*Integral Transforms and Special Functions*, 2013
Vol. 24, No. 2, 84–97, http://dx.doi.org/10.1080/10652469.2012.656919





# Summation identities involving certain classes of polynomials

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(*Received 12 October 2011; final version received 21 December 2011*)

In some recent investigations involving differential operators for a general family of Lagrange polynomials, Chen *et al.* [*Some new results for the Lagrange polynomials in several variables*, ANZIAM J. 49 (2007), pp. 243–258] encountered and proved a certain summation identity for the Chan–Chyan–Srivastava polynomials. With a view to extending and generalizing the aforementioned summation identity, we derive the corresponding results for one class of hypergeometric polynomials, the Jacobi polynomials, the extended Jacobi polynomials, the Laguerre polynomials, the Hermite polynomials, the Lagrange–Hermite polynomials and the Erkuş–Srivastava polynomials.

**Keywords:** Hypergeometric polynomials; Jacobi and extended Jacobi polynomials; Laguerre and Hermite polynomials; Lagrange polynomials; Chan–Chyan–Srivastava polynomials; Srivastava–Singhal polynomials; Lagrange–Hermite polynomials; Erkuş–Srivastava polynomials

*2010 Mathematics Subject Classification*: primary 33C05; 33C45; secondary 33B15

## 1. Introduction, definitions and preliminaries

The familiar (*two-variable*) polynomials $g_n^{(\alpha,\beta)}(x,y)$ generated by

$$\sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x,y)z^n = (1-xz)^{-\alpha}(1-yz)^{-\beta} \qquad (|z| < \min\{|x|^{-1},|y|^{-1}\}) \tag{1}$$

are known as the Lagrange polynomials which occur in certain problems in statistics (see [9, p. 267]; see also [14, pp. 441–442]). A *multivariable* extension of the Lagrange polynomials $g_n^{(\alpha,\beta)}(x,y)$ (popularly known as the *Chan–Chyan–Srivastava polynomials*), generated by

$$\prod_{j=1}^{r}\{(1-x_jz)^{-\alpha_j}\} = \sum_{n=0}^{\infty} g_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)z^n \qquad \left(|z| < \min\{|x_1|^{-1},\cdots,|x_r|^{-1}\}\right), \tag{2}$$

was introduced recently and investigated systematically by Chan *et al.* [3].

We begin by recalling an interesting summation identity involving the Chan–Chyan–Srivastava polynomials $g_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)$ as Theorem 1 below.

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THEOREM 1 (see [7, p. 249, Theorem 2.3]) *For every polynomial $\mathcal{P}_m(x)$ of degree m in x,*

$$\sum_{k=0}^{p} \mathcal{P}_m(k) g_k^{(-\alpha_1-n_1,\cdots,-\alpha_r-n_r)}(x_1,\cdots,x_r) g_{p-k}^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)$$

$$= \delta_{p,n_1+\cdots+n_r} \prod_{j=1}^{r} \{(-x_j)^{n_j}\} \mathcal{P}_m(\alpha_1+\cdots+\alpha_r+p)$$

$$+ \frac{(-1)^m}{m!} \mathcal{P}_m^{(m)}(0) \sum_{s=1}^{r} \left[ \delta_{n_s,m-1} (\alpha_s)_m \, x_s^p \prod_{1\leqq j\leqq r \ (j\neq s)} \left\{ \left(1-\frac{x_j}{x_s}\right)^{n_j} \right\} \right], \tag{3}$$

$$\big(m,p,n_1,\cdots,n_r \in \mathbb{N}_0;\ n_j \geqq m-1\ (j=1,\cdots,r);\ p \geqq n_1+\cdots+n_r\big).$$

Several summation identities as well as limit (and other) relationships exist for such hypergeometric polynomials as the Jacobi and the extended Jacobi polynomials, the Laguerre and Hermite polynomials, the Lagrange–Hermite polynomial, the Erkuş–Srivastava polynomials, the Chan–Chyan–Srivastava polynomials, and the Srivastava–Singhal polynomials (see, for details, [1,2,4,5,7,10,12,13]; see also [14,15]). The main object of this sequel to some of the aforementioned investigations is to derive a number of new summation identities for a family of hypergeometric polynomials, the Jacobi polynomials, the extended Jacobi polynomials, the Laguerre polynomials, the Hermite polynomials, the Lagrange–Hermite polynomial and the Erkuş–Srivastava polynomials.

## 2. Summation identities for a family of hypergeometric polynomials

In terms of the Gaussian hypergeometric function ${}_2F_1(a,b;c;z)$, the hypergeometric polynomials $\mathcal{S}_n^{(\alpha,\beta)}(x)$ are defined *explicitly* by [6, p. 3296, Eq. (5)]

$$\mathcal{S}_n^{(\alpha,\beta)}(x) := \binom{\alpha+n-1}{n} {}_2F_1(-n,\beta;\alpha;x) \qquad (x,\alpha,\beta \in \mathbb{C}) \tag{4}$$

or, equivalently, by means of the following generating function [14, p. 293, Eq. (12)]:

$$\sum_{n=0}^{\infty} \mathcal{S}_n^{(\alpha,\beta)}(x) z^n = (1-z)^{-\alpha} \left(1+\frac{xz}{1-z}\right)^{-\beta} = (1-z)^{\beta-\alpha}[1-(1-x)z]^{-\beta} \tag{5}$$

$$(|z| < \min\{1, |1-x|^{-1}\}).$$

By appealing to (1) and (5), we have the following relationship between the hypergeometric polynomials $\mathcal{S}_n^{(\alpha,\beta)}(x)$ and the Lagrange polynomials $g_n^{(\alpha,\beta)}(x,y)$:

$$\mathcal{S}_n^{(\alpha,\beta)}(x) = g_n^{(\alpha-\beta,\beta)}(1,1-x). \tag{6}$$

Upon setting

$$r=2, \quad x_1=1 \quad \text{and} \quad x_2=1-x$$

in the general result (3), if we make use of the relationship (6), we obtain the following corollary.

