y, t) : 0 \le y \le 1,\ t \in [\tau, \tau+1]\}$ for any $\tau \ge 1$, where $\alpha \in (0, 1)$. Note that this bound is independent of τ. Hence there exists a positive constant C such that

$$\|s'\|_{C^{\frac{\alpha}{2}}[1,\infty)} \le C, \tag{3.2}$$

by using the boundary condition (1.8).

Now, for contradiction, suppose that there exists a sequence $\{t_n\}$ such that $t_n \to \infty$ and $s'(t_n) \to \sigma$ as $n \to \infty$ for some $\sigma > 0$. Due to (3.2) we can find $\varepsilon > 0$ small enough such that $s'(t) \ge \sigma/2$ for all $t \in [t_n - \varepsilon, t_n + \varepsilon]$ for all n. Then we obtain

$$s_\infty = s_0 + \int_0^\infty s'(t)dt \ge s_0 + \sum_{n=1}^{\infty} \int_{t_n-\varepsilon}^{t_n+\varepsilon} \frac{\sigma}{2} dt = \infty,$$

a contradiction. Hence Lemma 3.3 follows. □

Lemma 3.4 *Let* (u, v, s) *be a solution of* **(FBP)**. *If* $s_\infty > s^*$, *then* $s_\infty = +\infty$.

Proof We divide our discussions into three cases: (i) $D < r$ (ii) $D > r$ (iii) $D = r$.

Case (i): In this case, we have

$$s^* = \frac{\pi}{2}\sqrt{\frac{D}{r}} \frac{1}{\sqrt{1-h}}.$$

For contradiction, we assume that $s_\infty \in (s^*, +\infty)$. Then one can choose $l \in (s^*, s_\infty)$ which is sufficiently close to s_∞ and $\epsilon > 0$ small enough such that

$$l > \frac{\pi}{2}\sqrt{\frac{D}{r}} \left[\sqrt{1 - h - \epsilon - \frac{1}{rD}\left(\frac{\epsilon}{2}\right)^2}\,\right]^{-1} > s^*, \tag{3.3}$$

$$\left(\frac{s_\infty}{l}\right) s'(t) \le \epsilon \quad \forall\, t \ge s^{-1}(l). \quad \text{(Due to Lemma 3.3)} \tag{3.4}$$

For such a fixed l, we define w as in (3.1) with $\kappa := \epsilon/D$ and

$$(\bar{u}, \underline{v})(x,t) := \left(1+\epsilon\,,\, \delta w\left(\frac{lx}{s(t)}\right)\right),$$

where $\delta > 0$ is to be determined.

We shall compare (u, v) with $(\bar{u}, \underline{v})$ over Ω_T, where

$$\Omega_T := \{(x,t):\ 0 \le x \le s(t),\ t \ge T\} \text{ for some } T \gg 1.$$

To do so, we first prove that there exists $T_0 > 0$ such that

$$u(x,t) \le 1+\epsilon,\ \forall\,(x,t) \in [0, s(t)] \times [T_0, +\infty). \tag{3.5}$$

Let $\eta(t)$ be the solution of $\eta_t = \eta(1-\eta)$ with $\eta(0) = \|u_0\|_{C[0,s_0]}$. By the comparison principle, $u(x,t) \le \eta(t)$ for all $x \in [0, s(t)]$ and $t \ge 0$. Letting $t \to +\infty$ yields that $\limsup_{t\to+\infty} u(x,t) \le 1$ uniformly for $x \in [0, s_\infty)$. Hence (3.5) follows.

We next prove that there exist $\theta > 0$ and $T_1 > T_0$ such that $v(0,t) \ge \theta$ for all $t \ge T_1$. For this, recall from (3.3) that

$$l > \frac{\pi}{2}\sqrt{\frac{D}{r}}\frac{1}{\sqrt{1-h-\epsilon}}.$$

Using Lemma 3.1 one can find $\phi > 0$ satisfying

$$D\phi'' + r\phi[(1-h-\epsilon)-\phi] = 0,\ 0 < x < l,\ \phi'(0) = 0 \text{ and } \phi(l) = 0.$$

By using (3.5) and choosing $\nu > 0$ sufficiently small, one can compare $\nu\phi$ with v to obtain $v \ge \nu\phi$ for all $x \in [0, l]$ and $t \in [T_1, \infty)$ for some $T_1 > T_0$ large enough. Hence we obtain that $v(0,t) \ge \theta := \nu\phi(0) > 0$ for all $t \ge T_1$.

We now fix $T > \max\{s^{-1}(l), T_1\}$ and choose $0 < \delta \ll 1$ such that $v(x,T) \ge \delta\underline{v}(x,T)$ for all $x \in [0, s(T)]$ and $v(0,t) \ge \theta \ge \delta\underline{v}(0,t)$ for all $t \ge T$. Also, note that $\underline{v}(s(t),t) = 0 = v(s(t),t)$ and $\bar{u}_t \ge \bar{u}_{xx} + \bar{u}(1-\bar{u}-k\underline{v})$. To compare $(\bar{u}, \underline{v})$ with (u, v) over Ω_T, it suffices to show that

$$\underline{v}_t \le D\underline{v}_{xx} + r\underline{v}(1-\underline{v}-h\bar{u}). \tag{3.6}$$

Direct calculation yields that

$$\begin{aligned}
&\underline{v}_t - D\underline{v}_{xx} - r\underline{v}(1-\underline{v}-h\bar{u})\\
&= -\delta\left(\frac{s'(t)lx}{s^2(t)}\right)w' - \delta D\left(\frac{l}{s(t)}\right)^2 w'' - \delta r w[(1-h-\epsilon)-\delta w]\\
&\le \delta\left(\frac{l}{s(t)}\right)^2\left[\epsilon - \left(\frac{s_\infty}{l}\right)s'(t)\right]w' + \delta w\,[r\delta w + D\lambda_1(\kappa) - r(1-h-\epsilon)]
\end{aligned}$$

for all $(x,t) \in \Omega_T$.

