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Scalar Spheroidal Harmonics in Five Dimensional Kerr-(A)dS

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We rewrite expressions for a general five dimensional metric on a Kerr-(A)dS black hole background, based on the derivation given be Chen, Lü and Pope [W. Chen, H. Lu and C. N. Pope, Class. Quantum Grav. **23** (2006), 5323, hep-th/0604125]. The Klein-Gordon equation is explicitly separated using this form and we show that the angular part of the wave equation leads to just one spheroidal wave equation. We then present results for the perturbative expansion of the angular eigenvalue in powers of the rotation parameters up to 6th order and compare numerically with the continued fraction method.

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§1. Introduction

Rotating black holes in higher dimensions were first discussed in the seminal paper of Myers and Perry.[2] One surprising feature of the Myers-Perry (MP) solution is that in general $\lfloor \frac{D-1}{2} \rfloor$ spin parameters are required. The first asymptotically non-flat five dimensional MP metric was given in Ref. 3), where subsequent generalizations to arbitrary dimension, with multiple rotation parameters was done by Gibbons, Lü, Pope and Page (GLPP),[4,5] which gave a formal proof of the solution in Ref. 3). The most general Kerr-(A)dS-NUT metric was found by Chen, Lü and Pope (CLP).[1] These two papers represent a fundamental step to developing separable equations of motion.

Deriving wave equations for higher dimensional rotating black hole spacetimes[6] relies crucially on the method of separation of variables from a Hamilton-Jacobi equation (see also Ref. 7)). Using the CLP metric, Frolov, Krtous and Kubiznak[8] were able to separate the geodesic equation and find the Klein-Gordon equation in the most general setting (this was generalized to charged cases in Ref. 9) for a zero valued NUT parameter). The actual separation is due to the presence of hidden symmetries in the form of Killing tensors,[10,11,12] where a whole tower of Killing tensors and symmetry operators[13] can be constructed with the help of Killing-Yano and conformal Killing-Yano tensors. These then guarantee the separability of the geodesic equation, the Klein-Gordon equation and also the Dirac equation.[14]

Unfortunately the separation of the graviton does not appear to be possible based on these hidden symmetries.[11)]

In the literature research has been largely directed toward solutions with only one rotation parameter, the so-called simply-rotating case (e.g., see Ref. 15)). However, recently, solutions with two rotation parameters in $D \geq 6$ have been investigated.[16),17)] In five dimensions, some work has been performed in the asymptotically flat case,[6),18)] and in this paper we extend these results to non-asymptotically flat cases. These are relevant to the AdS/CFT correspondence, where black hole solutions in five dimensional minimal gauged supergravity models require rotating solutions to avoid closed timelike curves (CTCs).[19),9),20)] The most general SUGRA solution in five dimensions for arbitrary rotation parameters (a_1, a_2) was found recently in the work of Chong and collaborators.[19)] The important point about the spheroidal harmonics in these cases is that they do not depend on the charge and hence will be the same as that for Kerr-(A)dS.

This article is a follow up to Ref. 17) in which the Klein Gordon equation for doubly rotating black holes in $D \geq 6$ dimensions was investigated. Superficially, $D = 5$ would not appear to be significantly different from $D \geq 6$; however, it turns out that in five dimensions only a single angular equation occurs, thereby making the analysis different in these two cases.

In this work we will, after separating the Klein-Gordon equation on the GLPP background (with two rotations in five dimensions) investigate the (scalar) spheroidal harmonics, where in particular we show in detail how to apply perturbation theory to obtain the angular eigenvalues (separation constant) of the spheroidal harmonics. This is in the spirit of Fackerell and Crossman, and Seidel,[21),22)] which used the properties of Jacobi functions.

There are various reasons for studying this case perturbatively. One is that in five dimensional flat space the rotation parameters are bounded $(|a_1| + |a_2|)^2 \leq 1$ (with units of black hole mass $M = 1$) and hence a perturbative expansion of the eigenvalue (for low frequencies) is well suited to speed up numerics for QNMs or Hawking emission greybody factors. Furthermore, as we mentioned, because there is only one spheroidal equation for a $D = 5$ doubly rotating black hole, the perturbative expansion is much simpler.

The structure of the paper is as follows: In the next section (§2), we discuss the general metric for Kerr-(A)dS black holes with two rotations in five dimensions. The corresponding Klein-Gordon equation is separated into a radial and angular equation in §3. In §4 we explain the perturbative method. Then in §5 (as a sanity check) we compare our analytics with the continued fraction method. Conclusions are then given in §6.

§2. CLP metric in five dimensions

Many of the steps for the separation follow closely to that for $D \geq 6$ doubly rotating black holes,[17)] and hence, we shall only briefly outline the steps. As we are interested in the five dimensional case, we take the CLP metric to be[1)]

$$ds^2 = \frac{U_1}{X_1}dy^2 - \frac{U_2}{X_2}dr^2 + \frac{X_1}{U_1}\left[\frac{\tilde{W}}{1-g^2y^2}d\tilde{t} - \frac{a_1^2\tilde{\gamma}_1}{a_1^2-y^2}d\tilde{\phi}_1 - \frac{a_2^2\tilde{\gamma}_2}{a_2^2-y^2}d\tilde{\phi}_2\right]^2$$
$$+\frac{X_2}{U_2}\left[\frac{\tilde{W}}{1+g^2y^2}d\tilde{t} - \frac{a_1^2\tilde{\gamma}_1}{a_1^2+r^2}d\tilde{\phi}_1 - \frac{a_2^2\tilde{\gamma}_2}{a_2^2+r^2}d\tilde{\phi}_2\right]^2$$
$$+\frac{a_1^2a_2^2}{y^2r^2}\left[\tilde{W}d\tilde{t} - \tilde{\gamma}_1 d\tilde{\phi}_1 - \tilde{\gamma}_2 d\tilde{\phi}_2\right]^2 , \tag{2·1}$$

where

$$U_1 = -(y^2+r^2) , \tag{2·2}$$
$$U_2 = y^2+r^2 , \tag{2·3}$$
$$X_1 = \frac{1}{y^2}(1-g^2y^2)(a_1^2-y^2)(a_2^2-y^2) , \tag{2·4}$$
$$X_2 = -\frac{1}{r^2}(1+g^2r^2)(a_1^2+r^2)(a_2^2+r^2) + 2M ,^{*)} \tag{2·5}$$
$$\tilde{W} = (1-g^2y^2)(1+g^2r^2) , \tag{2·6}$$
$$\tilde{\gamma}_1 = (a_1^2-y^2)(a_1^2+r^2) , \tag{2·7}$$
$$\tilde{\gamma}_2 = (a_2^2-y^2)(a_2^2+r^2) , \tag{2·8}$$

and

$$t = \tilde{t}(1-g^2a_1^2)(1-g^2a_2^2) , \tag{2·9}$$
$$\phi_1 = \tilde{\phi}_1 a_1(1-g^2a_1^2)(a_1^2-a_2^2) , \tag{2·10}$$
$$\phi_2 = \tilde{\phi}_2 a_2(1-g^2a_2^2)(a_2^2-a_1^2) . \tag{2·11}$$

