Proceedings of 2012 CACS International Automatic Control Conference
National Formosa University, Yunlin, Taiwan, Nov. 30, 2012-Dec. 2, 2012

H-infinity Decentralized Output Feedback Synthesis in Finite Frequency Domain for Continuous-Time Linear Systems

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***Abstract*—This work extends the current results of centralized H-infinity control in finite frequency domain to the subject of decentralized control. Solvability conditions of the problems of decentralized dynamic and static output feedback for continuous-time linear systems are derived in terms of linear matrix inequalities (LMIs) in a unified manner. A numerical example is given that establishes the efficacy of the proposed controller sysnthesis.**
Keywords*—*H-infinity, decentralized control, LMI, GKYP

I. INTRODUCTION

Over the last three decades we have witnessed the tremendous progress of H-infinity control theory, see e.g., [1,2] and references therein. Widespread applications have been found in various engineering disciplines. Nevertheless, a naive application of the H-infinity control method does not necessarily yield to satisfactory performance. In some cases, frequency-dependent weighting functions are often introduced into the design procedure for performance improvement, for instance the problem of band-limit noises attenuation. The technique is known as H-infinity loop shaping. Indeed, the weighting functions play important role in this sort of designs; however, two common criticisms associated with the loop shaping method are that it is difficult (e.g., no general guidelines exist) to search for an appropriate weighting function, and the introduced weights increase the controller order. This motivates the birth of the new control method via the generalized Kalman-Yakubovich-Popov (GKYP) lemma [3,4,5], in which no weights are used.

At the early stage of its development [4], the use of the new method is restricted due to the lack of methods to solve the general control synthesis problem. It was until 2007, the problem was partially solved by the multiplier expansion method [5]. The improved method produces full-order *centralized* H-infinity controllers. For the sake of theoretical interest and the potentially profound applications of decentralized control, it's the purpose of this work to extend the results of GKYP control to the subject of decentralized control. To the knowledge of the authors, no such results have appeared in the open literature. The rest of the work is organized as follows: Section II covers the problem statement and preliminaries. Section III presents the main results. A comprehensive numerical example is given in Section IV. Some of the proofs of the new results are in Section V. Section VI is Conclusions.

II. PROBLEM STATEMENT AND PRELIMINARIES

Notation: Let $\mathbb{R}$ be the set of real numbers, and $\mathbb{R}^{p\times m}$ denotes the set of all real $p\times m$ matrices. For a matrix G, G^T and G^*, denote its transpose and complex conjugate, respectively. The Hermitian part of a square matrix G is denoted by $He\{G\}:=G+G^*$. RH_∞ is the set of real-rational proper transfer functions with poles in the open left half complex plane. Let Ω be a closed interval in $\mathbb{R}$ and X be a complex-valued function of a single complex variable, $X^\sim(s):=X^T(-s)$; $\|X\|_\Omega:=\sup_{\theta\in\Omega}\bar{\sigma}(X(j\omega))$, where $\bar{\sigma}(\bullet)$ denotes the largest singular value of the argument. A transfer function X is called inner if $X\in RH_\infty$ and $X^\sim X=I$; $X^\perp$ is called a complementary inner factor (CIF) of X if $[X\quad X^\perp]$ is square and is inner. A square function $X\in RH_\infty$ is called strictly positive real (SPR) if $He\{X(j\omega)\}>0$ for all $\omega\in\mathbb{R}\cup\infty$. Symbol $*$ in a matrix inequality is readily inferred by symmetry.

Consider an L-channel linear time-invariant system P described by

$$\begin{aligned}\dot{x}&=Ax+B_1w+\sum_{i=1}^{L}B_{2i}u_i\\ z&=C_1x+D_{11}w+\sum_{i=1}^{L}D_{12i}u_i\\ y_i&=C_{2i}x+D_{21i}w+\sum_{j=1}^{L}D_{22ij}u_j,\, i,j=1,2,\ldots,L.\end{aligned}\tag{1}$$

where $x(t)\in\mathbb{R}^n$ is the state, $w(t)\in\mathbb{R}^{m_1}$ is the exogenous input, $z(t)\in\mathbb{R}^{p_1}$ is the observed output, and $u_i(t)\in\mathbb{R}^{m_{2i}}$ and $y_i(t)\in\mathbb{R}^{p_{2i}}$ represent the control input and measurement output of channel i,(i=1,2,…,L), respectively. The matrices $A, B_1, B_{2i}, C_1, C_2, D_{11}, D_{12i}, D_{21i}$, and D_{22ij} are constant and of appropriate dimensions. To expedite calculations involving transfer functions, we shall use the following notation:

$$P(s)=\begin{bmatrix}P_{11}(s) & P_{12}(s)\\ P_{21}(s) & P_{22}(s)\end{bmatrix}:=\left[\begin{array}{c|cc}A & B_1 & B_2\\ \hline C_1 & D_{11} & D_{12}\\ C_2 & D_{21} & D_{22}\end{array}\right]\tag{2}$$

where

$$\begin{aligned}&B_2=\left[B_{21}\cdots B_{2L}\right],C_2=\left[C_{21}^T\cdots C_{2L}^T\right]^T,D_{12}=\left[D_{121}\cdots D_{12L}\right],\\ &D_{21}=\left[D_{211}^T\cdots D_{21L}^T\right]^T,D_{22}=\left[D_{22ij}\right],i,j=1,2,\ldots,L.\end{aligned}\tag{3}$$

Let $m_2=\sum_{i=1}^{L}m_{2i}$ and $p_2=\sum_{i=1}^{L}p_{2i}$. Throughout this paper we assume plant P satisfies the following assumptions:

(i) There is no unstable fixed mode with respect to the triplet (C_2, A, B_2) (see e.g., [6]).

(ii) $P_{12}^{\sim}P_{12}(j\omega) > 0 \quad \forall \omega \in \mathbb{R} \cup \infty$.

(iii) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all $\omega \in \mathbb{R}$.

The objective of this work is to determine decentralized control strategies that ensure the resulting closed-loop system T_{zw} stable and satisfies a given H-infinity gain level within prescribed restricted frequency range $\Omega = [0, \varpi_l]$. Both dynamic and static controllers are discussed. Specifically, the problems are stated as follows.

Problem 1: Let γ and ϖ_l be given positive values. Consider plant (1), find a decentralized dynamic output feedback controller $K=diag(K_1,\dots,K_L)$ with

$$K_i : \begin{cases} \dot{\hat{x}}_i = \hat{A}_i \hat{x}_i + \hat{B}_i y_i \\ u_i = \hat{C}_i \hat{x}_i + \hat{D}_i y_i, \quad i = 1, 2, \cdots, L. \end{cases} \tag{4}$$

where $\hat{x}_i(t) \in \mathbb{R}^{n_i}, \hat{A}_i \in \mathbb{R}^{n_i \times n_i}, \hat{B}_i \in \mathbb{R}^{n_i \times p_{2i}}, \hat{C}_i \in \mathbb{R}^{m_{2i} \times n_i}, \hat{D}_i \in \mathbb{R}^{m_{2i} \times p_{2i}}$ with n_i's being specified positive integers satisfying $\sum_{i=1}^{L} n_i = n$, that stabilizes the plant and ensures $\|T_{zw}\|_{\Omega} < \gamma$.

Problem 2: Let γ and ϖ_l be given positive values. Consider plant (1), find a decentralized static output feedback control law of the form (5)

$$u_i = \hat{D}_i y_i, \quad i = 1, 2, \cdots, L. \tag{5}$$

where $\hat{D}_i \in \mathbb{R}^{m_{2i} \times p_{2i}}$, that stabilizes the plant and ensures $\|T_{zw}\|_{\Omega} < \gamma$.

With a slight abuse of notation, the transfer function of the closed-loop system (1) and (4) (or (5)), denoted by T_{zw} can be calculated by the following formula:

$$T_{zw} = P_{11} + P_{12}(I - KP_{22})^{-1} K P_{21}. \tag{6}$$

For ease of exposition we also define the notations:

$$\mathcal{S}_{\mathcal{A}} := \left\{ diag\left(\hat{A}_1, \hat{A}_2, \cdots, \hat{A}_L\right) : \hat{A}_i \in \mathbb{R}^{n_i \times n_i} \right\},$$

$$\mathcal{S}_{\mathcal{B}} := \left\{ \begin{matrix} \left(diag\left(\hat{B}_{M1}, \hat{B}_{M2}, \cdots, \hat{B}_{ML}\right), diag\left(\hat{B}_{N1}, \hat{B}_{N2}, \cdots, \hat{B}_{NL}\right) \right) : \\ \hat{B}_{Mi} \in \mathbb{R}^{n_i \times m_{2i}}, \hat{B}_{Ni} \in \mathbb{R}^{n_i \times p_{2i}} \end{matrix} \right\},$$