Using (3.4) and the fact that $w' < 0$ for all $x \in [0, l]$, to derive (3.6) it suffices to show that

$$r\delta w + D\lambda_1(\kappa) - r(1-h-\epsilon) \le 0 \quad \text{in } \Omega_T.$$

Note that (3.3) is equivalent to

$$D\lambda_1(\kappa) - r(1-h-\epsilon) < 0,$$

so (3.6) holds if necessary we choose a smaller δ. Then the comparison principe yields that $\underline{v}(x,t) \le v(x,t)$ for $(x,t) \in \Omega_T$. Moreover, we see from $\underline{v}(s(t),t) = 0 = v(s(t),t)$ that

$$\underline{v}_x(s(t),t) \ge v_x(s(t),t) \quad \forall\, t > T. \tag{3.7}$$

By taking $t \to +\infty$ it follows that

$$0 \le \frac{\delta l}{s_\infty} w'(l) < 0,$$

a contradiction. Thus we conclude that $s_\infty = \infty$ if $s > s^*$.

Case (ii) can be proved by a similar argument as in Case (i), so we omit here. For Case (iii), we may assume, without loss of generality,

$$\frac{\pi}{2}\frac{1}{\sqrt{1-h}} < \frac{\pi}{2}\frac{1}{\sqrt{1-k}}.$$

Then the proof can be done by using the same argument as in Case (i). □

Recall that $s_* = \min\left\{\frac{\pi}{2}, \frac{\pi}{2}\sqrt{\frac{D}{r}}\right\}$.

Lemma 3.5 *When $D \ne r$, $s_\infty \notin \left(s_*, \max\left\{\frac{\pi}{2}, \frac{\pi}{2}\sqrt{\frac{D}{r}}\right\}\right]$.*

Proof We argue by contradiction. Since the proof of $D < r$ and $D > r$ are similar, so we only consider the case for $D < r$. Hence we assume that $s_\infty \in (s_*, \pi/2]$. Our goal is to show that $v(\cdot,t)$ converges to some function in $C^2([0, s_\infty))$ to reach a contradiction. Such idea is from [9].

Let (u, v, s) be the solution of **(FBP)** and $\bar{u}$ be the solution of

$$\begin{cases} \bar{u}_t = \bar{u}_{xx} + \bar{u}(1-\bar{u}), & 0 < x < \pi/2, \quad t > 0, \\ \bar{u}_x(0,t) = 0, \quad \bar{u}(\pi/2, t) = 0, & t > 0, \\ \bar{u}(x,0) = \begin{cases} u(x,0), & \text{if } x \in [0, s_0], \\ 0, & \text{if } x \in [s_0, \pi/2]. \end{cases} \end{cases} \tag{3.8}$$

Then, by Lemma 3.1,

$$\lim_{t\to+\infty} \|\bar{u}(\cdot,t)\|_{C([0,\pi/2])} \to 0 \text{ as } t \to +\infty. \tag{3.9}$$

Comparing $(\bar{u}, 0)$ with (u, v) yields that

$$\bar{u}(x,t) \ge u(x,t) \quad \text{for all } x \in [0, s(t)], \quad t > 0. \tag{3.10}$$

On the other hand, let $\bar{v}$ be the solution of

$$\begin{cases} \bar{v}_t = D\bar{v}_{xx} + r\bar{v}(1-\bar{v}), & 0 < x < s_\infty, \quad t > 0, \\ \bar{v}_x(0,t) = 0, \quad \bar{v}(s_\infty, t) = 0, & t > 0, \\ \bar{v}(x,0) = \begin{cases} v(x,0), & \text{if } x \in [0, s_0], \\ 0, & \text{if } x \in [s_0, s_\infty]. \end{cases} \end{cases}$$

Again, by Lemma 3.1, we have

$$\lim_{t\to+\infty} \|\bar{v}(\cdot,t) - v_\infty(\cdot)\|_{C([0,s_\infty])} = 0 \text{ as } t \to +\infty, \tag{3.11}$$

where $v_\infty > 0$ satisfies

$$\begin{cases} Dv''_\infty + rv_\infty(1 - v_\infty) = 0, & 0 < x < s_\infty, \\ v'_\infty(0) = v_\infty(s_\infty) = 0. \end{cases} \tag{3.12}$$

Comparing $(0, \bar{v})$ with (u, v) implies that

$$\bar{v}(x,t) \geq v(x,t) \quad \text{for all } x \in [0, s(t)], \quad t > 0. \tag{3.13}$$

Combining (3.11) with (3.13), we obtain

$$\limsup_{t\to+\infty} v(x,t) \leq v_\infty(x) \quad \text{for } x \in [0, s_\infty). \tag{3.14}$$

Next, we shall estimate $\liminf_{t\to+\infty} v(x,t)$. Choose $s_n \in (\frac{\pi}{2}\sqrt{\frac{D}{r}}, s_\infty)$ with $s_n \uparrow s_\infty$ as $n \to +\infty$ and fix s_1 such that s_1 is close enough to s_∞, then $\{s_n\}$ can have the following property:

$$s_n > \frac{\pi}{2}\sqrt{\frac{D}{r}}\frac{1}{\sqrt{1-(s_\infty - s_n)}} > 0 \quad \text{for all } n \in \mathbb{N}.$$

Thus, thanks to Lemma 3.1, for each n there exists a unique $v_n(x) > 0$ satisfying

$$\begin{cases} Dv''_n + rv_n[1 - (s_\infty - s_n) - v_n] = 0, & 0 < x < s_n, \\ v'_n(0) = v_n(s_n) = 0. \end{cases}$$

For each $j \in \mathbb{N}$, since v_n is bounded in $C^{2+\alpha}([0, s_j])$ for all $n \geq j$, by the Arzela–Ascoli Theorem and the diagonal process, we obtain that $v_n \to v_\infty$ in $C^2_{loc}([0, s_\infty))$ as $n \to \infty$ (up to a subsequence), where v_∞ satisfies (3.12).