Corollary 1 *For every polynomial $\mathcal{P}_m(x)$ of degree $m$ in $x$,*

$$\sum_{k=0}^{p} \mathcal{P}_m(k)\mathcal{S}_k^{(-\alpha_1-\alpha_2-n_1-n_2,-\alpha_2-n_2)}(x)\mathcal{S}_{p-k}^{(\alpha_1+\alpha_2,\alpha_2)}(x)$$
$$= \delta_{p,n_1+n_2}(-1)^{n_1}(x-1)^{n_2}\mathcal{P}_m(\alpha_1+\alpha_2+p)$$
$$+ \frac{(-1)^m}{m!}\,\mathcal{P}_m^{(m)}(0)\Big[\delta_{n_1,m-1}(\alpha_1)_m\, x^{n_2} + \delta_{n_2,m-1}(\alpha_2)_m\,(1-x)^{p-n_1}(-x)^{n_1}\Big] \tag{7}$$

$$\big(m,p,n_1,n_2 \in \mathbb{N}_0;\ n_1,n_2 \geqq m-1;\ p \geqq n_1+n_2\big).$$

*Remark 1* For a given nonnegative integer $m \in \mathbb{N}_0$, if we let $r \geqq m$ and $n \geqq m+r$, and then set

$$\alpha_1 \mapsto -\alpha+\beta-n+r+1,\ \alpha_2 \mapsto -\beta-r+1,\ n_1 \mapsto n-r-1 \quad \text{and} \quad n_2 \mapsto r-1$$

in (7), we are led to the following summation identity:

$$\sum_{k=0}^{p} \mathcal{P}_m(k)\mathcal{S}_k^{(\alpha,\beta)}(x)\mathcal{S}_{p-k}^{(-\alpha-n+2,-\beta-r+1)}(x)$$
$$= \delta_{p,n-2}(-1)^n(1-x)^{r-1}\mathcal{P}_m(-\alpha-n+p+2)$$
$$+ \frac{\mathcal{P}_m^{(m)}(0)}{m!}\left[\delta_{n-r,m}(\alpha-\beta)_m\, x^{r-1} + \delta_{r,m}(\beta)_m\,(1-x)^{p-n+r+1}(-x)^{n-r-1}\right] \tag{8}$$

$$\big(m,n,r,p \in \mathbb{N}_0;\ r \geqq m;\ n \geqq m+r;\ p \geqq n-2\big).$$

By considering its *further* special case when $p \geqq n-1$, the formula (8) would yield the following summation identity for the hypergeometric polynomials $\mathcal{S}_n^{(\alpha,\beta)}(x)$ defined by (4):

$$\sum_{k=0}^{p} \mathcal{P}_m(k)\mathcal{S}_k^{(\alpha,\beta)}(x)\mathcal{S}_{p-k}^{(-\alpha-n+2,-\beta-r+1)}(x)$$
$$= \frac{\mathcal{P}_m^{(m)}(0)}{m!}\Big[\delta_{n-r,m}(\alpha-\beta)_m\, x^{r-1} + \delta_{r,m}(\beta)_m\,(1-x)^{p-n+r+1}(-x)^{n-r-1}\Big] \tag{9}$$

$$(m,n,r,p \in \mathbb{N}_0;\ r \geqq m;\ n \geqq m+r;\ p \geqq n-1).$$

Clearly, this last result (9) is equivalent to the following known summation identity due to Chen and Srivastava [6, p. 3298, Theorem 2]:

$$\sum_{k=j}^{\ell} \mathcal{P}_m(k)\mathcal{S}_{k-j}^{(\alpha,\beta)}(x)\mathcal{S}_{\ell-k}^{(-\alpha-n+2,-\beta-r+1)}(x)$$
$$= \frac{\mathcal{P}_m^{(m)}(0)}{m!}\Big[\delta_{n,r+m}(\alpha-\beta)_m\, x^{r-1} + \delta_{r,m}(\beta)_m\,(1-x)^{\ell-j+m-n+1}(-x)^{n-m-1}\Big] \tag{10}$$

$$\big(j,\ell,m,n,r \in \mathbb{N}_0;\ \ell-j+1 \geqq n \geqq m+r;\ r \geqq m\big),$$

which happens to be one of the *main* results proven in [6]. Obviously, therefore, our summation identity (8) and hence also the summation identity (7) asserted by Corollary 1 would generalize the Chen–Srivastava result (10).

## 3. Summation identities for the extended Jacobi polynomials

The classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ of degree $n$ in $x$ are defined by the Rodrigues formula:

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n(1-x)^{-\alpha}(1+x)^{-\beta}}{2^n n!} D_x^n\{(1-x)^{n+\alpha}(1+x)^{n+\beta}\} \qquad \left(D_x := \frac{d}{dx}\right). \tag{11}$$

Fujiwara [11] studied the so-called extended Jacobi polynomials $F_n^{(\alpha,\beta)}(x;a,b,c)$ by defining them by means of the following Rodrigues formula:

$$F_n^{(\alpha,\beta)}(x;a,b,c) = \frac{(-c)^n}{n!}(x-a)^{-\alpha}(b-x)^{-\beta} D_x^n\{(x-a)^{n+\alpha}(b-x)^{n+\beta}\} \qquad (c>0). \tag{12}$$