$$\mathcal{S}_{\mathcal{C}} := \left\{ diag\left(\hat{C}_1, \hat{C}_2, \cdots, \hat{C}_L\right) : \hat{C}_i \in \mathbb{R}^{m_{2i} \times n_i} \right\},$$

$$\mathcal{S}_{\mathcal{D}} := \left\{ \begin{matrix} \left(diag\left(\hat{D}_{M1}, \hat{D}_{M2}, \cdots, \hat{D}_{ML}\right), diag\left(\hat{D}_{N1}, \hat{D}_{N2}, \cdots, \hat{D}_{NL}\right) \right) : \\ \hat{D}_{Mi} \in \mathbb{R}^{m_{2i} \times m_{2i}}, \hat{D}_{Ni} \in \mathbb{R}^{m_{2i} \times p_{2i}} \end{matrix} \right\},$$

$$\mathcal{S}_{\mathcal{D}}(RH_\infty) := \left\{ \begin{matrix} \left(diag\left(\hat{D}_{M1}, \hat{D}_{M2}, \cdots, \hat{D}_{ML}\right), diag\left(\hat{D}_{N1}, \hat{D}_{N2}, \cdots, \hat{D}_{NL}\right) \right) : \\ \hat{D}_{Mi} \in RH_\infty^{m_{2i} \times m_{2i}}, \hat{D}_{Ni} \in RH_\infty^{m_{2i} \times p_{2i}} \end{matrix} \right\}. \tag{7}$$

The following lemmas are given, which are useful for the later developments.

Lemma 1: Given a positive value γ, let H be a transfer function which has a real-valued state-space realization (A,B,C,D) and has no poles on the $j\omega$ axis. Then under the assumption $D^T D - \gamma^2 I < 0$, the following statements are equivalent.

(i) $\bar{\sigma}(H(j\omega)) < \gamma \quad \forall \omega \in [0, \varpi_l]$.

(ii) There exist real symmetry matrices P and Q satisfying $Q>0$ and

$$\begin{bmatrix} -A^T QA + PA + A^T P + \varpi_l^2 Q & -A^T QB + PB & C^T \\ * & -B^T QB - \gamma^2 I & D^T \\ * & * & -I \end{bmatrix} < 0 \tag{8}$$

(iii) There exist real symmetry matrices P, Q and real matrices G, W satisfying $Q>0$ and

$$\begin{bmatrix} -Q + G^T + G & -P - W^T + GA & -GB & 0 \\ * & \varpi_l^2 Q - He\{WA\} & WB & C^T \\ * & * & -\gamma I & -D^T \\ * & * & * & -\gamma I \end{bmatrix} < 0 \tag{9}$$

Lemma 2: Let H be a transfer function which has a real-valued state-space realization (A,B,C,D) and has no poles on the $j\omega$ axis. Then under the assumption $He\{D\}>0$, the following statements are equivalent.

(i) $He\{H(j\omega)\} > 0 \quad \forall \omega \in [0, \varpi_l]$.

(ii) There exist real symmetry matrices P and Q satisfying $Q>0$ and

$$\begin{bmatrix} -A^T QA + PA + A^T P + \varpi_l^2 Q & -A^T QB + PB - C^T \\ * & -B^T QB - D^T - D \end{bmatrix} < 0 \tag{10}$$

(iii) There exist real symmetry matrices P, Q and real matrices G, W satisfying $Q>0$ and

$$\begin{bmatrix} -Q + G^T + G & -P - W^T + GA & -GB \\ * & \varpi_l^2 Q - He\{WA\} & C^T + WB \\ * & * & -He\{D\} \end{bmatrix} < 0 \tag{11}$$

Lemma 3: Let H be a transfer function with all poles in the open left half complex plane and has a real-valued, stabilizable and detectable state-space realization (A,B,C,D). Then the following statements are equivalent.

(i) H is SPR.

(ii) There exist real symmetric matrix P satisfying $P>0$ and

$$\begin{bmatrix} A^T P + PA & PB - C^T \\ * & -D - D^T \end{bmatrix} < 0 \tag{12}$$

(iii) There exist real symmetric matrix P and real matrices G, W satisfying $P>0$ and

$$\begin{bmatrix} G^T + G & -P - W^T + GA & -GB \\ * & -He\{WA\} & C^T + WB \\ * & * & -He\{D\} \end{bmatrix} < 0 \tag{13}$$

Notice that, while the equivalence between conditions (i) and (ii) of Lemma J (J=1,2,3) is known [1,3,4], the equivalence with condition (iii) in each lemma is new.