For each n, combing (3.9), one can find $T_n > 0$ such that

$$h\bar{u} \leq s_\infty - s_n \quad \text{for all } x \in [0, s(t)], \quad t \in [T_n, \infty). \tag{3.15}$$

Let $\underline{v}_n(x,t)$ be the solution of

$$\begin{cases} (\underline{v}_n)_t = D(\underline{v}_n)_{xx} + r\underline{v}_n[1 - (s_\infty - s_n) - \underline{v}_n], & 0 < x < s_n, \quad t > 0, \\ (\underline{v}_n)_x(0,t) = 0, \ (\underline{v}_n)(s_n, t) = 0, & t > T_n, \\ \underline{v}_n(x, T_n) = \begin{cases} v(x, T_n), & \text{if } x \in [0, s_0], \\ 0, & \text{if } x \in [s_0, s_n]. \end{cases} \end{cases}$$

Also, we see from (3.15) that

$$\begin{aligned} &(\underline{v}_n)_t - D(\underline{v}_n)_{xx} - r\underline{v}_n(1 - \underline{v}_n - h\bar{u}) \\ &\leq (\underline{v}_n)_t - D(\underline{v}_n)_{xx} - r\underline{v}_n[1 - (s_\infty - s_n) - \underline{v}_n] = 0, \end{aligned}$$

for all $x \in [0, s_n]$ and $t \in [T_n, \infty)$. Hence, together with (3.8) and (3.10) we can compare $(\bar{u}, \underline{v}_n)$ with (u, v) over

$$\{(x,t) : (x,t) \in [0, s_n] \times [T_n, \infty)\},$$

which yields $v \geq \underline{v}_n$ for all $(x,t) \in [0, s_n] \times [T_n, \infty)$. Again, using Lemma 3.1 yields $\underline{v}_n \to v_n$ in $C([0, s_n])$ as $t \to +\infty$. Thus, we obtain that for each n,

$$\liminf_{t\to+\infty} v(x,t) \geq v_n(x) \quad \text{for } x \in [0, s_n].$$

Taking $n \to +\infty$ we have

$$\liminf_{t\to+\infty} v(x,t) \geq v_\infty(x) \quad \text{for } x \in [0, s_\infty), \tag{3.16}$$

where v_∞ satisfies (3.12).

From (3.14) and (3.16) we see that $\lim_{t\to+\infty} v(x,t) = v_\infty(x)$ for $x \in [0, s_\infty)$. Finally, following the process of Lemma 2.2 in [9] we can derive

$$\lim_{t\to+\infty} \|v(\cdot,t) - v_\infty(\cdot)\|_{C^2([0,s(t)])} = 0,$$

which implies that $v_x(s(t),t) \to v'_\infty(s_\infty) < 0$ as $t \to +\infty$. Hence we can find $\beta > 0$ such that $s'(t) \geq \beta$ for all large t by using (1.8). But this contradicts Lemma 3.3. Hence we complete the proof of Lemma 3.5. □

4 Long Time Behavior of Solutions when $s_\infty = \infty$

In this section, we shall derive (1.14) when $s_\infty = \infty$. Firstly, the persistence for the two species can be established.

Lemma 4.1 *Let (u, v, s) be a solution of* **(FBP)** *with $s_\infty = +\infty$. Then*

(i) $\limsup_{t\to+\infty} u(x,t) \leq 1$ *and* $\limsup_{t\to+\infty} v(x,t) \leq 1$ *uniformly in* $x \in [0, +\infty)$,
(ii) $\liminf_{t\to+\infty} u(x,t) \geq 1-k$ *and* $\liminf_{t\to+\infty} v(x,t) \geq 1-h$ *uniformly in any compact subset of* $[0, +\infty)$.

Proof Let $\bar{u}$ be the solution of $\bar{u}_t = \bar{u}(1-\bar{u})$ with $\bar{u}(0) = \|u_0\|_{C([0,s_0])}$. Then it follows that $u(x,t) \leq \bar{u}(t)$ for all $x \in [0, s(t)]$, $t \geq 0$. Taking $t \to +\infty$, we obtain that $\limsup_{t\to+\infty} u(x,t) \leq 1$. Similarly, we have $\limsup_{t\to+\infty} v(x,t) \leq 1$ and so part (i) holds.

We now prove (ii). For any $\varepsilon \in (0,1)$ such that $1 - k(1+\varepsilon) > 0$, we fix l so that

$$l > \left\{ \frac{\pi}{2} \left(\frac{1}{\sqrt{1-k(1-\varepsilon)}} \right), s_0 \right\}.$$

Since $s_\infty = \infty$ and using (i), one can find $T_l > 0$ such that $s(T_l) = l$ and $v(x,t) \leq 1+\varepsilon$ for $(x,t) \in [0,l] \times [T_l, \infty)$. Let $\underline{u}^l$ be the solution of

$$\begin{cases} \underline{u}_t = \underline{u}^l_{xx} + \underline{u}^l[1-k(1+\varepsilon) - \underline{u}^l], & 0 < x < l, \quad t > T_l, \\ \underline{u}^l(l,t) = \underline{u}^l_x(0,t) = 0, & t > T_l, \\ \underline{u}^l(x, T_l) = u(x, T_l), & 0 \leq x \leq l. \end{cases}$$

Comparing $(\underline{u}^l, 1+\varepsilon)$ with (u, v) yields that $u \geq \underline{u}^l$ for $(x,t) \in [0,l] \times [T_l, \infty)$. By Lemma 3.1, $\underline{u}^l(x,t) \to u^l_*(x)$ in $C([0,l])$ as $t \to +\infty$, where $u^l_* > 0$ satisfies

$$\begin{cases} (u^l_*)_{xx} + u^l_*[1-k(1+\varepsilon) - u^l_*] = 0, & 0 < x < l, \\ (u^l_*)_x(0) = 0, \quad u^l_*(l) = 0. \end{cases}$$

Thus $\liminf_{t\to+\infty} u(x,t) \geq u^l_*(x)$ uniformly in $[0,l]$.