The polynomials $F_n^{(\alpha,\beta)}(x;a,b,c)$ are essentially those that where considered by Szegö himself [15, p. 58], who indeed showed that these polynomials are just a constant multiple of the classical Jocobi polynomials $P_n^{(\alpha,\beta)}(x)$ defined by (11). As a matter of fact, by comparing the Rodrigues representations (11) and (12), it is not difficult to rewrite Szegö's observation [15, p. 58, Eq. (4.1.2)] in the following form (cf., e.g. [14, p. 388, Problem 11] and [13]):

$$F_n^{(\alpha,\beta)}(x;a,b,c) = \{c(a-b)\}^n P_n^{(\alpha,\beta)}\left(\frac{2(x-a)}{a-b}+1\right) \qquad (a \neq b). \tag{13}$$

On the other hand, it is known that [14, p. 442, Eq. 8.5(17)]

$$g_n^{(\alpha,\beta)}(x,y) = (y-x)^n P_n^{(-\alpha-n,-\beta-n)}\left(\frac{x+y}{x-y}\right). \tag{14}$$

By making use of (13) and (14), we have the following relationship between the Lagrange polynomials $g_n^{(\alpha,\beta)}(x,y)$ and the extended Jacobi polynomials $F_n^{(\alpha,\beta)}(x;a,b,c)$ defined by (1) and (12), respectively:

$$g_n^{(\alpha,\beta)}(x,y) = \left(\frac{y-x}{c(a-b)}\right)^n F_n^{(-\alpha-n,-\beta-n)}\left(\frac{ax-by}{x-y};a,b,c\right). \tag{15}$$

By appropriately combining (3) and (15), we obtain a summation identity for the extended Jacobi polynomials $F_n^{(\alpha,\beta)}(x;a,b,c)$, which is asserted by Theorem 2 below.

THEOREM 2 *For every polynomial $\mathcal{P}_m(x)$ of degree $m$ in $x$,*

$$\begin{aligned}
&\sum_{k=0}^{p} \mathcal{P}_m(k) F_k^{(\alpha+n_1-k,\beta+n_2-k)}(x;a,b,c) F_{p-k}^{(-\alpha-p+k,-\beta-p+k)}(x;a,b,c) \\
&\quad = \delta_{p,n_1+n_2} c^p (x-a)^{n_2}(x-b)^{n_1}\mathcal{P}_m(\alpha+\beta+p) + \frac{(-1)^{p+m}}{m!}\, c^p \mathcal{P}_m^{(m)} \\
&\qquad \cdot \left[\delta_{n_1,m-1}(\alpha)_m (a-b)^{n_2}(x-b)^{p-n_2} + \delta_{n_2,m-1}(\beta)_m (b-a)^{n_1}(x-a)^{p-n_1}\right] \\
&\qquad\quad \left(m,p,n_1,n_2 \in \mathbb{N}_0;\ n_1,n_2 \geqq m-1;\ p \geqq n_1+n_2\right).
\end{aligned} \tag{16}$$

*Proof* It is easily seen from the relationship (15) that

$$\sum_{k=0}^{p} \mathcal{P}_m(k) F_k^{(\alpha+n_1-k,\beta+n_2-k)}\left(\frac{ax-by}{x-y};a,b,c\right) F_{p-k}^{(-\alpha-p+k,-\beta-p+k)}\left(\frac{ax-by}{x-y};a,b,c\right)$$
$$= \sum_{k=0}^{p} \mathcal{P}_m(k)\left(\frac{c(a-b)}{y-x}\right)^p g_k^{(-\alpha-n_1,-\beta-n_2)}(x,y) g_{p-k}^{(\alpha,\beta)}(x,y)$$
$$= \left(\frac{c(a-b)}{y-x}\right)^p \Bigg\{\delta_{p,n_1+n_2}(-x)^{n_1}(-y)^{n_2}\mathcal{P}_m(\alpha+\beta+p) + \frac{(-1)^m}{m!}\,\mathcal{P}_m^{(m)}(0)$$
$$\cdot\left[\delta_{n_1,m-1}(\alpha)_m x^p\left(1-\frac{y}{x}\right)^{n_2} + \delta_{n_2,m-1}(\beta)_m y^p\left(1-\frac{x}{y}\right)^{n_1}\right]\Bigg\}. \tag{17}$$

Upon setting

$$y \mapsto \frac{x(x-a)}{x-b},$$

in the last member of (17), the proof of Theorem 2 is completed. ∎

Since

$$F_n^{(\alpha,\beta)}\left(x;1,-1,\frac{1}{2}\right) = P_n^{(\alpha,\beta)}(x), \tag{18}$$

upon setting

$$a=1,\quad b=-1 \quad \text{and} \quad c=\frac{1}{2}$$

in the assertion (16) of Theorem 2, we get the following result.

COROLLARY 2 (see [7, p. 250, Corollary 2.5]) *For every polynomials $\mathcal{P}_m(x)$ of degree m in x,*

$$\sum_{k=0}^{p} \mathcal{P}_m(k) P_k^{(\alpha+n_1-k,\beta+n_2-k)}(x) P_{p-k}^{(-\alpha-p+k,-\beta-p+k)}(x)$$
$$= \delta_{p,n_1+n_2}\left(\frac{x+1}{2}\right)^{n_1}\left(\frac{x-1}{2}\right)^{n_2}\mathcal{P}_m(\alpha+\beta+p)$$
$$+ \frac{(-1)^m}{m!}\,\mathcal{P}_m^{(m)}(0)\left[\delta_{n_1,m-1}(\alpha)_m(-1)^p\left(\frac{x+1}{2}\right)^{p-n_2} + \delta_{n_2,m-1}(\beta)_m\left(\frac{1-x}{2}\right)^{p-n_1}\right]$$
$$\left(m,p,n_1,n_2 \in \mathbb{N}_0;\ n_1,n_2 \geqq m-1;\ p \geqq n_1+n_2\right). \tag{19}$$

Chen *et al.* [7] presented many other summation identities for the classical Jocobi polynomials $P_n^{(\alpha,\beta)}(x)$. Each of their results involving the classical Jocobi polynomials $P_n^{(\alpha,\beta)}(x)$ can also be restated in terms of the extended Jacobi polynomials $F_n^{(\alpha,\beta)}(x;a,b,c)$ by making use of the relationship (13) in a rather straightforward manner. The details involved in such obviously trivial translations of known results are, therefore, skipped here.