III. MAIN RESULTS

In this section, we present solvability conditions of the finite frequency decentralized output feedback control problems as stated in Section II.

A. Frequency Domain Solvability Conditions

The following two lemmas are useful for deriving the conditions of Theorem 1 and Corollary 1.

Lemma 4 Suppose there exist a positive value α and a square matrix W satisfying $He\{W - \alpha I\} > 0$. Then W^{-1} exists and $\bar{\sigma}(W^{-1}) < \alpha^{-1}$.

Lemma 5 Under Assumptions (i)-(iii), there exists a right coprime factorization $[P_{12}^T \quad P_{22}^T]^T = [N_{12}^T \quad N_{22}^T]^T M_{22}^{-1}$ for $[P_{12}^T \quad P_{22}^T]^T$ with N_{12} being inner.

Notice that implicit in Lemma 5 is the requirement that $p_1 \geq m_2$. For the case $p_1 > m_2$, since N_{12} is inner, there exists a CIF of N_{12} such that $U := [N_{12} \quad N_{12}^{\perp}]$ is square and is inner [1]. In this case, we define the notation $[R_1^T \quad R_2^T]^T = U^{\sim} P_{11}$, where R_1 and R_2 are $m_2 \times m_1$ and $(p_1 - m_2) \times m_1$ real-rational proper transfer functions, respectively. On the other hand, let $R_1 := N_{12}^{\sim} P_{11}$, for the case $p_1 = m_2$.

Theorem 1: Assume $p_1 > m_2$. With notations of $N_{22}, M_{22}, R_1,$ and R_2 defined above for the case, let γ be given positive value and Ω be a closed interval in $\mathbb{R}$. Suppose that there exist a positive value α and function $[\tilde{M}_K \quad \tilde{N}_K] \in \mathcal{S}_{\mathcal{D}}(RH_\infty)$ (resp. constant matrix $[\tilde{M}_K \quad \tilde{N}_K] \in \mathcal{S}_{\mathcal{D}}$), and real-rational proper transfer function V, satisfying the following conditions:

(i) $$\bar{\sigma}\left(\begin{bmatrix} S_{cl}R_1 + \tilde{N}_K P_{21} \\ VR_2 \end{bmatrix}(j\omega)\right) < \alpha\gamma \quad \forall \omega \in \Omega. \tag{14}$$

(ii) $$He\{S_{cl}(j\omega) - \alpha I\} > 0 \quad \forall \omega \in \Omega. \tag{15}$$

(iii) $$He\{V(j\omega) - \alpha I\} > 0 \quad \forall \omega \in \Omega. \tag{16}$$

(iv) S_{cl} is SPR. (17)

where $S_{cl} = \tilde{M}_K M_{22} - \tilde{N}_K N_{22}$. Then the decentralized dynamic (resp. static) controller is determined by $K = \tilde{M}_K^{-1}\tilde{N}_K$, which stabilizes plant (1) and ensures $\|T_{zw}\|_\Omega < \gamma$.

Corollary1: Assume $p_1 = m_2$. With notations of $N_{22}, M_{22},$ and R_1 defined above for the case, let γ be given positive value and Ω be a closed interval in $\mathbb{R}$. Suppose that there exist a positive value α and function $[\tilde{M}_K \quad \tilde{N}_K] \in \mathcal{S}_{\mathcal{D}}(RH_\infty)$ (resp. constant matrix $[\tilde{M}_K \quad \tilde{N}_K] \in \mathcal{S}_{\mathcal{D}}$), satisfying the following conditions:

(i) $$\bar{\sigma}\left(\left(S_{cl}R_1 + \tilde{N}_K P_{21}\right)(j\omega)\right) < \alpha\gamma \quad \forall \omega \in \Omega. \tag{18}$$

(ii) $$He\{S_{cl}(j\omega) - \alpha I\} > 0 \quad \forall \omega \in \Omega. \tag{19}$$

(iii) S_{cl} is SPR. (20)

where $S_{cl} = \tilde{M}_K M_{22} - \tilde{N}_K N_{22}$. Then the decentralized dynamic (resp. static) controller is determined by $K = \tilde{M}_K^{-1}\tilde{N}_K$, which stabilizes plant (1) and ensures $\|T_{zw}\|_\Omega < \gamma$.