Here we shall only consider the Kerr-(A)dS case, so all NUT charges have been set to zero, where g^2 is related to the five dimensional cosmological constant with $R_{\mu\nu} = -3g^2 g_{\mu\nu}$. Moreover, this metric is already in the Boyer-Lindquist form because there is no cross term for dr. The ingenuity of the CLP metric is the introduction of the coordinate y, which is related to the original direction cosines coordinates. In five dimensions these are $\mu_1 = \sin\theta$ and $\mu_2 = \cos\theta$, with

$$\mu_1^2 = \frac{a_1^2-y^2}{a_1^2-a_2^2}, \quad \mu_2^2 = \frac{a_2^2-y^2}{a_2^2-a_1^2} , \tag{2·12}$$

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*) Solutions to the equation $X_2 = 0$ define event horizon(s). In the $g \geq 0$ regime, the parameter space for which the solutions represent bonafide black hole solutions can be found by analytic techniques described in Ref. 16). In particular, it can be shown that the solution has a black hole event horizon if $M \geq \frac{\Pi(\tilde{l}_h)}{2\tilde{l}_h}$ where $\Pi(\tilde{l}_h) = (1+g^2\tilde{l}_h)(a_1^2+\tilde{l}_h)(a_2^2+\tilde{l}_h)$, and $\tilde{l}_h$ is defined as the positive solution to the cubic equation $0 = \tilde{l}_h\partial_l\Pi(\tilde{l}_h) - \Pi(\tilde{l}_h)$. Unfortunately, in the DeSitter case ($g^2 < 0$) the analysis is complicated by the existence of the cosmological horizon in addition to the black hole horizon(s) and the same method does not seem to yield analytic expressions. However, the existence of solutions in the flat space case would be enough to argue the existence of black hole solutions in the negative g^2 case, for sufficiently small g^2.

or

$$y^2 = a_1^2 \cos^2\theta + a_2^2 \sin^2\theta \ . \tag{2·13}$$

As one might worry about the coordinate singularity for the degenerate case $a_1 = a_2$, we should note at this point that the angular equation that shall arise, Eq. (3·9), has no singularities in the limit $a_1 = a_2$ (because the numerator cancels a factor of $(1 - g^2 a_1^2)$ in the denominator in the last term on the second line in Eq. (3·9)). In this limit the equation has a standard analytic solution (e.g. see Refs. 6) and 9)) and no perturbative expansion is needed, unlike that in §4.

This metric in Eq. (2·1) can be further simplified via the relations:[1)]

$$\begin{aligned}\frac{\tilde{W}}{1-g^2y^2}d\tilde{t} - \frac{a_1^2\tilde{\gamma}_1}{a_1^2-y^2}d\tilde{\phi}_1 - \frac{a_2^2\tilde{\gamma}_2}{a_2^2-y^2}d\tilde{\phi}_2 &= (d\tilde{t} - a_1^4 d\tilde{\phi}_1 - a_2^4 d\tilde{\phi}_2) \\ &\quad -r^2(-g^2 d\tilde{t} + a_1^2 d\tilde{\phi}_1 + a_2^2 d\tilde{\phi}_2) \\ &\equiv d\psi_0 - r^2 d\psi_1 \ ,\end{aligned} \tag{2·14}$$

$$\frac{\tilde{W}}{1+g^2y^2}d\tilde{t} - \frac{a_1^2\tilde{\gamma}_1}{a_1^2+r^2}d\tilde{\phi}_1 - \frac{a_2^2\tilde{\gamma}_2}{a_2^2+r^2}d\tilde{\phi}_2 = d\psi_0 + y^2 d\psi_1 \ , \tag{2·15}$$

and

$$\tilde{W}d\tilde{t} - \tilde{\gamma}_1 d\tilde{\phi}_1 - \tilde{\gamma}_2 d\tilde{\phi}_2 = d\psi_0 + (y^2 - r^2)d\psi_1 - y^2r^2 d\psi_2 \ . \tag{2·16}$$

In the above the linear relations are

$$\psi_0 = \tilde{t} - a_1^4\tilde{\phi}_1 - a_2^4\tilde{\phi}_2 \ , \tag{2·17}$$

$$\psi_1 = -g^2\tilde{t} + a_1^2\tilde{\phi}_1 + a_2^2\tilde{\phi}_2 \, , \tag{2·18}$$

$$\psi_2 = g^4\tilde{t} - \tilde{\phi}_1 - \tilde{\phi}_2 \ , \tag{2·19}$$

or

$$t = \psi_0 + (a_1^2 + a_2^2)\psi_1 + a_1^2 a_2^2 \psi_2 \ , \tag{2·20}$$

$$\frac{\phi_1}{a_1} = \psi_1 + a_2^2\psi_2 + g^2(\psi_0 + a_2^2\psi_1) \ , \tag{2·21}$$

$$\frac{\phi_2}{a_2} = \psi_1 + a_1^2\psi_2 + g^2(\psi_0 + a_1^2\psi_1) \ . \tag{2·22}$$

The metric in Eq. (2·1) becomes

$$\begin{aligned}ds^2 = \frac{U_1}{X_1}dy^2 - \frac{U_2}{X_2}dr^2 + \frac{X_1}{U_1}\left[d\psi_0 - r^2 d\psi_1\right]^2 + \frac{X_2}{U_2}\left[d\psi_0 + y^2 d\psi_1\right]^2 \\ + \frac{a_1^2a_2^2}{y^2r^2}\left[d\psi_0 + (y^2-r^2)d\psi_1 - y^2r^2 d\psi_2\right]^2 \ .\end{aligned} \tag{2·23}$$