## 4. Summation identities for the Laguerre polynomials

The Laguerre polynomials $L_n^{(\alpha)}(x)$, given *explicitly* by

$$L_n^{(\alpha)}(x) = \binom{\alpha+n}{n}\, {}_1F_1(-n; 1+\alpha; x), \tag{20}$$

possess the following generating function:

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x) z^n = (1-z)^{-\alpha-1} \exp\left(-\frac{xz}{1-z}\right). \tag{21}$$

The Laguerre polynomials $L_n^{(\alpha)}(x)$ and the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are indeed related by means of the following familiar limit relationship [14, p. 131, Eq. 2.5(1)]:

$$L_n^{(\alpha)}(x) = \lim_{|\beta|\to\infty} \left\{ P_n^{(\alpha,\beta)}\left(1 - \frac{2x}{\beta}\right)\right\}. \tag{22}$$

Making use of the limit relationship (22), we find from (19) with

$$x \mapsto 1 - \frac{2x}{\beta}$$

that

$$\begin{aligned}
&\sum_{k=0}^{p} \mathcal{P}_m(k) P_k^{(\alpha+n_1-k,\ \beta+n_2-k)}\left(1-\frac{2x}{\beta}\right) P_{p-k}^{(-\alpha-p+k,\ -\beta-p+k)}\left(1-\frac{2x}{\beta}\right) \\
&\quad = \delta_{p,n_1+n_2}\left(1-\frac{x}{\beta}\right)^{n_1}\left(-\frac{x}{\beta}\right)^{n_2} \mathcal{P}_m(\alpha+\beta+p) \\
&\qquad + \frac{(-1)^m}{m!}\, \mathcal{P}_m^{(m)}(0)\left[\delta_{n_1,m-1}(\alpha)_m(-1)^p\left(1-\frac{x}{\beta}\right)^{p-n_2} + \delta_{n_2,m-1}(\beta)_m\left(\frac{x}{\beta}\right)^{p-n_1}\right]
\end{aligned} \tag{23}$$

$$\big(m, p, n_1, n_2 \in \mathbb{N}_0;\ n_1, n_2 \geqq m-1;\ p \geqq n_1+n_2\big).$$

For fixed nonnegative integers $n_1$ and $n_2$, if we let $|\beta| \to \infty$ in (23), we will obtain several interesting summation identities for the Laguerre polynomials. For example, upon letting

$$n_2 = m \qquad \text{and} \qquad |\beta| \to \infty$$

in (23), we get

$$\begin{aligned}
&\sum_{k=0}^{p} \mathcal{P}_m(k) L_k^{(\alpha+n_1-k)}(x) L_{p-k}^{(-\alpha-p+k)}(-x) \\
&\quad = \delta_{p,n_1+m}\frac{\mathcal{P}_m^{(m)}(0)}{m!}(-x)^m + \delta_{n_1,m-1}\,(\alpha)_m \frac{(-1)^{m+p}}{m}\, \mathcal{P}_m^{(m)}(0)
\end{aligned} \tag{24}$$

$$\big(p, m, n_1 \in \mathbb{N}_0;\ p \geqq n_1+m;\ n_1 \geqq m-1\big),$$

where, *and in what follows*, $(\lambda)_\nu$ denotes the Pochhammer symbol or the *shifted factorial*, since

$$(1)_n = n! \qquad (n \in \mathbb{N}_0),$$

which is defined (for $\lambda, \nu \in \mathbb{C}$ and in terms of the Gamma function) by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C} \setminus \{0\}), \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}), \end{cases} \tag{25}$$

it being understood *conventionally* that $(0)_0 := 1$.

For $n_1 = m$, the summation identity (24) yields the following known result due to Chen *et al.* [7, p. 253]:

$$\sum_{k=0}^{p} \mathcal{P}_m(k) L_k^{(\alpha+m-k)}(x) L_{p-k}^{(-\alpha-p+k)}(-x) = \delta_{p,2m} \frac{\mathcal{P}_m^{(m)}(0)}{m!}(-x)^m \qquad (p, m \in \mathbb{N}_0;\ p \geqq 2m). \tag{26}$$

On the other hand, if we let

$$n_2 = m+1 \qquad \text{and} \qquad |\beta| \to \infty$$

in (23), we get

$$\sum_{k=0}^{p} \mathcal{P}_m(k) L_k^{(\alpha+n_1-k)}(x) L_{p-k}^{(-\alpha-p+k)}(-x) = \delta_{n_1,m-1}\, (\alpha)_m \frac{(-1)^{m+p}}{m!}\, \mathcal{P}_m^{(m)}(0) \tag{27}$$

$$\big(p, m, n_1 \in \mathbb{N}_0;\ p \geqq n_1+m+1;\ n_1 \geqq m-1\big),$$

which, upon setting $m = 0$ in (27), yields the following result:

$$\sum_{k=0}^{p} L_k^{(\alpha+n-k)}(x) L_{p-k}^{(-\alpha-p+k)}(-x) = 0 \qquad \big(p, n \in \mathbb{N}_0;\ p \geqq n+1\big). \tag{28}$$