Remark 1. For single band case, the variable α in Theorem 1 and Corollary 1 can without loss of generality be assumed to be one because it can be absorbed into $\tilde{N}_K, \tilde{M}_K$ and V. However, it makes difference for multi-band problems, in which the solvability conditions can be relaxed by employing different α's.

B. State-Space Solutions

To convert the frequency-domain conditions of Theorem 1 into a state-space form, we proceed by writing the relevant transfer functions in terms of linear fractional transformation [1] as follows:

$\Pi_H := \begin{bmatrix} S_{cl}R_1 + \tilde{N}_K P_{21} \\ VR_2 \end{bmatrix} = F_l(P_H, K_H)$ where

$$P_H = \left[\begin{array}{c|cc} 0 & I & 0 \\ 0 & 0 & I \\ \hline M_{22}R_1 & 0 & 0 \\ P_{21} - N_{22}R_1 & 0 & 0 \\ R_2 & 0 & 0 \end{array}\right] \leftrightarrow \left[\begin{array}{c|c:c} A_H & B_{H1} & 0 \\ \hline 0 & 0 & D_{H12} \\ \hdashline C_{H2} & D_{H21} & 0 \end{array}\right],$$
$$K_H = \left[\begin{array}{cc|c} \tilde{M}_K & \tilde{N}_K & 0 \\ 0 & 0 & V \end{array}\right] \leftrightarrow \left[\begin{array}{c|c} A_{KH} & B_{KH} \\ \hline C_{KH} & D_{KH} \end{array}\right] := \tag{21}$$
$$\begin{bmatrix} T_H^{-1} & 0 \\ 0 & I \end{bmatrix} \left[\begin{array}{cc:ccc} A_{MN} & 0 & B_M & B_N & 0 \\ 0 & A_V & 0 & 0 & B_V \\ \hdashline C_{MN} & 0 & D_M & D_N & 0 \\ 0 & C_V & 0 & 0 & D_V \end{array}\right] \begin{bmatrix} T_H & 0 \\ 0 & I \end{bmatrix}.$$

$\Pi_S := S_{cl} - \alpha I = F_l(P_S, K_{MN})$ where

$$P_S = \left[\begin{array}{c:c} -\alpha I & I \\ \hdashline M_{22} & 0 \\ -N_{22} & 0 \end{array}\right] \leftrightarrow \left[\begin{array}{c|c:c} A_S & B_{S1} & 0 \\ \hline 0 & -\alpha I & D_{S12} \\ \hdashline C_{S2} & D_{S21} & 0 \end{array}\right],$$
$$K_{MN} = \begin{bmatrix} \tilde{M}_K & \tilde{N}_K \end{bmatrix} \leftrightarrow \begin{bmatrix} T_S^{-1} & 0 \\ 0 & I \end{bmatrix} \left[\begin{array}{c:cc} A_{MN} & B_M & B_N \\ \hdashline C_{MN} & D_M & D_N \end{array}\right] \begin{bmatrix} T_S & 0 \\ 0 & I \end{bmatrix}. \tag{22}$$

$\Pi_V := V - \alpha I = F_l(P_V, K_V)$ where

$$P_V = \left[\begin{array}{c:c} -\alpha I & I \\ \hdashline I & 0 \end{array}\right] \leftrightarrow \left[\begin{array}{c|c:c} A_V & B_{V1} & 0 \\ \hline 0 & -\alpha I & D_{V12} \\ \hdashline C_{V2} & D_{V21} & 0 \end{array}\right],$$
$$K_V = V \leftrightarrow \begin{bmatrix} T_V^{-1} & 0 \\ 0 & I \end{bmatrix} \left[\begin{array}{c:c} A_V & B_V \\ \hdashline C_V & D_V \end{array}\right] \begin{bmatrix} T_V & 0 \\ 0 & I \end{bmatrix}. \tag{23}$$