Note that, comparing with the CLP metric (see Eq. (22) in Ref. 1)), we have

$$\begin{aligned}&A_1^{(0)} = 1 \quad , \quad A_1^{(1)} = -r^2 \quad , \quad A_2^{(0)} = 1 \quad , \quad A_2^{(1)} = y^2 \quad , \\ &A^{(0)} = 1 \quad , \quad A^{(1)} = -r^2 + y^2 \quad , \quad A^{(2)} = -y^2r^2 \quad .\end{aligned} \tag{2·24}$$

§3. Separation of the Klein-Gordon equation

Using the results of Frolov, Krtous, and Kubiznak[8)] and considering their equation (4.2), the Klein-Gordon field in five dimensions has the following ansatz:

$$\Phi = R_1(y)R_2(r)e^{i\Psi_0\psi_0+i\Psi_1\psi_1+i\Psi_2\psi_2}\ . \tag{3·1}$$

The separated equations for R_1 and R_2 then follow

$$\frac{d}{dy}\left(X_1\frac{dR_1}{dy}\right)+\frac{X_1}{y}\frac{dR_1}{dy}-\frac{R_1}{X_1}\left(-y^2\Psi_0+\Psi_1-\frac{1}{y^2}\Psi_2\right)^2+\left(-b_1+\frac{1}{a_1^2a_2^2y^2}\Psi_2^2\right)R_1 \\ =0\ , \tag{3·2}$$

$$-\frac{d}{dr}\left(X_2\frac{dR_2}{dr}\right)-\frac{X_2}{r}\frac{dR_2}{dr}-\frac{R_2}{X_2}\left(r^2\Psi_0+\Psi_1+\frac{1}{r^2}\Psi_2\right)^2-\left(b_1+\frac{1}{a_1^2a_2^2y^2}\Psi_2^2\right)R_2 \\ =0\ , \tag{3·3}$$

where b_1 is the separation constant.

By using the relationship between t, ϕ_1, ϕ_2 and ψ_0, ψ_1, ψ_2 in Eqs. (2·20)–(2·22), we can obtain a relationship between the eigenvalues:

$$\begin{aligned}\Psi_0\psi_0+\Psi_1\psi_1+\Psi_2\psi_2 &= -\omega t+m_1\phi_1+m_2\phi_2\\ &=[-\omega+g^2(m_1a_1+m_2a_2)]\psi_0\\ &\quad+[-\omega(a_1^2+a_2^2)+m_1a_1(1+g^2a_2^2)+m_2a_2(1+g^2a_1^2)]\psi_1\\ &\quad+[-\omega a_1^2a_2^2+m_1a_1a_2^2+m_2a_1^2a_2]\psi_2\ ,\end{aligned} \tag{3·4}$$

and

$$\begin{aligned}-y^2\Psi_0+\Psi_1-\frac{1}{y^2}\Psi_2=\frac{1}{y^2}\Big[&\omega(y^2-a_1^2)(y^2-a_2^2)+(1-g^2y^2)\big(m_1a_1(y^2-a_2^2)\\ &+m_2a_2(y^2-a_1^2)\big)\Big]\ .\end{aligned} \tag{3·5}$$

We then find:

$$\begin{aligned}&y\frac{d}{dy}\left[\left(\frac{1}{y}\right)(1-g^2y^2)(a_1^2-y^2)(a_2^2-y^2)\frac{dR_y}{dy}\right]+\Bigg\{\left[a_1^2a_2^2-\frac{(a_1^2-y^2)(a_2^2-y^2)}{1-g^2y^2}\right]\omega^2\\ &+\left[\frac{a_2^2}{a_1^2}-\frac{(1-g^2y^2)(a_2^2-y^2)}{a_1^2-y^2}\right]m_1^2a_1^2+\left[\frac{a_1^2}{a_2^2}-\frac{(1-g^2y^2)(a_1^2-y^2)}{a_2^2-y^2}\right]m_2^2a_2^2\\ &+\left[-b_1-2\omega(m_1a_1+m_2a_2)+2g^2m_1a_1m_2a_2\right]y^2\Bigg\}R_y=0\ ,\end{aligned} \tag{3·6}$$

with $0\le\theta\le\pi/2$ or $a_2\le y\le a_1$, where b_1 is a constant of separation. In the next section we shall effectively work with the variable y.

To be able to compare our spheroidal harmonic with other work in the literature, it is also interesting to write an expression for the angular wave equation in terms of the latitude variable θ. From the angular equation in (3·2) above the derivative terms, with $y^2 = a_1^2\cos^2\theta + a_2^2\sin^2\theta$, become

$$\frac{d}{dy}\left(X_1\frac{dR_1}{dy}\right) + \frac{X_1}{y}\frac{dR_1}{dy} = \frac{1}{y}\frac{d}{dy}\left(yX_1\frac{dR_1}{dy}\right)$$
$$= -\frac{1}{\sin\theta\cos\theta}\frac{d}{d\theta}\left[\left(1 - g^2(a_1^2\cos^2\theta + a_2^2\sin^2\theta)\right)\sin\theta\cos\theta\frac{dR_1}{d\theta}\right] . \qquad (3\cdot7)$$

The non-derivative term can be simplified to

$$-\frac{1}{X_1}\left(-y^2\Psi_0 + \Psi_1 - \frac{1}{y^2}\Psi_2\right)^2 + \left(-b_1 + \frac{1}{a_1^2a_2^2y^2}\Psi_2^2\right)$$
$$= -b_1 + \omega^2\left[\frac{1}{g^2} - \frac{(1-g^2a_1^2)(1-g^2a_2^2)}{g^2(1-g^2(a_1^2\cos^2\theta + a_2^2\sin^2\theta))}\right] + \omega\left[-2(m_1a_1 + m_2a_2)\right]$$
$$+ \left[g^2(m_1a_1+m_2a_2)^2 + \frac{m_1^2}{\sin^2\theta}(1-g^2a_1^2) + \frac{m_2^2}{\cos^2\theta}(1-g^2a_2^2)\right] . \qquad (3\cdot8)$$

Finally, the angular equation in terms of θ becomes

$$\frac{d}{d\theta}\left[\left(1 - g^2(a_1^2\cos^2\theta + a_2^2\sin^2\theta)\right)\sin\theta\cos\theta\frac{dR_1}{d\theta}\right] + \left[b_1 - \frac{\omega^2}{g^2} + 2\omega(m_1a_1+m_2a_2)\right.$$
$$-g^2(m_1a_1+m_2a_2)^2 + \frac{\omega^2(1-g^2a_1^2)(1-g^2a_2^2)}{g^2(1-g^2(a_1^2\cos^2\theta + a_2^2\sin^2\theta))}$$
$$\left.-\frac{m_1^2}{\sin^2\theta}(1-g^2a_1^2) - \frac{m_2^2}{\cos^2\theta}(1-g^2a_2^2)\right]\sin\theta\cos\theta R_1 = 0 . \qquad (3\cdot9)$$

This agrees with the angular equation in the general charged five dimensional Kerr-(A)dS case with two rotations that was found in Ref. 9) (also see Ref. 6) for the flat $g = 0$ case).