In particular, by setting $n = 0$ in (28) or, equivalently, by setting $m = 0$ in (26), we obtain [7, p. 253]

$$\sum_{k=0}^{p} L_k^{(\alpha-k)}(x) L_{p-k}^{(-\alpha-p+k)}(-x) = 0 \qquad (p \in \mathbb{N}). \tag{29}$$

If we let

$$n_1 = m-1 \qquad \text{and} \qquad \mathcal{P}_m(k) = k^m$$

in (27), we get

$$\sum_{k=0}^{p} k^m L_k^{(\alpha+m-k-1)}(x) L_{p-k}^{(-\alpha-p+k)}(-x) = (-1)^{m+p}(\alpha)_m \qquad \big(p, m \in \mathbb{N}_0;\ p \geqq 2m\big). \tag{30}$$

Moreover, in its special case when

$$\mathcal{P}_m(k) = k^m \qquad (m \in \mathbb{N}_0), \tag{31}$$

the summation identity (26) yields

$$\sum_{k=0}^{p} k^m L_k^{(\alpha+m-k)}(x) L_{p-k}^{(-\alpha-p+k)}(-x) = \delta_{p,2m}(-x)^m, \qquad \big(p, m \in \mathbb{N}_0;\ p \geqq 2m\big). \tag{32}$$

*Remark 2* It is known that [7, p. 252, Eq. (2.18)]

$$\sum_{k=0}^{p} \mathcal{P}_m(k) P_k^{(-\alpha-\beta-n_1-n_2-1,\, \beta+n_2-k)}(x) P_{p-k}^{(\alpha+\beta-1,\, -\beta-p+k)}(x)$$

$$= \delta_{p,\, n_1+n_2} (-1)^p \left(\frac{x+1}{2}\right)^{n_1} \mathcal{P}_m(\alpha+\beta+p) + \frac{(-1)^m}{m!}\, \mathcal{P}_m^{(m)}(0)$$

$$\cdot \left[ \delta_{n_1,\, m-1} (\alpha)_m \left(\frac{x+1}{2}\right)^p \left(\frac{x-1}{x+1}\right)^{n_2} + \delta_{n_2,\, m-1} (\beta)_m \left(\frac{1-x}{2}\right)^{n_1} \right] \tag{33}$$

$$\big(m, p, n_1, n_2 \in \mathbb{N}_0;\ n_1, n_2 \geqq m-1;\ p \geqq n_1+n_2\big),$$

which, upon setting

$$\alpha \mapsto -\alpha-\beta-p-j,\ n_1 \mapsto m,\ \ n_2 \mapsto p-m-1,\ \ x \mapsto 1-\frac{2x}{\beta} \quad \text{and}$$

$$\mathcal{P}_m(k) = k^m \qquad (m \in \mathbb{N}_0),$$

and by letting $|\beta| \to \infty$, leads us to the following result:

$$\sum_{k=0}^{p} k^m L_k^{(\alpha+j)}(x) L_{p-k}^{(-\alpha-p-j-1)}(-x) = \delta_{p,2m}(-x)^m \qquad \big(p, j, m \in \mathbb{N}_0;\ p \geqq 2m\big) \tag{34}$$

or, equivalently,

$$\sum_{k=j}^{\ell} k^m L_{k-j}^{(\alpha+j)}(x) L_{\ell-k}^{(-\alpha-\ell-1)}(-x) = \delta_{\ell,2m+j}(-x)^m \qquad \big(\ell, j, m \in \mathbb{N}_0;\ \ell \geqq 2m+j\big). \tag{35}$$

This last summation identity (35) happens to be the *main* result of the earlier work [2]. As already remarked by Bavinck [2, p. L279], it was encountered in connection with certain differential operators for the Laguerre polynomials $L_n^{(\alpha)}(x)$.

*Remark 3* Given $m \in \mathbb{N}_0$, if we take $r \geqq m$ and $n \geqq m+r$, and then set

$$\alpha \longmapsto -\alpha-\beta-r,\ \ \beta \longmapsto \beta-n+r+1,\ \ n_1 = r-1 \qquad \text{and} \qquad n_2 = n-r-1$$

in (33), we obtain

$$\sum_{k=0}^{p} \mathcal{P}_m(k) P_k^{(\alpha,\beta-k)}(x) P_{p-k}^{(-\alpha-n,-\beta+n-r+k-p-1)}(x)$$

$$= \delta_{p,n-2}(-1)^n \left(\frac{x+1}{2}\right)^{r-1} P_m(-\alpha-1) + \frac{\mathcal{P}_m^{(m)}(0)}{m!} \left[ \delta_{n-r,m}(-\beta)_m \left(\frac{1-x}{2}\right)^{r-1} \right.$$

$$\left. + \ \delta_{r,m}(\alpha+\beta+1)_m \left(\frac{x+1}{2}\right)^p \left(\frac{x-1}{x+1}\right)^{n-m-1} \right] \tag{36}$$

$$\big(p, m, n, r \in \mathbb{N}_0;\ p+2 \geqq n \geqq m+r;\ r \geqq m\big).$$

Upon setting

$$n \longmapsto n+2, \quad r = m+1 \quad \text{and} \quad x \longmapsto 1 - \frac{2x}{\beta}$$

in (36), if we let $|\beta| \to \infty$, we obtain

$$\sum_{k=0}^{p} \mathcal{P}_m(k) L_k^{(\alpha)}(x) L_{p-k}^{(-\alpha-n-2)}(-x) = \delta_{p,n}(-1)^n \mathcal{P}_m(-\alpha-1) + \delta_{n,2m-1} \frac{\mathcal{P}_m(0)}{m!}(-x)^m \tag{37}$$