On the other hand, $u^l_*(x) \to 1 - k(1+\varepsilon)$ uniformly in any compact subset of $[0,\infty)$ as $l \to +\infty$ (cf. Lemma 2.2 of Du and Ma [13]), which implies that $\liminf_{t\to+\infty} u(x,t) \geq 1 - k(1+\varepsilon)$ uniformly in any compact subset of $[0,\infty)$. Letting $\varepsilon \to 0^+$, it follows that $\liminf_{t\to+\infty} u(x,t) \geq 1-k$ uniformly in any compact subset of $[0,\infty)$. By a similar argument, we can prove that $\liminf_{t\to+\infty} v(x,t) \geq 1-h$ uniformly in any compact subset of $[0,\infty)$. This completes the proof of Lemma 4.1. □

Lemma 4.2 *Assume that* $0 < h, k < 1$.

(i) *Consider two sequences* $\{\bar{u}_n\}_{n\in\mathbb{N}}$ *and* $\{\underline{v}_n\}_{n\in\mathbb{N}}$ *defined as follows:*

$$(\bar{u}_1, \underline{v}_1) := (1, 1-h), \quad (\bar{u}_{n+1}, \underline{v}_{n+1}) := (1 - k\underline{v}_n, 1 - h(1 - k\underline{v}_n)).$$

Then $\bar{u}_n > \bar{u}_{n+1} > 0$ *and* $\underline{v}_n < \underline{v}_{n+1} < 1$ *for all* $n \in \mathbb{N}$. *Moreover,*

$$(\bar{u}_n, \underline{v}_n) \to \left(\frac{1-k}{1-hk}, \frac{1-h}{1-hk}\right) \text{ as } n \to +\infty.$$

(ii) *Consider two sequences* $\{\underline{u}_n\}_{n\in\mathbb{N}}$ *and* $\{\bar{v}_n\}_{n\in\mathbb{N}}$ *defined as follows:*

$$(\underline{u}_1, \bar{v}_1) := (1-k, 1), \quad (\underline{u}_{n+1}, \bar{v}_{n+1}) := (1 - k(1 - h\underline{u}_n), 1 - h\underline{u}_n).$$

Then $\underline{u}_n < \underline{u}_{n+1} < 1$ *and* $\bar{v}_n > \bar{v}_{n+1} > 0$ *for all* $n \in \mathbb{N}$. *Moreover,*

$$(\underline{u}_n, \bar{v}_n) \to \left(\frac{1-k}{1-hk}, \frac{1-h}{1-hk}\right) \text{ as } n \to +\infty.$$

Proof The proof of (i) and (ii) are similar, we only show (i). By induction, it is easy to see that $\bar{u}_n > \bar{u}_{n+1} > 0$ and $\underline{v}_n < \underline{v}_{n+1} < 1$ for all $n \in \mathbb{N}$. Hence $u_\infty := \lim_{n\to+\infty} u_n$ and $v_\infty := \lim_{n\to+\infty} v_n$ are well-defined and are finite. From $(u_\infty, v_\infty) = (1 - v_\infty, 1 - h(1 - kv_\infty))$ we can see that $(u_\infty, v_\infty) = (\frac{1-k}{1-hk}, \frac{1-h}{1-hk})$. The proof of Lemma 4.2 is completed. □

The symbols $\bar{u}_n, \underline{v}_n, \underline{u}_n, \bar{v}_n, u_\infty$ and v_∞ defined in Lemma 4.2 will be always used in this section.

Lemma 4.3 *Let* (u, v, s) *be a solution of* **(FBP)** *with* $s_\infty = +\infty$. *Then*

$$\underline{u}_2 \le \liminf_{t\to+\infty} u(x,t) \le \limsup_{t\to+\infty} u(x,t) \le \bar{u}_2,$$
$$\underline{v}_2 \le \liminf_{t\to+\infty} v(x,t) \le \limsup_{t\to+\infty} v(x,t) \le \bar{v}_2,$$

uniformly in any compact subset of $[0, +\infty)$.

Proof We first prove that $\limsup_{t\to+\infty} u(x,t) \le \bar{u}_2$ uniformly in any compact subset of $[0, +\infty)$. For any given $\varepsilon \in (0, \frac{1-h}{3h})$, by Lemma 4.1, there exists $T_\varepsilon \gg 1$ such that

$$\begin{cases} v(x,t) \ge 1 - h - 3h\varepsilon = \underline{v}_1 - 3h\varepsilon > 0, & (x,t) \in [0, 2S_\varepsilon] \times [T_\varepsilon, \infty) \\ u(x,t) \le 1 + \varepsilon = \bar{u}_1 + \varepsilon, & (x,t) \in [0, 2S_\varepsilon] \times [T_\varepsilon, \infty), \end{cases}$$

where $S_\varepsilon := \frac{1}{2\varepsilon}\sqrt{\frac{b_\varepsilon \pi}{\alpha}}$, $\alpha := \frac{1}{2}(1 - \frac{1}{\pi})$ and $b_\varepsilon > 0$ is to be determined.