For completeness, we also present the separation of the radial part, Eq. (3·3). We have

$$-\frac{d}{dr}\left(X_2\frac{dR_2}{dr}\right) - \frac{X_2}{r}\frac{dR_2}{dr} = -\frac{1}{r}\frac{d}{dr}\left[rX_2\frac{dR_2}{dr}\right] = \frac{1}{r}\frac{d}{dr}\left(\frac{\Delta}{r}\frac{dR_2}{dr}\right) , \qquad (3\cdot10)$$

for the derivative terms, where we defined

$$\Delta = (1+g^2r^2)(r^2+a_1^2)(r^2+a_2^2) - 2Mr^2 . \qquad (3\cdot11)$$

In a similar way to the angular equation, we can simplify the non-derivative term in the radial equation as

$$-\frac{1}{X_2}\left(r^2\Psi_0 + \Psi_1 + \frac{1}{r^2}\Psi_2\right)^2 - \left(b_1 + \frac{1}{a_1^2a_2^2r^2}\Psi_2^2\right)$$
$$= -b_1 + \omega^2\left[\frac{1}{g^2} - \frac{(1-g^2a_1^2)(1-g^2a_2^2)}{g^2(1+g^2r^2)}\right] + \omega\left[-2(m_1a_1+m_2a_2)\right]$$
$$+ \left[g^2(m_1a_1+m_2a_2)^2 + \frac{m_1^2(a_1^2-a_2^2)(1-g^2a_1^2)}{r^2+a_1^2} + \frac{m_2^2(a_2^2-a_1^2)(1-g^2a_2^2)}{r^2+a_2^2}\right]$$
$$+\frac{2M}{\Delta}(r^2+a_1^2)(r^2+a_2^2)(1+g^2r^2)\left[\frac{\omega}{1+g^2r^2} - \frac{m_1a_1}{r^2+a_1^2} - \frac{m_2a_2}{r^2+a_2^2}\right]^2 . \qquad (3\cdot12)$$

The radial equation then becomes

$$\frac{\Delta}{r}\frac{d}{dr}\left(\frac{\Delta}{r}\frac{dR_2}{dr}\right)+\Bigg\{\Delta\left[-b_1+\frac{\omega^2}{g^2}-2\omega(m_1a_1+m_2a_2)+g^2(m_1a_1+m_2a_2)^2\right.$$
$$\left.-\frac{\omega^2(1-g^2a_1^2)(1-g^2a_2^2)}{g^2(1+g^2r^2)}+\frac{m_1^2(a_1^2-a_2^2)(1-g^2a_1^2)}{r^2+a_1^2}+\frac{m_2^2(a_2^2-a_1^2)(1-g^2a_2^2)}{r^2+a_2^2}\right]$$
$$+2M(r^2+a_1^2)(r^2+a_2^2)(1+g^2r^2)\left[\frac{\omega}{1+g^2r^2}-\frac{m_1a_1}{r^2+a_1^2}-\frac{m_2a_2}{r^2+a_2^2}\right]^2\Bigg\}R_2=0\,. \tag{3·13}$$

This radial equation could be used to investigate QNMs, for example, on a five dimensional Kerr-(A)dS background.

§4. Eigenvalue expansion for $D = 5$

In §3 we have given a general analysis of the massless scalar equation in a Kerr-(A)dS spacetime with two rotations and $D = 5$ for general g. As we saw, the angular equation in this case has two rotation parameters, but only one angular equation, because there is only one latitude variable θ (cf. Eq. (2·13)). This simplifies things a great deal and hence we shall now work out the perturbative expansion of the angular eigenvalue using techniques similar to the works of Refs. 21) and 22) (also see Ref. 23)).

To develop a perturbative expansion for the eigenvalue b_1, it is convenient to make the change of variables

$$y^2=\frac{1}{2}\left(a_1^2+a_2^2\right)-\frac{1}{2}\left(a_1^2-a_2^2\right)x\,, \tag{4·1}$$

with $-1\le x\le 1$. Then in terms of x, the angular equation becomes

$$-\left(1-x^2\right)\left[1-\frac{1}{2}g^2a_1^2(1-x)-\frac{1}{2}g^2a_2^2(1+x)\right]\frac{d^2R}{dx^2}$$
$$+\left[2x-\frac{1}{2}g^2a_1^2(1-x)(1+3x)-\frac{1}{2}g^2a_2^2(1+x)(-1+3x)\right]\frac{dR}{dx}$$
$$+\frac{1}{4}\left[\frac{\omega^2}{g^2}-\frac{\omega^2(1-g^2a_1^2)(1-g^2a_2^2)}{g^2\left(1-\frac{1}{2}g^2a_1^2(1-x)-\frac{1}{2}g^2a_2^2(1+x)\right)}\right.$$
$$\left.+\frac{2m_1^2(1-g^2a_1^2)}{1+x}+\frac{2m_2^2(1-g^2a_2^2)}{1-x}\right]R=\frac{B}{4}R\,, \tag{4·2}$$

where the constant $B=b_1+2\omega(m_1a_1+m_2a_2)-g^2(m_1a_1+m_2a_2)^2$. Note that for simplicity we have dropped the subscript on R.