$$\big(p, m, n \in \mathbb{N}_0;\ p \geqq n \geqq 2m-1\big),$$

which is equivalent to the following known result [4, p. 416, Theorem 3]:

$$\sum_{k=j}^{\ell} \mathcal{P}_m(k) L_{k-j}^{(\alpha)}(x) L_{\ell-k}^{(-\alpha-n-2)}(-x) = \delta_{\ell,n+j}(-1)^n \mathcal{P}_m(j-\alpha-1) + \delta_{n,2m-1} \frac{\mathcal{P}_m(0)}{m!}(-x)^m \tag{38}$$

$$\big(j, \ell, m, n \in \mathbb{N}_0;\ \ell - j \geqq n \geqq 2m-1\big).$$

## 5. Summation identities for the Hermite polynomials

The Hermite polynomials $H_n(x)$, defined *explicitly* by

$$H_n(x) = \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k n! (2x)^{n-2k}}{k!(n-2k)!}, \tag{39}$$

possess the following generating function:

$$\sum_{n=0}^{\infty} H_n(x) \frac{z^n}{n!} = e^{2xz-z^2}. \tag{40}$$

As long ago as 1939, Palamà [12] made use of the second-order homogeneous linear differential equations satisfied by the two orthogonal polynomials $L_n^{(\alpha)}(x)$ and $H_n(x)$ in order to prove the following limit relationship (see also [5, p. 76, Eq. (4)]):

$$H_n(x) = (-1)^n 2^{\frac{n}{2}} n! \lim_{\alpha\to\infty} \left\{ \alpha^{-\frac{n}{2}} L_n^{(\alpha)}\left(\alpha + x\sqrt{2\alpha}\right) \right\}. \tag{41}$$

Here, in this section, we first find another limit relationship between the Laguerre polynomials $L_n^{(\alpha)}(x)$ and the Hermite polynomials $H_n(x)$.

LEMMA *The following limit relationship holds true*:

$$\lim_{\alpha\to\infty} \left\{ \alpha^{-\frac{n}{2}} L_n^{(-\alpha)}\left(-\alpha - x\sqrt{2\alpha}\right) \right\} = \left(-\frac{i}{\sqrt{2}}\right)^n \frac{H_n(ix)}{n!}. \tag{42}$$

*Proof* Upon setting

$$\alpha \mapsto -\alpha, \quad x \mapsto -\alpha - x\sqrt{2\alpha} \quad \text{and} \quad z \mapsto \frac{z}{\sqrt{\alpha}}$$

in (21), we get

$$\sum_{n=0}^{\infty} \alpha^{-\frac{n}{2}} L_n^{(-\alpha)}\left(-\alpha - x\sqrt{2\alpha}\right) z^n = e^{\varphi(\alpha,z)} \quad \left(|z| < \sqrt{\alpha};\ \alpha > 0\right), \tag{43}$$

where

$$\begin{aligned}\varphi(\alpha,z) &= (\alpha-1)\log\left(1-\frac{z}{\sqrt{\alpha}}\right) + \frac{(\alpha + x\sqrt{2\alpha})z}{\sqrt{\alpha}-z} \\ &= (1-\alpha)\left(\frac{z}{\sqrt{\alpha}} + \frac{z^2}{2\alpha} + \frac{z^3}{3\alpha^{\frac{3}{2}}} + \cdots\right) + \frac{\alpha z + xz\sqrt{2\alpha}}{\sqrt{\alpha}-z}.\end{aligned} \tag{44}$$

It follows from (44) that

$$\lim_{\alpha\to\infty} \varphi(\alpha,z) = \frac{z^2}{2} + \sqrt{2}\,xz, \tag{45}$$

so that the equation (43) assumes the following form:

$$\begin{aligned}&\sum_{n=0}^{\infty} \lim_{\alpha\to\infty}\left\{\alpha^{-\frac{n}{2}} L_n^{(-\alpha)}\left(-\alpha - x\sqrt{2\alpha}\right)\right\} z^n \\ &\quad = e^{\frac{z^2}{2}+\sqrt{2}\,xz} = \sum_{n=0}^{\infty}\left(-\frac{i}{\sqrt{2}}\right)^n \frac{H_n(ix)}{n!} z^n,\end{aligned} \tag{46}$$

by means of the Hermite generating function (40). Obviously, our demonstration of the Lemma is completed by comparing the coefficients of $z^n$ in (46). ■

By setting $x \mapsto \alpha + x\sqrt{2\alpha}$ in (26) and using (41) and (42), we have the following summation identity for the Hermite polynomials.

THEOREM 3 *For every polynomial $\mathcal{P}_m(x)$ of degree $m$ in $x$,*

$$\sum_{k=0}^{p} \mathcal{P}_m(k)\binom{p}{k}(-i)^k H_k(x) H_{p-k}(ix) = \delta_{p,2m}\mathcal{P}_m^{(m)}(0) 8^m \left(\frac{1}{2}\right)_m \quad (p, m \in \mathbb{N}_0;\ p \geqq 2m). \tag{47}$$

By setting $m = 0$ in the assertion (47) of Theorem 3, we obtain the following simpler summation formula:

$$\sum_{k=0}^{p}\binom{p}{k}(-i)^k H_k(x) H_{p-k}(ix) = 0 \quad (p \in \mathbb{N}). \tag{48}$$

## 6. Summation identities for the Erkuş–Srivastava polynomials

Altın and Erkuş [1] presented a multivariable extension of the so-called Lagrange–Hermite polynomials generated by

$$\prod_{j=1}^{r}\{(1-x_jz^j)^{-\alpha_j}\}=\sum_{n=0}^{\infty}h_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)z^n \tag{49}$$

$$\left(\alpha_j\in\mathbb{C}\quad (j=1,\cdots,r);\ |z|<\min\{|x_1|^{-1},\cdots,|x_r|^{-1/r}\}\right).$$

The case $r=2$ of the polynomials given by (49) corresponds to the familiar (two-variable) Lagrange-Hermite polynomials considered by Dattoli *et al.* [8].