To compare with (u, v) we need to construct a suitable supersolution $(\bar{U}(x,t), \underline{V}(x,t))$. To do so, let

$$a_\varepsilon := \bar{u}_2 + 3hk\varepsilon = 1 - k(\underline{v}_1 - 3h\varepsilon) > 0,$$
$$b_\varepsilon := \bar{u}_1 + \varepsilon - a_\varepsilon = 1 + \varepsilon - a_\varepsilon > 0.$$

Then, we define $(\bar{U}(x,t), \underline{V}(x,t)) = (\phi(t) + \psi(x) + \varepsilon, \underline{v}_1 - 3h\varepsilon)$, where ϕ satisfies $\phi_t = \phi(a_\varepsilon - \phi)$ with $\phi(T_\varepsilon) = 1 + \varepsilon$, and

$$\psi(x) := \begin{cases} 0, & x \in [0, S_\varepsilon], \\ \dfrac{b_\varepsilon}{2\alpha S_\varepsilon}\left[x - S_\varepsilon - \dfrac{2S_\varepsilon}{\pi}\sin\left(\dfrac{(x - S_\varepsilon)\pi}{2S_\varepsilon}\right)\right], & x \in [S_\varepsilon, 2S_\varepsilon]. \end{cases}$$

Note that $\phi \downarrow a_\varepsilon$ as $t \to +\infty$ and it is easy to see $\bar{U} \in C^{2,1}(\Omega_T)$, where

$$\Omega_T := \{(x,t) : x \in [0, 2S_\varepsilon], \ t \geq T_\varepsilon\}.$$

By direct computation, we have

$$\underline{V}_t - D\underline{V}_{xx} - r\underline{V}(1 - \underline{V} - h\bar{U}) = -rh\underline{V}(1 + 3\varepsilon - \bar{U}) \leq 0 \quad \text{if } \bar{U} \leq 1 + 3\varepsilon. \tag{4.1}$$

For $(x,t) \in [0, S_\varepsilon] \times [T_\varepsilon, \infty)$, $\bar{U}(x,t) = \phi(t) + \varepsilon$. Since $\phi(t) > a_\varepsilon$ for all $t \geq T_\varepsilon$, it is easy to see that

$$\bar{U}_t - \bar{U}_{xx} - \bar{U}(1 - \bar{U} - k\underline{V}) > \varepsilon(\phi(t) - a_\varepsilon) > 0 \quad \text{for } (x,t) \in [0, S_\varepsilon] \times [T_\varepsilon, \infty). \tag{4.2}$$

For $(x,t) \in [S_\varepsilon, 2S_\varepsilon] \times [T_\varepsilon, \infty)$, since $\psi_{xx}(x) \leq \frac{b_\varepsilon \pi}{4\alpha S_\varepsilon^2}$ for all $x \in [S_\varepsilon, 2S_\varepsilon]$, we have

$$\begin{aligned} \bar{U}_t - \bar{U}_{xx} - \bar{U}(1 - \bar{U} - k\underline{V}) &\geq \varepsilon^2 - \psi_{xx} + (\psi + \varepsilon)(\phi - a_\varepsilon) \\ &\geq \varepsilon^2 - \psi_{xx} \geq \varepsilon^2 - \frac{b_\varepsilon \pi}{4\alpha S_\varepsilon^2} = 0. \end{aligned} \tag{4.3}$$

Because (4.1) holds only for those (x,t) satisfying $\bar{U} \leq 1 + 3\varepsilon$, we need to adjust the region for applying the comparison principle. Let $x = L(t)$, $t \geq T_\varepsilon$, be the curve so that $\bar{U}(L(t), t) = 1 + 3\varepsilon$. Set $\widetilde{\Omega}_T := \{(x,t) : s \in [0, L(t)], \ t \geq T_\varepsilon\}$. Then, it is not hard to see

$$\{(x,t) : x \in [0, S_\varepsilon], \ t \geq T_\varepsilon\} \subset \widetilde{\Omega}_T \cap \Omega_T. \tag{4.4}$$

On the other hand, we also have the following:

$$\begin{aligned} &\bar{U}(L(t), t) = 1 + 3\varepsilon > u(L(t), t), \quad \text{for } t \geq T_\varepsilon, \\ &\bar{U}(2S_\varepsilon, t) = \phi(t) + \psi(2S_\varepsilon) + \varepsilon \geq a_\varepsilon + b_\varepsilon + \varepsilon = 1 + 2\varepsilon > u(2S_\varepsilon, t), \quad \text{for } t \geq T_\varepsilon, \\ &\underline{V}(2S_\varepsilon, t) = \underline{v}_1 - 3h\varepsilon \leq v(2S_\varepsilon, t) \quad \text{for } t \geq T_\varepsilon, \\ &\bar{U}_x(0, t) = \underline{V}_x(0, t) = 0 \quad \text{for } t \geq T_\varepsilon, \\ &\bar{U}(x, T_\varepsilon) \geq u(x, T_\varepsilon), \ \underline{V}(x, T_\varepsilon) \leq v(x, T_\varepsilon) \quad \text{for } x \in [0, 2S_\varepsilon]. \end{aligned}$$

Together with (4.1), (4.2) and (4.3), the comparison principle yields that $\bar{U} \geq u$ in $\widetilde{\Omega}_T \cap \Omega_T$. In particular, $\phi(t) + \varepsilon \geq u$ in $(x,t) \in [0, S_\varepsilon] \times [T_\varepsilon, \infty)$ because of (4.4). Thus

$$\limsup_{t \to +\infty} u(x,t) \leq a_\varepsilon + \varepsilon = \bar{u}_2 + (3hk + 1)\varepsilon, \quad \text{for } x \in [0, S_\varepsilon].$$

Taking $\varepsilon \to 0$ ($S_\varepsilon \to +\infty$), we obtain that $\limsup_{t \to +\infty} u(x,t) \leq \bar{u}_2$ uniformly in any compact subset of $[0, +\infty)$.