Now comes the perturbative part of the method, which is essentially identical to time-independent perturbation theory in quantum mechanics.[23)] Expanding in powers of a_1 and a_2, which we assume to be small, we can schematically write

$$\left(\mathcal{O}_0+\mathcal{O}_2+\mathcal{O}_4+\mathcal{O}_6+\cdots\right)\left(R_0+R_2+R_4+R_6+\cdots\right)$$

$$= \frac{1}{4}\left(B_0 + B_2 + B_4 + B_6 + \cdots\right)\left(R_0 + R_2 + R_4 + R_6 + \cdots\right), \tag{4·3}$$

where

$$\mathcal{O}_0 = -(1-x^2)\frac{d^2}{dx^2} + 2x\frac{d}{dx} + \frac{1}{4}\left[\frac{2m_1^2}{1+x} + \frac{2m_2^2}{1-x}\right], \tag{4·4}$$

$$\begin{aligned}\mathcal{O}_2 = &\frac{1}{2}(1-x^2)\left[g^2a_1^2(1-x) + g^2a_2^2(1+x)\right]\frac{d^2}{dx^2} \\ &-\frac{1}{2}\left[g^2a_1^2(1-x)(1+3x) - g^2a_2^2(1+x)(1-3x)\right]\frac{d}{dx} \\ &+\frac{1}{4}\left[\frac{1}{2}\omega^2a_1^2(1+x) + \frac{1}{2}\omega^2a_2^2(1-x) - \frac{2m_1^2g^2a_1^2}{1+x} - \frac{2m_2^2g^2a_2^2}{1-x}\right],\end{aligned} \tag{4·5}$$

$$\mathcal{O}_4 = \frac{1}{16}(a_1^2 - a_2^2)^2g^2\omega^2(1-x^2)\ , \tag{4·6}$$

$$\mathcal{O}_6 = \frac{1}{2}g^2\left[(a_1^2 + a_2^2) - (a_1^2 - a_2^2)x\right]\mathcal{O}_4\ . \tag{4·7}$$

The regularity of the form of the operators after $\mathcal{O}_4$ makes it possible to evaluate the eigenvalue iteratively as we shall do in the following.

We start with the zeroth order in Eq. (4·3),

$$\mathcal{O}_0R_0 = \frac{1}{4}B_0R_0\ , \tag{4·8}$$

which has regular solution

$$(R_0)_{lm_1m_2} = c_{lm_1m_2}(1-x)^{|m_2|/2}(1+x)^{|m_1|/2}P^{(|m_2|,|m_1|)}_{\frac{1}{2}(l-|m_1|-|m_2|)}(x)\ , \tag{4·9}$$

$$(B_0)_{lm_1m_2} = l(l+2)\ , \tag{4·10}$$

where $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial.[24] The normalization constant is

$$c_{lm_1m_2} = \left\{\frac{(l+1)\Gamma\left[\frac{1}{2}(l-|m_1|-|m_2|)+1\right]\Gamma\left[\frac{1}{2}(l+|m_1|+|m_2|)+1\right]}{2^{|m_1|+|m_2|-1}\Gamma\left[\frac{1}{2}(l+|m_1|-|m_2|)+1\right]\Gamma\left[\frac{1}{2}(l-|m_1|+|m_2|)+1\right]}\right\}^{1/2}, \tag{4·11}$$

with normalization

$$\frac{1}{4}\int_{-1}^{1}dx\,(R_0)^2 = 1\ . \tag{4·12}$$

Before moving on we will also need to consider how the normalization condition above affects the higher order terms:

$$\begin{aligned}&\frac{1}{4}\int_{-1}^{1}dx\,(R_0 + R_2 + R_4 + R_6 + \cdots)(R_0 + R_2 + R_4 + R_6 + \cdots) = 1 \\ &\Rightarrow \int_{-1}^{1}dx\ R_0R_2 = 0 \quad ; \quad \int_{-1}^{1}dx\ R_0R_4 = -\frac{1}{2}\int_{-1}^{1}dx\ R_2R_2 \quad ; \\ &\qquad \int_{-1}^{1}dx\ R_0R_6 = -\int_{-1}^{1}dx\ R_2R_4\ .\end{aligned} \tag{4·13}$$

Now we consider the next order in Eq. (4·3),

$$\mathcal{O}_2 R_0 + \mathcal{O}_0 R_2 = \frac{1}{4}\left(B_0 R_2 + B_2 R_0\right). \tag{4·14}$$

Contracting with $(R_0)_{lm_1m_2}$ and making use of the fact that the operators are self-adjoint, we have

$$(B_2)_{lm_1m_2} = \int_{-1}^{1} dx\, (R_0)_{lm_1m_2} \mathcal{O}_2 (R_0)_{lm_1m_2}\ . \tag{4·15}$$

From the properties of the Jacobi polynomial,[24] we obtain

$$\begin{aligned}\mathcal{O}_2(R_0)_{lm_1m_2} = {} & (O_2)_{l,l+2}\frac{c_{lm_1m_2}}{c_{(l+2)m_1m_2}}(R_0)_{(l+2)m_1m_2} + (O_2)_{l,l}(R_0)_{lm_1m_2} \\ & +(O_2)_{l,l-2}\frac{c_{lm_1m_2}}{c_{(l-2)m_1m_2}}(R_0)_{(l-2)m_1m_2}\ ,\end{aligned} \tag{4·16}$$

with

$$(O_2)_{l,l+2} = \frac{1}{16}(a_1^2 - a_2^2)\left[\omega^2 + g^2 l(l+4)\right]\frac{\left[(l+2)^2 - (|m_1| + |m_2|)^2\right]}{(l+2)(l+1)}\ , \tag{4·17}$$

$$\begin{aligned}(O_2)_{l,l} = {} & \frac{1}{8}(a_1^2 + a_2^2)\omega^2 + \frac{1}{8}(a_1^2 - a_2^2)\omega^2\left[\frac{m_1^2 - m_2^2}{l(l+2)}\right] - \frac{1}{8}g^2(a_1^2 + a_2^2)l(l+2) \\ & -\frac{1}{8}g^2(a_1^2 - a_2^2)(m_1^2 - m_2^2)\left[\frac{l^2 + 2l + 4}{l(l+2)}\right],\end{aligned} \tag{4·18}$$

$$(O_2)_{l,l-2} = \frac{1}{16}(a_1^2 - a_2^2)\left[\omega^2 + g^2 (l^2 - 4)\right]\frac{\left[l^2 - (|m_1| - |m_2|)^2\right]}{l(l+1)}\ . \tag{4·19}$$

With this result, Eq. (4·15) gives

$$\begin{aligned}(B_2)_{lm_1m_2} &= 4(O_2)_{l,l} \\ &= \frac{a_1^2 + a_2^2}{2}\left[\omega^2 - g^2 l(l+2)\right] + \frac{(a_1^2 - a_2^2)(m_1^2 - m_2^2)}{2l(l+2)}\left[\omega^2 - g^2(l^2 + 2l + 4)\right].\end{aligned} \tag{4·20}$$

To go on to the next order we now need to express $(R_2)_{lm_1m_2}$ in terms of the $(R_0)_{l'm_1m_2}$ with the coefficients $(d_2)_{ll'}$. Contracting Eq. (4·14) for $(R_0)_{l'm_1m_2}$ with $l' \neq l$, we can obtain the coefficients $(d_2)_{l,l'}$ in the following series expansion:

$$(R_2)_{lm_1m_2} = (d_2)_{l,l+2}\frac{c_{lm_1m_2}}{c_{(l+2)m_1m_2}}(R_0)_{(l+2)m_1m_2} + (d_2)_{l,l-2}\frac{c_{lm_1m_2}}{c_{(l-2)m_1m_2}}(R_0)_{(l-2)m_1m_2}\ , \tag{4·21}$$

with

$$(d_2)_{l,l+2} = \frac{4}{(B_0)_{lm_1m_2} - (B_0)_{(l+2)m_1m_2}}(O_2)_{l,l+2}\ , \tag{4·22}$$

$$(d_2)_{l,l-2} = \frac{4}{(B_0)_{lm_1m_2} - (B_0)_{(l-2)m_1m_2}}(O_2)_{l,l-2}\ . \tag{4·23}$$

From Eq. (4·13), one has $(d_2)_{l,l} = 0$.