$\Pi_{SA} := S_{cl} = F_l(P_{SA}, K_{MN})$ where

$$P_{SA} = \left[\begin{array}{c:c} 0 & I \\ \hdashline M_{22} & 0 \\ -N_{22} & 0 \end{array}\right] \leftrightarrow \left[\begin{array}{c|c:c} A_{SA} & B_{SA1} & 0 \\ \hline 0 & 0 & D_{SA12} \\ \hdashline C_{SA2} & D_{SA21} & 0 \end{array}\right],$$
$$K_{MN} = \begin{bmatrix} \tilde{M}_K & \tilde{N}_K \end{bmatrix} \leftrightarrow \begin{bmatrix} T_{SA}^{-1} & 0 \\ 0 & I \end{bmatrix} \left[\begin{array}{c:cc} A_{MN} & B_M & B_N \\ \hdashline C_{MN} & D_M & D_N \end{array}\right] \begin{bmatrix} T_{SA} & 0 \\ 0 & I \end{bmatrix}. \tag{24}$$

where T_H, T_S, T_V and T_{SA} are nonsingular matrices which play the role of coordinate transformation. With these, the conditions (iii) of Lemmas 1, 2, and 3 are invoked to convert the frequency-domain conditions of Theorem 1 into LMI conditions (when the parameter λ is specified).

Theorem 2: Assume $p_1 > m_2$. Let γ and ϖ_l be given positive values. Suppose that there exist scalars $\lambda \in \mathbb{R}, \alpha > 0$, real matrices $Z_1 \in \mathcal{S}_{\mathcal{A}}, Z_2 \in \mathcal{S}_{\mathcal{B}}, Z_3 \in \mathcal{S}_{\mathcal{C}}, Z_4 \in \mathcal{S}_{\mathcal{D}}, \Sigma_2 \in \mathcal{S}_{\mathcal{A}}, Z_{i1}, Z_{i2}, Z_{i3}, Z_{i4}, \Sigma_{i2}$ $(i = H, V)$, $R_j, \bar{R}_j, \Sigma_{j1}, \bar{\Sigma}_{j1}$ $(j = H, S, V, SA)$, real symmetric matrices

$P_j = \begin{bmatrix} P_{j11} & P_{j12} \\ P_{j12}^T & P_{j22} \end{bmatrix}, (j = H,S,V)$, and positive definite matrices $Q_j = \begin{bmatrix} Q_{j11} & Q_{j12} \\ Q_{j12}^T & Q_{j22} \end{bmatrix}, (j = H,S,V)$, and $P_{SA} = \begin{bmatrix} P_{SA11} & P_{SA12} \\ P_{SA12}^T & P_{SA22} \end{bmatrix}$, with $Z_{H1} = diag(Z_1, Z_{V1}), Z_{H2} = diag(Z_2, Z_{V2}), Z_{H3} = diag(Z_3, Z_{V3}),$ $Z_{H4} = diag(Z_4, Z_{V4}), \Sigma_{H2} = diag(\Sigma_2, \Sigma_{V2}),$ satisfying the following matrix inequalities:

$$\begin{bmatrix} \Theta_{H11} & \Theta_{H12} & \Theta_{H13} & \Theta_{H14} & -\Sigma_{H1}B_{H1} & 0 \\ * & \Theta_{H22} & \Gamma_{23} & \Gamma_{24} & \Gamma_{25} & 0 \\ * & * & \Theta_{H33} & \Gamma_{34} & \bar{\Sigma}_{H1}B_{H1} & \Gamma_{36} \\ * & * & * & \Gamma_{44} & \Gamma_{45} & C_{H2}^T Z_{H4}^T D_{H12}^T \\ * & * & * & * & -\alpha\gamma I & -D_{H21}^T Z_{H4}^T D_{H12}^T \\ * & * & * & * & * & -\alpha\gamma I \end{bmatrix} < 0 \quad (25)$$

$$\begin{bmatrix} \Theta_{S11} & \Theta_{S12} & \Theta_{S13} & \Theta_{S14} & -\Sigma_{S1}B_{S1} \\ * & \Theta_{S22} & \Phi_{23} & \Phi_{24} & -R_S B_{S1} - Z_2 D_{S21} \\ * & * & \Theta_{S33} & \Phi_{34} & \Phi_{35} \\ * & * & * & \Phi_{44} & \Phi_{45} \\ * & * & * & * & -He\{-\alpha I + D_{S12} Z_4 D_{S21}\} \end{bmatrix} < 0 \quad (26)$$