Next, we can prove that $\liminf_{t \to +\infty} u(x,t) \geq \underline{v}_2$ uniformly in any compact subset of $[0, \infty)$ by using the argument similar to the proof of Lemma 4.1(ii). Indeed, we replace $1 + \varepsilon$ by $\bar{u}_2 + \varepsilon$ in the proof of Lemma 4.1(ii), then the result follows.

Using an argument similar to the above we can prove that $\limsup_{t \to +\infty} v(x,t) \leq \bar{v}_2$ and $\liminf_{t \to +\infty} u(x,t) \geq \underline{u}_2$ uniformly in any compact subset of $[0, +\infty)$. We omit the details here. Thus, we complete the proof of Lemma 4.3. □

Indeed, we can continue the strategy as in the proof of Lemma 4.3 to obtain the following Corollary.

Corollary 2 *Let (u, v, s) be a solution of* **(FBP)** *with $s_\infty = +\infty$. Then for each $n \in \mathbb{N}$,*

$$\underline{u}_n \le \liminf_{t\to+\infty} u(x,t) \le \limsup_{t\to+\infty} u(x,t) \le \bar{u}_n,$$
$$\underline{v}_n \le \liminf_{t\to+\infty} v(x,t) \le \limsup_{t\to+\infty} v(x,t) \le \bar{v}_n,$$

uniformly in any compact subset of $[0, +\infty)$.

5 Proofs of the Main Theorems

This section is devoted to the proofs of the main theorems stated in Sect. 1.

At the beginning, we state a comparison principle for the free boundary problem **(FBP)**. Indeed, we will find some suitable functions w_1, w_2 and σ such that we can compare (w_1, w_2, σ) with (u, v, s), the solution of (**FBP**). The proof can be modified by the comparison principle for the free boundary problem in a scalar equation (see Lemma 3.5 of [11]). For reader's convenience, we also give a proof here.

Lemma 5.1 *Let (u, v, s) be a solution of* **(FBP)**. *Also assume that $(w_1, w_2, \sigma) \in C^{2,1}(\mathcal{D}) \times C^{2,1}(\mathcal{D}) \times C^1([0,\infty))$, where $\mathcal{D} := \{(x,t) : 0 \le x \le \sigma(t),\ t > 0\}$, satisfying the following:*

$$w_{1,t} \ge w_{1,xx} + w_1(1 - w_1) \text{ in } \mathcal{D}, \tag{5.1}$$
$$w_{2,t} \ge D w_{2,xx} + r w_2(1 - w_2) \text{ in } \mathcal{D}, \tag{5.2}$$
$$w_{i,x}(0,t) \le 0, \quad w_i(\sigma(t),t) = 0, \quad t > 0, \quad i = 1, 2, \tag{5.3}$$
$$\sigma'(t) \ge -\mu(1+\rho) w_{i,x}(\sigma(t),t), \quad t > 0, \quad i = 1, 2. \tag{5.4}$$

If $w_1(x,0) \ge u_0(x)$, $w_2(x,0) \ge v_0(x)$ for all $x \in [0, s_0]$ and $\sigma(0) \ge s_0$, then $\sigma(t) \ge s(t)$ for all $t \ge 0$, $w_1(x,t) \ge u(x,t)$ and $w_2(x,t) \ge v(x,t)$ for all $x \in [0, s(t)]$, $t \ge 0$.

Proof We first consider that $\sigma(0) > s_0$. Then $\sigma(t) > s(t)$ for small t. We can derive that $\sigma(t) > s(t)$ for all $t \ge 0$. If this is not true, there exists $T > 0$ such that $\sigma(T) = s(T)$, $\sigma(t) > s(t)$ for all $t \in (0, T)$. Thus,

$$s'(T) \ge \sigma'(T). \tag{5.5}$$

Set $\Omega_T := \{(x,t) : 0 < x < s(t),\ t \in (0, T]\}$. If $u_x(s(T),T) \le v_x(s(T),T)$, by (5.1), (5.3) and $w_1(x,0) \ge u_0(x)$ for $x \in [0, s_0]$, the strong maximal principle implies that $w_1 > u$ in Ω_T. Due to $w_1(s(T),T) = u(s(T),T)$, we obtain $w_{1,x}(s(T),T) < u_x(s(T),T)$. However, it follows from (5.4) that

$$\begin{aligned}\sigma'(T) &\ge -\mu(1+\rho) w_{1,x}(s(T),T) > -\mu(1+\rho) u_x(s(T),T)\\ &\ge -\mu(u_x(s(T),T) + \rho v_x(s(T),T)) = s'(T),\end{aligned}$$

a contradiction to (5.5). If $u_x(s(T),T) \ge v_x(s(T),T)$, similarly, using (5.2), (5.3), (5.4) and $w_2(x,0) \ge v_0(x)$ for $x \in [0, s_0]$, we can reach a contradiction again. Thus we obtain that $\sigma(t) > s(t)$ for all $t \ge 0$.

From this, by comparing $(\bar{u}, \underline{v}) := (w_1, 0)$ with (u, v), and $(\underline{u}, \bar{u}) := (0, w_2)$ with (u, v) over Ω_T for any $T > 0$, respectively, we obtain that $w_1(x,t) \ge u(x,t)$ and $w_2(x,t) \ge v(x,t)$ for all $x \in [0, s(t)]$, $t \ge 0$.

For the general case that $\sigma(0) \ge s_0$, we can construct some suitable function $(u_\varepsilon, v_\varepsilon, s_\varepsilon)$ solving (1.5)–(1.7) and $s'_\varepsilon(t) = -\mu(1-\varepsilon)(u_x(s_\varepsilon(t),t) + \rho v_x(s_\varepsilon(t),t))$ for $t > 0$, with suitable initial data $(u_{\varepsilon,0}, v_{\varepsilon,0}, s_{\varepsilon,0})$ such that $\sigma(0) > s_{\varepsilon,0}$ for each $\varepsilon > 0$ and $(u_\varepsilon, v_\varepsilon, s_\varepsilon) \to (u, v, s)$ as $\varepsilon \to +0$. Then the lemma follows by taking $\varepsilon \to +0$. □

We are ready to prove our main results.