The next order in Eq. (4·3) is

$$\mathcal{O}_4 R_0 + \mathcal{O}_2 R_2 + \mathcal{O}_0 R_4 = \frac{1}{4}\left(B_4 R_0 + B_2 R_2 + B_0 R_4\right), \qquad (4{\cdot}24)$$

where contracting with $(R_0)_{lm_1m_2}$, we have

$$(B_4)_{lm_1m_2} = \int_{-1}^{1} dx\ \left[(R_0)_{lm_1m_2}\mathcal{O}_4(R_0)_{lm_1m_2} + (R_0)_{lm_1m_2}\mathcal{O}_2(R_2)_{lm_1m_2}\right]\ . \qquad (4{\cdot}25)$$

Thus, we need to consider the term $\mathcal{O}_4(R_0)_{lm_1m_2}$, and to do that we use the Jacobi functional relation:[24)]

$$\begin{aligned} x\,(R_0)_{lm_1m_2} &= X_{l,l+2}\frac{c_{lm_1m_2}}{c_{(l+2)m_1m_2}}(R_0)_{(l+2)m_1m_2} + X_{l,l}(R_0)_{lm_1m_2} \\ &\quad + X_{l,l-2}\frac{c_{lm_1m_2}}{c_{(l-2)m_1m_2}}(R_0)_{(l-2)m_1m_2}\ . \end{aligned} \qquad (4{\cdot}26)$$

From the recurrence relation of the Jacobi polynomials, we have

$$\begin{aligned} X_{l,l+2} &= \frac{[(l+2)^2 - (|m_1| + |m_2|)^2]}{2(l+1)(l+2)}\ , \\ X_{l,l} &= \frac{(m_1^2 - m_2^2)}{l(l+2)}\ , \\ X_{l,l-2} &= \frac{[l^2 - (|m_1| - |m_2|)^2]}{2l(l+1)}\ . \end{aligned} \qquad (4{\cdot}27)$$

Then writing

$$\begin{aligned} \mathcal{O}_4(R_0)_{lm_1m_2} &= (O_4)_{l,l+4}\frac{c_{lm_1m_2}}{c_{(l+4)m_1m_2}}(R_0)_{(l+4)m_1m_2} \\ &\quad + (O_4)_{l,l+2}\frac{c_{lm_1m_2}}{c_{(l+2)m_1m_2}}(R_0)_{(l+2)m_1m_2} \\ &\quad + (O_4)_{l,l}(R_0)_{lm_1m_2} + (O_4)_{l,l-2}\frac{c_{lm_1m_2}}{c_{(l-2)m_1m_2}}(R_0)_{(l-2)m_1m_2} \\ &\quad + (O_4)_{l,l-4}\frac{c_{lm_1m_2}}{c_{(l-4)m_1m_2}}(R_0)_{(l-4)m_1m_2}\ , \end{aligned} \qquad (4{\cdot}28)$$

and using the result in Eq. (4·27), we have

$$(O_4)_{l,l+4} = -\frac{1}{16}(a_1^2 - a_2^2)^2 g^2\omega^2 X_{l,l+2}X_{l+2,l+4}\ , \qquad (4{\cdot}29)$$

$$(O_4)_{l,l+2} = -\frac{1}{16}(a_1^2 - a_2^2)^2 g^2\omega^2 (X_{l,l+2}X_{l+2,l+2} + X_{l,l}X_{l,l+2})\ , \qquad (4{\cdot}30)$$

$$(O_4)_{l,l} = \frac{1}{16}(a_1^2 - a_2^2)^2 g^2\omega^2 (1 - X_{l,l+2}X_{l+2,l} - X_{l,l}X_{l,l} - X_{l,l-2}X_{l-2,l})\ , \qquad (4{\cdot}31)$$

$$(O_4)_{l,l-2} = -\frac{1}{16}(a_1^2 - a_2^2)^2 g^2\omega^2 (X_{l,l-2}X_{l-2,l-2} + X_{l,l}X_{l,l-2})\ , \qquad (4{\cdot}32)$$

$$(O_4)_{l,l-4} = -\frac{1}{16}(a_1^2 - a_2^2)^2 g^2\omega^2 X_{l,l-2}X_{l-2,l-4}\ . \qquad (4{\cdot}33)$$

Putting these into Eq. (4·25), we have

$$
\begin{aligned}
&(B_4)_{lm_1m_2} \\
&= 4\left[(O_4)_{l,l} + (d_2)_{l,l+2}(O_2)_{l+2,l} + (d_2)_{l,l-2}(O_2)_{l-2,l}\right] \\
&= \frac{1}{64}(a_1^2 - a_2^2)^2 \left\{ \frac{8g^2\omega^2[l^4 + 4l^3 + 2l^2(m_1^2 + m_2^2) + 4l(m_1^2 + m_2^2 - 2) - 3(m_1^2 - m_2^2)^2]}{(l-1)l(l+2)(l+3)} \right. \\
&\qquad - \frac{[\omega^2 + g^2 l(l+4)]^2[(l+2)^2 - (|m_1| + |m_2|)^2][(l+2)^2 - (|m_1| - |m_2|)^2]}{(l+1)(l+2)^3(l+3)} \\
&\qquad \left. + \frac{[\omega^2 + g^2(l^2-4)]^2[l^2 - (|m_1| + |m_2|)^2][l^2 - (|m_1| - |m_2|)^2]}{l^3(l^2-1)} \right\}. \qquad (4·34)
\end{aligned}
$$