$$\begin{bmatrix} \Theta_{V11} & \Theta_{V12} & \Theta_{V13} & \Theta_{V14} & -\Sigma_{V1}B_{V1} \\ * & \Theta_{V22} & \Psi_{23} & \Psi_{24} & -R_V B_{V1} - Z_{V2} D_{V21} \\ * & * & \Theta_{V33} & \Psi_{34} & \Psi_{35} \\ * & * & * & \Psi_{44} & \Psi_{45} \\ * & * & * & * & -He\{-\alpha I + D_{V12} Z_{V4} D_{V21}\} \end{bmatrix} < 0 \quad (27)$$

where

$\Theta_{j11} = -Q_{j11} + He\{\Sigma_{j1}\}, \Theta_{j12} = -Q_{j12} + R_j^T + \Sigma_{j2}^T + \Sigma_{j1},$
$\Theta_{j13} = -P_{j11} - \bar{\Sigma}_{j1}^T + \Sigma_{j1}A_j, \Theta_{j14} = -P_{j12} - \bar{R}_j^T - \lambda\Sigma_{j2}^T + \Sigma_{j1}A_j,$
$\Theta_{j22} = -Q_{j22} + He\{R_j^T\}, \Theta_{j33} = \varpi_l^2 Q_{j11} - He\{\bar{\Sigma}_{j1}A_j\},$
$j = H,S,V.$
$\Gamma_{23} = -P_{H12}^T - \bar{\Sigma}_{H1}^T + R_H A_H + Z_{H2}C_{H2} + Z_{H1},$
$\Gamma_{24} = -P_{H22} - \bar{R}_H^T + R_H A_H + Z_{H2}C_{H2},$
$\Gamma_{34} = \varpi_l^2 Q_{H12} - \bar{\Sigma}_{H1}A_H - A_H^T \bar{R}_H^T - \lambda C_{H2}^T Z_{H2}^T - \lambda Z_{H1}^T,$
$\Gamma_{44} = \varpi_l^2 Q_{H22} - He\{\bar{R}_H A_H + \lambda Z_{H2}C_{H2}\},$
$\Gamma_{25} = -R_H B_{H1} - Z_{H2}D_{H21},$
$\Gamma_{45} = \bar{R}_H B_{H1} + \lambda Z_{H2}D_{H21},$
$\Gamma_{36} = C_{H2}^T Z_{H4}^T D_{H12}^T + Z_{H3}^T D_{H12}^T,$
$\Phi_{23} = -P_{S12}^T - \bar{\Sigma}_{S1}^T + R_S A_S + Z_2 C_{S2} + Z_1,$
$\Phi_{24} = -P_{S22} - \bar{R}_S^T + R_S A_S + Z_2 C_{S2},$
$\Phi_{34} = \varpi_l^2 Q_{S12} - \bar{\Sigma}_{S1}A_S - A_S^T \bar{R}_S^T - \lambda C_{S2}^T Z_2^T - \lambda Z_1^T,$
$\Phi_{44} = \varpi_l^2 Q_{S22} - He\{\bar{R}_S A_S + \lambda Z_2 C_{S2}\},$
$\Phi_{35} = C_{S2}^T Z_4^T D_{S12}^T + Z_3^T D_{S12}^T + \bar{\Sigma}_{S1}B_{S1},$
$\Phi_{45} = C_{S2}^T Z_4^T D_{S12}^T + \bar{R}_S B_{S1} + \lambda Z_2 D_{S21},$
$\Psi_{23} = -P_{V12}^T - \bar{\Sigma}_{V1}^T + R_V A_V + Z_{V2}C_{V2} + Z_{V1},$
$\Psi_{24} = -P_{V22} - \bar{R}_V^T + R_V A_V + Z_{V2}C_{V2},$
$\Psi_{34} = \varpi_l^2 Q_{V12} - \bar{\Sigma}_{V1}A_V - A_V^T \bar{R}_V^T - \lambda C_{V2}^T Z_{V2}^T - \lambda Z_{V1}^T,$
$\Psi_{44} = \varpi_l^2 Q_{V22} - He\{\bar{R}_V A_V + \lambda Z_2 C_{V2}\},$
$\Psi_{35} = C_{V2}^T Z_{V4}^T D_{V12}^T + Z_{V3}^T D_{V12}^T + \bar{\Sigma}_{V1}B_{V1},$
$\Psi_{45} = C_{V2}^T Z_{V4}^T D_{V12}^T + \bar{R}_V B_{V1} + \lambda Z_{V2} D_{V21}.$