5.1 Proof of Theorem 2

Choose $l \in [s_\infty, s_*]$. Let $\bar{u}$ be the unique solution for $u_t = u_{xx} + u(1-u)$, $(x,t) \in (0,l) \times (0,+\infty)$ with the boundary condition $u_x(0,t) = u(l,t) = 0$ for $t > 0$ and the initial data

$$u(x,0) = \begin{cases} u_0(x) & \text{if } x \in [0, s_0], \\ 0 & \text{if } x \in [s_0, l]. \end{cases}$$

Also, let $\bar{v}$ be the unique solution for $v_t = Dv_{xx} + rv(1-v)$, $(x,t) \in (0,l) \times (0,+\infty)$ with the boundary condition $v_x(0,t) = v(l,t) = 0$ for $t > 0$ and the initial data

$$v(x,0) = \begin{cases} v_0(x) & \text{if } x \in [0, s_0], \\ 0 & \text{if } x \in [s_0, l]. \end{cases}$$

Due to Lemma 3.1,

$$\lim_{t\to+\infty} \|u(\cdot,t)\|_{C([0,l])} = \lim_{t\to+\infty} \|v(\cdot,t)\|_{C([0,l])} = 0. \tag{5.6}$$

Comparing $(\bar{u}, 0)$ with (u, v) and $(0, \bar{v})$ with (u, v) respectively, over

$$\Omega := \{(x,t) \in \mathbb{R}^2 : \ 0 \le x \le s(t), \ t \ge 0\},$$

we obtain $0 \le u \le \bar{u}$ and $0 \le v \le \bar{v}$ in Ω. Together with (5.6) we complete the proof of Theorem 2 (i). Part (ii) follows from Lemmas 3.4 and 4.1. □

5.2 Proof of Theorem 3

To prove this, it suffices to show that $s_\infty = +\infty$ if $s_\infty > s_*$. Indeed, when $(D, r, h, k, \mu, \rho) \in A \cup B$, we have

$$s^* \in \left(s_*, \ \max\left\{ \frac{\pi}{2}, \frac{\pi}{2}\sqrt{\frac{D}{r}} \right\} \right].$$

By Lemma 3.5, we see that $s_\infty > s^*$, if $s_\infty > s_*$. Thus Theorem 3 follows from Theorem 2 (ii). □

5.3 Proof of Corollary 1

(i) Since $s'(t) > 0$ for all $t > 0$, $s_\infty > s^*$ if $s_0 \ge s^*$. Then Corollary 1 (i) follows from Theorem 2.

(ii) Again, using that $s'(t) > 0$ for all $t > 0$, we have $s_\infty > s_*$ if $s_0 \ge s_*$. So Corollary 1 (ii) follows from Theorem 3.

(iii) To do so, we shall use the argument from Ricci and Tarzia [36] and adopt the following functions constructed by Du and Lin [11]:

$$\sigma(t) := s_0\left(1+\delta-\frac{\delta}{2}e^{-\alpha t}\right), \quad t \geq 0,$$
$$w(x,t) := Me^{-\alpha t}V\left(\frac{x}{\sigma(t)}\right), \quad 0 \leq x \leq \sigma(t),$$
$$V(y) := \cos\left(\frac{\pi}{2}y\right), \quad 0 \leq y \leq 1,$$

where

$$\delta := \frac{1}{2}\left[\frac{s_*}{s_0}-1\right] > 0 \quad (\text{since } s_0 < s_*),$$

and α, $M > 0$ are to be determined. To apply Lemma 5.1, we need to confirm (5.1)–(5.4).

Since $s_0(1+\delta) < s_*$, we have

$$\alpha := \frac{1}{2}\min\left\{\left(\frac{\pi}{2}\right)^2\frac{D}{(1+\delta)^2 s_0^2}-r, \left(\frac{\pi}{2}\right)^2\frac{1}{(1+\delta)^2 s_0^2}-1\right\} > 0. \tag{5.7}$$

It follows from direct computation and (5.7) that

$$w_t - w_{xx} - w(1-w) \geq MVe^{-\alpha t}\left[\left(\frac{\pi}{2}\right)^2\frac{1}{(1+\delta)^2 s_0^2}-1-\alpha\right] \geq 0,$$
$$w_t - Dw_{xx} - rw(1-w) \geq MVe^{-\alpha t}\left[\left(\frac{\pi}{2}\right)^2\frac{D}{(1+\delta)^2 s_0^2}-r-\alpha\right] \geq 0.$$

By choosing $M := \max\{\|u_0\|_{L^\infty}, \|v_0\|_{L^\infty}\}/\cos(\frac{\pi}{2+\delta})$, we have $w(x,0) \geq \max\{u_0(x), v_0(x)\}$ for all $x \in [0, s_0]$.

When (1.13) holds, we have

$$\begin{aligned}\sigma'(t) + \mu(1+\rho)w_x(\sigma(t),t) &= \frac{\delta}{2}s_0\alpha e^{-\alpha t} - (1+\rho)\mu M e^{-\alpha t}\sigma^{-1}(t)\frac{\pi}{2}\\ &\geq \frac{\delta s_0\alpha e^{-\alpha t}}{2}\left(1-\frac{s_0(2+\delta)}{2\sigma(t)}\right) \quad \left(\text{using } \sigma(0) = s_0\left(1+\frac{\delta}{2}\right)\right)\\ &= \frac{\delta s_0\alpha e^{-\alpha t}}{2}\left(1-\frac{\sigma(0)}{\sigma(t)}\right) \geq 0,\end{aligned}$$

the last equality holds because $\sigma'(t) > 0$ for all t. Thus we obtain (5.4). By Lemma 5.1, $\sigma(t) \geq s(t)$ for all $t \geq 0$. Taking $t \to +\infty$ and using that $s_0(1+\delta) < s_*$,

$$s_\infty \leq \sigma(+\infty) = s_0(1+\delta) < s_*.$$

Then Corollary 1 (iii) follows from Theorem 2 (i). □

5.4 Proof of Theorem 4

Letting $n \to +\infty$ in Corollary 2 and applying Lemma 4.2, Theorem 4 is proved.