Using the same iterative procedure one can obtain the angular eigenvalue B to higher orders in a_1 and a_2, where for completeness we present the full expression up to 6th order in B:

$$
\begin{aligned}
B &= l(l+2) + \frac{a_1^2 + a_2^2}{2}\left[\omega^2 - g^2 l(l+2)\right] + \frac{(a_1^2 - a_2^2)(m_1^2 - m_2^2)}{2l(l+2)}\left[\omega^2 - g^2(l^2 + 2l + 4)\right] \\
&+ \frac{1}{64}(a_1^2 - a_2^2)^2 \left\{ \frac{8g^2\omega^2[l^4 + 4l^3 + 2l^2(m_1^2 + m_2^2) + 4l(m_1^2 + m_2^2 - 2) - 3(m_1^2 - m_2^2)^2]}{(l-1)l(l+2)(l+3)} \right. \\
&\qquad - \frac{[\omega^2 + g^2 l(l+4)]^2[(l+2)^2 - (|m_1| + |m_2|)^2][(l+2)^2 - (|m_1| - |m_2|)^2]}{(l+1)(l+2)^3(l+3)} \\
&\qquad \left. + \frac{[\omega^2 + g^2(l^2-4)]^2[l^2 - (|m_1| + |m_2|)^2][l^2 - (|m_1| - |m_2|)^2]}{l^3(l^2-1)} \right\} \times \left(1 + \frac{1}{2}g^2(a_1^2 + a_2^2)\right) \\
&\quad + \frac{1}{128}(a_1^2 - a_2^2)^3(m_1^2 - m_2^2) \times \\
&\left\{ \frac{8g^4\omega^2[l^4 + 4l^3 - 2l^2(3(m_1^2 + m_2^2) - 4) - 4l(3(m_1^2 + m_2^2) - 2) + (5(m_1^2 - m_2^2)^2 + 8(m_1^2 + m_2^2) - 16)]}{(l^2-4)(l-1)l(l+3)(l+4)} \right. \\
&\quad - \frac{[\omega^4 - g^2\omega^2(3l^2 + 12l + 20)][\omega^2 + g^2 l(l+4)][(l+2)^2 - (|m_1| + |m_2|)^2][(l+2)^2 - (|m_1| - |m_2|)^2]}{l(l+1)(l+2)^5(l+3)(l+4)} \\
&\quad + \frac{4g^4 l(l+4)[\omega^2 + g^2 l(l+4)][(l+2)^2 - (|m_1| + |m_2|)^2][(l+2)^2 - (|m_1| - |m_2|)^2]}{l(l+1)(l+2)^5(l+3)(l+4)} \\
&\quad \left. - \frac{[-\omega^4 + g^2\omega^2(3l^2 + 8) + 4g^4(l^2-4)][\omega^2 + g^2(l^2-4)][l^2 - (|m_1| + |m_2|)^2][l^2 - (|m_1| - |m_2|)^2]}{l^5(l^2-1)(l^2-4)} \right\}.
\end{aligned}
\qquad (4·35)
$$

This is the main result of this paper,[*)] where some values are presented in Fig. 1 and Table I. These are compared with an exact numerical procedure developed in the next section.

Note that this power series expansion of B could be used to analyze the radial scalar perturbation equation for a general five dimensional Kerr-(A)dS spacetime. At this stage only the $g = 0$ case has been considered for two rotations (e.g., see Refs. 6) and 18)) (however, see Ref. 9)).

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*) A Mathematica notebook with results up to 8th higher order is available at `http://www-het.phys.sci.osaka-u.ac.jp/~naylor/AIM.html`.

§5. The continued fraction method

Unlike the case for $D \geq 6$,[17] the angular equation in five dimensions has four regular singular points. In this situation, the equation can be transformed into a Heun form and then the tried and tested method of continued fractions can be used.[25] Note, the asymptotic iteration method could also be applied[26),17] to solve for the angular eigenvalue.

To get a continued fraction we first transform the angular equation (3·6) according to

$$R(\zeta) = \zeta^{\frac{|m_2|}{2}}(\zeta-1)^{\frac{|m_1|}{2}}(\zeta-z_0)^{\frac{\omega}{2g}}y(\zeta)\ , \tag{5·1}$$

where $\zeta = (\cos(2\theta)+1)/2$ and

$$z_0 = \frac{1-g^2a_2^2}{g^2(a_1^2-a_2^2)}\ . \tag{5·2}$$

Then the angular mode equation can be written in Heun form:[25]

$$\left[\frac{d^2}{d\zeta^2} + \left(\frac{\gamma}{\zeta} + \frac{\delta}{\zeta-1} + \frac{\epsilon}{\zeta-z_0}\right)\frac{d}{d\zeta} + \frac{\alpha\beta\zeta - q}{\zeta(\zeta-1)(\zeta-z_0)}\right]y(\zeta) = 0\ , \tag{5·3}$$

with the constraint

$$\alpha+\beta+1 = \gamma+\delta+\epsilon\ , \tag{5·4}$$

where

$$\alpha = \frac{1}{2}(|m_1|+|m_2|+\omega/g)\ ,\quad \beta = \frac{1}{2}(|m_1|+|m_2|+\omega/g)+2\ ,\quad \gamma = |m_2|+1\ ,$$
$$\delta = |m_1|+1\ ,\quad \epsilon = \omega/g+1\ , \tag{5·5}$$

and

$$q = \frac{1}{4g^4}\frac{\omega^2-Bg^2}{a_1^2-a_2^2} - \frac{m_1^2}{4} + \frac{1}{4}(m_2+\omega/g)(m_2+\omega/g+2)$$
$$-\frac{1-g^2a_2^2}{4g^2(a_1^2-a_2^2)}(\omega^2/g^2-(m_1+m_2)(m_1+m_2+2))\ , \tag{5·6}$$