The multivariable (Erkuş–Srivastava) polynomials

$$\mathcal{U}_{n;\ell_1,\cdots,\ell_r}^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r),$$

defined by the following generating function:

$$\prod_{j=1}^{r}\{(1-x_jz^{\ell_j})^{-\alpha_j}\}=\sum_{n=0}^{\infty}\mathcal{U}_{n;\ell_1,\cdots,\ell_r}^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)z^n \tag{50}$$

$$\left(\alpha_j\in\mathbb{C}\ (j=1,\cdots,r);\ \ell_j\in\mathbb{N}\ (j=1,\cdots,r);\ |z|<\min\{|x_1|^{-1/\ell_1},\cdots,|x_r|^{-1/\ell_r}\}\right),$$

are a unification (and generalization) of several known families of multivariable polynomials including (for example) the Chan-Chyan-Srivastava polynomials

$$g_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)$$

defined by (2) (see, for details, [10]). Obviously, the Chan–Chyan–Srivastava polynomials

$$g_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)$$

follow as a special case of the Erkuş–Srivastava polynomials

$$\mathcal{U}_{n;\ell_1,\cdots,\ell_r}^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)$$

when

$$\ell_j=1\qquad (j=1,\cdots,r).$$

Moreover, the Lagrange–Hermite polynomials

$$h_n^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)$$

follow as a special case of the Erkuş–Srivastava polynomials

$$\mathcal{U}_{n;\ell_1,\cdots,\ell_r}^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)$$

when

$$\ell_j=j\qquad (j=1,\cdots,r).$$

First of all, by applying the generating functions (2) and (50), we derive the following relationship between the Erkuş-Srivastava polynomials and the Chan–Chyan–Srivastava polynomials:

$$\sum_{n=0}^{\infty}\mathcal{U}_{n;\ell_1,\cdots,\ell_r}^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)z^n$$

$$=\prod_{j=1}^{r}\left\{(1-x_jz^{\ell_j})^{-\alpha_j}\right\}=\prod_{j=1}^{r}\prod_{q=1}^{\ell_j}\{(1-\omega_{jq}z)^{-\alpha_j}\}$$

$$=\sum_{n=0}^{\infty}g_n^{(\alpha_1,\cdots,\alpha_1,\cdots,\alpha_r,\cdots,\alpha_r)}(\omega_{11},\cdots,\omega_{1\ell_1},\cdots,\omega_{r1},\cdots,\omega_{r\ell_r})z^n, \quad (51)$$

where we have *tacitly* assumed that the following set:

$$\{\omega_{jq}: 1 \leqq j \leqq r \quad \text{and} \quad 1 \leqq q \leqq \ell_j \quad (\ell_j \in \mathbb{N};\ j=1,\cdots,r)\}, \quad (52)$$

which depends upon the $\ell_j$ *distinct* values of the factor $x_j^{\frac{1}{\ell_j}}$ occurring in the expression:

$$1-\left(x_j^{\frac{1}{\ell_j}}\ z\right)^{\ell_j} \qquad (j=1,\cdots,r),$$

exists such that

$$(1-x_jz^{\ell_j})^{-\alpha_j}=\prod_{q=1}^{\ell_j}\left\{(1-\omega_{jq}z)^{-\alpha_j}\right\} \qquad (j=1,\cdots,r). \quad (53)$$

Hence, by the assertion (51), we obtain

$$\mathcal{U}_{n;\ell_1,\cdots,\ell_r}^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)=g_n^{(\alpha_1,\cdots,\alpha_1,\cdots,\alpha_r,\cdots,\alpha_r)}(\omega_{11},\cdots,\omega_{1\ell_1},\cdots,\omega_{r1},\cdots,\omega_{r\ell_r}), \quad (54)$$

which readily yields

$$\sum_{k=0}^{p}\mathcal{P}_m(k)\,\mathcal{U}_{k;\ell_1,\cdots,\ell_r}^{(-\alpha_1-n_1,\cdots,-\alpha_r-n_r)}(x_1,\cdots,x_r)\,\mathcal{U}_{p-k;\ell_1,\cdots,\ell_r}^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)$$

$$=\sum_{k=0}^{p}\mathcal{P}_m(k)g_k^{(-\alpha_1-n_1,\cdots,-\alpha_1-n_1,\cdots,-\alpha_r-n_r,\cdots,-\alpha_r-n_r)}(\omega_{11},\cdots,\omega_{1\ell_1},\cdots,\omega_{r1},\cdots,\omega_{r\ell_r})$$

$$\cdot\, g_{p-k}^{(\alpha_1,\cdots,\alpha_1,\cdots,\alpha_r,\cdots,\alpha_r)}(\omega_{11},\cdots,\omega_{1\ell_1},\cdots,\omega_{r1},\cdots,\omega_{r\ell_r}). \quad (55)$$

By using (55) in conjunction with the assertion (3) of Theorem 1, we obtain the following consequence.