5.5 Proof of Theorem 5

We shall apply Lemma 5.1 to prove

$$s(t) < \sigma(t) := \sigma_0 + c_{\min} \cdot t \quad \text{for all } t > 0, \tag{5.8}$$

where $\sigma_0 \gg 1$ is to be determined.

Let $U(\xi)$ and $V(\xi)$, $\xi := x - c_{\min} \cdot t$, with $U(0) = V(0) = 1/2$ be the solution of

$$\begin{aligned} &c_{\min} U' + U'' + U(1-U) = 0 \text{ in } \mathbb{R}, \quad c_{\min} V' + DV'' + rV(1-V) = 0 \text{ in } \mathbb{R}, \\ &(U, V)(-\infty) = (1, 1)\ (U, V)(\infty) = (0, 0), \quad U' < 0, \quad V' < 0 \text{ in } \mathbb{R}. \end{aligned}$$

Such U exists because $c_{\min} \geq 2$, V exists because $c_{\min} \geq 2\sqrt{rD}$ (c.f. [24]).

We now choose $\kappa > 1$ such that $\kappa U(\xi) > \|u_0\|_{L^\infty}$ and $\kappa V(\xi) > \|v_0\|_{L^\infty}$ for all $\xi \in [0, s_0]$. Next, fix $\sigma_0 > s_0$ depending on κ, D, r, μ, ρ such that

$$U(\sigma_0) < \min_{x \in [0,s_0]} \left[U(x) - \frac{u_0(x)}{\kappa} \right], \quad V(\sigma_0) < \min_{x \in [0,s_0]} \left[V(x) - \frac{v_0(x)}{\kappa} \right], \tag{5.9}$$

$$U(\sigma_0), \quad V(\sigma_0) \leq 1 - \frac{1}{\kappa}, \tag{5.10}$$

$$-\kappa(1+\rho)\mu \min\{U'(\sigma_0),\ V'(\sigma_0)\} < c_{\min}. \tag{5.11}$$

Now, set

$$\begin{aligned} w_1(x, t) &= \kappa U(x - c_{\min} t) - \kappa U(\sigma_0), \\ w_2(x, t) &= \kappa V(x - c_{\min} t) - \kappa V(\sigma_0). \end{aligned}$$

Then, using (5.9) and the monotonicity of U and V, we can see that (5.3) holds, $w_1(x, 0) \geq u_0(x)$ and $w_2(x, 0) \geq v_0(x)$ for $x \in [0, s_0]$. Also, direct calculation gives

$$w_{1,t} - w_{1,xx} - w_1(1 - w_1) = \kappa \left[(\kappa - 1) \left(U - \frac{\kappa U(\sigma_0)}{\kappa - 1} \right)^2 + \frac{\kappa - 1 - \kappa U(\sigma_0)}{\kappa(\kappa - 1)} \right] \geq 0,$$

the last inequality follows from (5.10), which implies (5.1) holds. Similarly, (5.2) also holds. Note that (5.4) follows from (5.11). Recall also $\sigma(0) = \sigma_0 > s_0$. Therefore, we can apply Lemma 5.1 to reach (5.8), and so

$$\limsup_{t \to +\infty} \frac{s(t)}{t} \leq \lim_{t \to +\infty} \frac{\sigma(t)}{t} = c_{\min}.$$

This completes the proof Theorem 5. □

6 Discussion

In this paper, we study a Lotka–Volterra type model with weak competition, i.e., $0 < h, k < 1$, and with a free boundary. The model describes that two species u and v competing with each other in a one-dimensional habitat. We envision that the species initially occupy the region $[0, s_0]$ and have a tendency to expand their territory together. Then we extend some results of [11] for one species case to two-species weak competition system.

We obtain several results for this setting. Theorem 2 provides a sufficient condition for spreading success and spreading failure via $s_\infty := \lim_{t \to +\infty} s(t)$. When the parameters $(D, h, k, r, \mu, \rho) \in A \cup B$, we can make sure that $s_\infty \notin (s_*, s^*]$, where sets A and B are defined in (1.11) and (1.12), respectively. Then a spreading–vanishing dichotomy can be established by using Theorem 2 and the critical length for the habitat can be characterize by s_* in the sense that the two species will spread successfully if the spreading front $x = s(t)$ can across the threshold s_*, while the two species will die out eventually if the spreading front stays within s_* (Theorem 3). However, if $(D, h, k, r, \mu, \rho) \notin A \cup B$, s_∞ may fall in $(s_*, s^*]$, we do not know much about the dynamics of u and v.

In Corollary 1, we provide some conditions on the initial data to distinguish the spreading and vanishing. If the size of initial habitat is small, and initial populations are small enough, it causes no population can survive eventually, while they can coexist if the size of habitat is large enough, regardless of initial population size. This phenomenon suggests that the size of the initial habitat is important to the survival for the two species. It is well-known that the effect of habitat size to the survival for species with Dirichlet boundary problem is quite important (see, for example, [5]).

Finally, Theorem 5 reveals that the asymptotic spreading speed (if exists) cannot be faster than the minimal speed for the traveling wave solutions corresponding to the model (1.3)-(1.4). It would be very interesting if one can realize how the asymptotic spreading speed depends on these parameters. We leave this issue for the future study.

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