A method identical to this was used by Kodama et al.[15] for the case of a singly rotating black hole in AdS. A key point of the Heun form is that it satisfies a three term recurrence relation:

$$\alpha_0 c_1 + \beta_0 c_0 = 0 \tag{5·7}$$
$$\alpha_p c_{p+1} + \beta_p c_p + \gamma_p c_{p-1} = 0\ , \qquad (p=1,2,\dots) \tag{5·8}$$

where

$$\alpha_p = -\frac{(p+1)(p+r-\alpha+1)(p+r-\beta+1)(p+\delta)}{(2p+r+2)(2p+r+1)}\ , \tag{5·9}$$
$$\beta_p = \frac{\epsilon p(p+r)(\gamma-\delta)+[p(p+r)+\alpha\beta][2p(p+r)+\gamma(r-1)]}{(2p+r+1)(2p+r-1)}$$

$$-z_0 p(p+r) - q \ , \tag{5·10}$$

$$\gamma_p = -\frac{(p+\alpha-1)(p+\beta-1)(p+\gamma-1)(p+r-1)}{(2p+r-2)(2p+r-1)} \ , \tag{5·11}$$

with

$$r = |m_1| + |m_2| + 1 \ . \tag{5·12}$$

The eigenvalue B can then be found (for a given ω) by solving a continued fraction of the form:[27),28)]

$$\beta_0 - \frac{\alpha_0\gamma_1}{\beta_1 -} \frac{\alpha_1\gamma_2}{\beta_2 -} \frac{\alpha_2\gamma_3}{\beta_3 -} \ldots = 0 \ . \tag{5·13}$$

We should finally mention that the limit $g = 0$ cannot be taken with the above method, because of the divergence in the exponent of Eq. (5·1).[*)] However, if the $g \to 0$ limit is taken from the start it can be considered as a separate case, because it removes one of the regular singular points (from four to three) and reduces to a normal continued fraction solution. Incidentally, the asymptotic iteration method can be used even without the additional factor $(\zeta - z_0)^{\omega/2g}$ in the scaling and one can find the angular eigenvalues at $g = 0$ (see Ref. 26) for $a_2 = 0$).

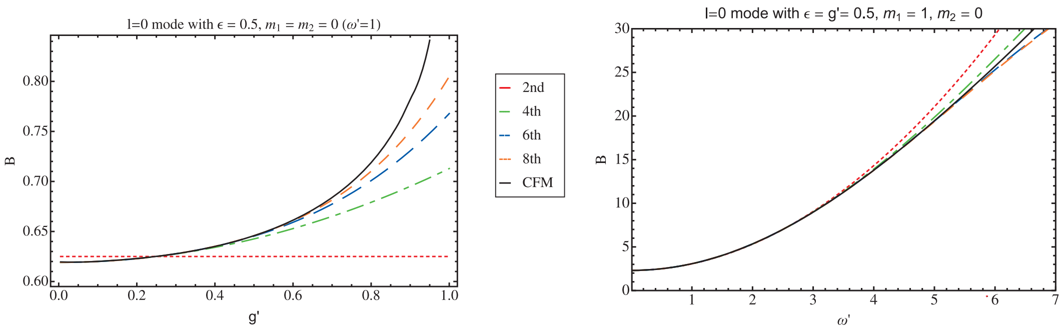

Fig. 1. Comparison of the perturbative expansion up to 8th order with the continued fraction method for $l = 0$ modes. In the above plots $\epsilon = a_2/a_1$, $\omega' = \omega a_1$ and $g' = g a_1$ ($a_1 = 1.0$).

Table I. Comparison between the small a_1 and a_2 expansion and the exact result for $l = 0, m_1 = 1$ and $m_2 = 0$ for given values of $\epsilon = a_1/a_2$, $\omega' = a_1\omega = 1$ and $g' = a_1 g = 0.01$. In the examples given below we found agreement between the numerical method and 6th and 8th order accurate to 6 s.f.

$(\epsilon,\ a_1)$	(0.1, 0.5)	(0.1, 1.0)	(0.5, 0.5)	(0.5, 1.0)	(0.9, 0.5)	(0.9, 1.0)
Numerical	3.16715	3.66516	3.18727	3.74711	3.23408	3.93621
2nd order	3.16743	3.66973	3.18743	3.74972	3.23409	3.93637
4th order	3.16715	3.66521	3.18727	3.74711	3.23408	3.93621

*) In the limit $a_1 = a_2$, the continued fraction terminates and results in an analytic solution (see the similar discussions for simply rotating black holes ($a_2 = 0$) when $a_1 \to 0$ in Ref. 28)).

§6. Results and discussion

In summary we have separated the Klein-Gordon wave equation on a five dimensional Kerr-(A)dS background with two rotation parameters and derived the radial and angular equations. The emphasis was on the scalar spheroidal harmonics, which are of relevance to charged and non-charged Kerr-(A)dS solutions.[9)] We developed perturbation theory and have presented the result for the angular eigenvalue, B, up to 6th order (see the footnote in §4). This now fills the gap, since a perturbative expansion for the angular eigenvalues in $D \geq 6$ has been studied in Ref. 17).

In Fig. 1 we show some plots for for varying ω' and g' for fixed values of m_1 and m_2. As expected we find improving agreement as the order of expansion increases. Note that in the left panel the 2nd order answer is only linear because for $l = m_1 = m_2 = 0$ at 2nd order there is no g dependence (cf. Eq. (4·35)). Table I also shows data for the $l = 0, m_1 = 1$ and $m_2 = 0$ mode, compared to the continued fraction method accurate to a precision of 6 s.f., which further confirms the improvement in increasing order.

It may be worth mentioning that we might also be able to form a perturbative expansion using inverted continued fractions; this was successfully applied to simply rotating cases ($a_2 = 0$) in Refs. 28) and 26), where the natural choice appears to be in powers of $\omega' = a_1\omega$ and $\alpha_1 = a_1^2 g^2$ (cf. Ref. 26)). However, in the case of two rotation parameters there are clearly two choices: one is an expansion in powers of $\omega' = a_1\omega$, $\alpha_1 = a_1^2 g^2$ and $\alpha_2 = a_2^2 g^2$; the other (used in this paper) is just an expansion in powers of a_1 and a_2. Preliminary work on an expansion of the inverted fraction for two rotations seems difficult to apply symbolically. However, it is possible to verify[*)] that a perturbative expansion in terms of ω', α_1 and α_2 (albeit more cumbersome) does agree with the inverted fraction second order answer[28),26)] for a simply rotating black hole ($a_2 = 0$), when $a_2, \alpha_2 \to 0$. We leave these issues for possible future works.

In conclusion, a perturbative analytic expression for the angular eigenvalue has been derived and could be used to find QNMs or absorption probabilities (greybody factors) for a massless scalar field separated on a Kerr-(A)dS background in five dimensions. It may also be worth mentioning that our angular equation (3·9) is identical to that found in Ref. 9) for charged Kerr-(A)dS black holes and thus has applications in these models. Furthermore, it would be interesting to confirm our expectation that the separation will result in the same angular equation, (3·9), for 5D Kerr-(A)dS-NUT black holes as well. Interesting future works might be to look at QNMs for general five dimensional solutions, particularly given their relevance to gauged supergravity models.

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