THEOREM 4 *For every polynomial $\mathcal{P}_m(x)$ of degree $m$ in $x$,*

$$\sum_{k=0}^{p}\mathcal{P}_m(k)\,\mathcal{U}_{k;\ell_1,\cdots,\ell_r}^{(-\alpha_1-n_1,\cdots,-\alpha_r-n_r)}(x_1,\cdots,x_r)\,\mathcal{U}_{p-k;\ell_1,\cdots,\ell_r}^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)$$

$$=\delta_{p,\ell_1n_1+\cdots+\ell_rn_r}\left(\prod_{j=1}^{r}\prod_{q=1}^{\ell_j}\left\{(-\omega_{jq})^{n_j}\right\}\right)\mathcal{P}_m(\ell_1\alpha_1+\cdots+\ell_r\alpha_r+p)$$

$$+\frac{(-1)^m}{m!}\,\mathcal{P}_m^{(m)}(0)\sum_{s=1}^{r}\left\{\delta_{n_s,m-1}(\alpha_s)_m\sum_{j=1}^{\ell_s}\left(\omega_{sj}^p\prod_{\varepsilon=1}^{r}\prod_{\substack{q=1\\(\varepsilon,q)\neq(s,j)}}^{\ell_\varepsilon}\left\{\left(1-\frac{\omega_{\varepsilon q}}{\omega_{sj}}\right)^{n_\varepsilon}\right\}\right)\right\} \tag{56}$$

$$\big(m,p,n_1,\cdots,n_r\in\mathbb{N}_0;\ \ell_j\in\mathbb{N},\ n_j\geqq m-1\quad(j=1,\cdots,r);\ p\geqq \ell_1n_1+\cdots+\ell_rn_r\big).$$

Upon setting

$$\ell_j=j\quad(j=1,\cdots,r)$$

in the assertion (56) of Theorem 4, we are led to the following corollary.

COROLLARY 3 *For every polynomial $\mathcal{P}_m(x)$ of degree $m$ in $x$,*

$$\sum_{k=0}^{p}\mathcal{P}_m(k)\,h_k^{(-\alpha_1-n_1,\cdots,-\alpha_r-n_r)}(x_1,\cdots,x_r)\,h_{p-k}^{(\alpha_1,\cdots,\alpha_r)}(x_1,\cdots,x_r)$$

$$=\delta_{p,n_1+2n_2+\cdots+rn_r}\left(\prod_{1\leqq q\leqq j\leqq r}\left\{(-\omega_{jq})^{n_j}\right\}\right)\mathcal{P}_m(\alpha_1+2\alpha_2+\cdots+r\alpha_r+p)$$

$$+\frac{(-1)^m}{m!}\,\mathcal{P}_m^{(m)}(0)\sum_{1\leqq j\leqq s\leqq r}\left\{\delta_{n_s,m-1}(\alpha_s)_m\,\omega_{sj}^p\left(\prod_{\substack{1\leqq q\leqq\varepsilon\leqq r\\(\varepsilon,q)\neq(s,j)}}\left\{\left(1-\frac{\omega_{\varepsilon q}}{\omega_{sj}}\right)^{n_\varepsilon}\right\}\right)\right\} \tag{57}$$

$$\big(m,p,n_1,\cdots,n_r\in\mathbb{N}_0;\ n_j\geqq m-1\ (j=1,\cdots,r);\ p\geqq n_1+2n_2+\cdots+rn_r\big).$$

Upon setting

$$r=2,\ x_1=x,\ x_2=y,\ \alpha_1=\alpha,\ \alpha_2=\beta,\ \omega_{11}=x,\ \omega_{21}=\sqrt{y}\quad\text{and}\quad\omega_{22}=-\sqrt{y}$$

in (57), we obtain

$$\sum_{k=0}^{p}\mathcal{P}_m(k)\,h_k^{(-\alpha-n_1,-\beta-n_2)}(x,y)\,h_{p-k}^{(\alpha,\beta)}(x,y)$$

$$=\delta_{p,n_1+2n_2}(-x)^{n_1}(-y)^{n_2}\mathcal{P}_m(\alpha+2\beta+p)+\frac{(-1)^m\mathcal{P}_m^{(m)}(0)}{m!}\left\{\delta_{n_1,m-1}(\alpha)_m\,x^p\left(1-\frac{y}{x^2}\right)^{n_2}\right.$$

$$\left.+\,\delta_{n_2,m-1}(\beta)_m\,2^{m-1}(\sqrt{y})^p\left[\left(1-\frac{x}{\sqrt{y}}\right)^{n_1}+(-1)^p\left(1+\frac{x}{\sqrt{y}}\right)^{n_1}\right]\right\} \tag{58}$$

$$\big(m,p,n_1,n_2\in\mathbb{N}_0;\ n_1,n_2\geqq m-1;\ p\geqq n_1+2n_2\big).$$

We next consider two special cases. Firstly, by letting $n_1=n_2=m$ in (58), we get

$$\sum_{k=0}^{p}\mathcal{P}_m(k)\,h_k^{(-\alpha-m,-\beta-m)}(x,y)\,h_{p-k}^{(\alpha,\beta)}(x,y)=\delta_{p,3m}(xy)^m\mathcal{P}_m(\alpha+2\beta+p) \tag{59}$$

$$\big(m,p\in\mathbb{N}_0;\ p\geqq 3m\big).$$

Secondly, upon setting $m=0$ in (59), we obtain

$$\sum_{k=0}^{p}h_k^{(-\alpha,-\beta)}(x,y)\,h_{p-k}^{(\alpha,\beta)}(x,y)=0\qquad(p\in\mathbb{N}). \tag{60}$$

## Acknowledgements

The present investigation was supported by the *National Science Council of the Republic of China* under Grants NSC 100-2115-M-033-004 and NSC 100-2811-M-033-013, and (in part) by the *National Science Council of the Republic of China* under Grant NSC 97-2115-M-032-003-MY2.

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