and

$$\begin{bmatrix} He\{\Sigma_{SA1}\} & R_{SA}^T + \Sigma_2^T + \Sigma_{SA1} & \Lambda_{13} & \Lambda_{14} & -\Sigma_{SA1}B_{SA1} \\ * & He\{R_{SA}^T\} & \Lambda_{23} & \Lambda_{24} & \Lambda_{25} \\ * & * & \Lambda_{33} & \Lambda_{34} & \Lambda_{35} \\ * & * & * & \Lambda_{44} & \Lambda_{45} \\ * & * & * & * & \Lambda_{55} \end{bmatrix} < 0 \quad (28)$$

where

$\Lambda_{13} = -P_{SA11} - \bar{\Sigma}_{SA1}^T + \Sigma_{SA1}A_{SA},$
$\Lambda_{23} = -P_{SA12}^T - \bar{\Sigma}_{SA1}^T + R_{SA}A_{SA} + Z_2 C_{SA2} + Z_1,$
$\Lambda_{33} = -He\{\bar{\Sigma}_{SA1}A_{SA}\},$
$\Lambda_{14} = -P_{SA12} - \bar{R}_{SA}^T - \lambda\Sigma_2^T + \Sigma_{SA1}A_{SA},$
$\Lambda_{24} = -P_{SA22} - \bar{R}_{SA}^T + R_{SA}A_{SA} + Z_2 C_{SA2},$
$\Lambda_{34} = -\bar{\Sigma}_{SA1}A_{SA} - A_{SA}^T \bar{R}_{SA}^T - \lambda C_{SA2}^T Z_2^T - \lambda Z_1^T,$
$\Lambda_{44} = -He\{\bar{R}_{SA}A_{SA} + \lambda Z_2 C_{SA2}\},$
$\Lambda_{25} = -R_{SA}B_{SA1} - Z_2 D_{SA21},$
$\Lambda_{35} = C_{SA2}^T Z_4^T D_{SA12}^T + Z_3^T D_{SA12}^T + \bar{\Sigma}_{SA1}B_{SA1},$
$\Lambda_{45} = C_{SA2}^T Z_4^T D_{SA12}^T + \bar{R}_{SA}B_{SA1} + \lambda Z_2 D_{SA21},$
$\Lambda_{55} = -He\{D_{SA12} Z_4 D_{SA21}\}.$

Then there exists a solution to Problem 1. The decentralized dynamic controller is determined by the formula

$$K = \hat{C}\left(sI - \hat{A}\right)^{-1}\hat{B} + \hat{D} \quad (29)$$

where

$$\hat{A} = A_{MN} - B_M D_M^{-1} C_{MN},\ \hat{B} = B_N - B_M D_M^{-1} D_N, \\ \hat{C} = D_M^{-1} C_{MN},\ \hat{D} = D_M^{-1} D_N \quad (30)$$

with

$$\left[\begin{array}{c|cc} A_{MN} & B_M & B_N \\ \hline C_{MN} & D_M & D_N \end{array}\right] = \left[\begin{array}{c|c} Z_1\Sigma_2^{-1} & Z_2 \\ \hline Z_3\Sigma_2^{-1} & Z_4 \end{array}\right]. \quad (31)$$

For the design of decentralized static output feedback, Lemmas 1(iii), 2(ii) and (iii), and 3(ii) are invoked to convert the frequency-domain conditions of Theorem 1 into LMI conditions (when the parameter λ is specified).

Theorem 3: Assume $p_1 > m_2$. Let γ and ϖ_l be given positive values. Suppose that there exist scalars $\lambda \in \mathbb{R}, \alpha > 0,$ real matrices $Z_4 \in \mathcal{S}_{\mathcal{D}}, Z_{i1}, Z_{i2}, Z_{i3}, Z_{i4},$ $\Sigma_{i2}\ (i = H,V), R_j, \bar{R}_j, \Sigma_{j1}, \bar{\Sigma}_{j1} (j = H,V)$, real symmetric matrices $P_j = \begin{bmatrix} P_{j11} & P_{j12} \\ P_{j12}^T & P_{j22} \end{bmatrix}, (j = H,V), P_S$, and positive definite matrices $Q_j = \begin{bmatrix} Q_{j11} & Q_{j12} \\ Q_{j12}^T & Q_{j22} \end{bmatrix} (j = H,V), Q_S, P_{SA},$ with $Z_{H1} = Z_{V1}, Z_{H2} = \begin{bmatrix} 0 & Z_{V2} \end{bmatrix},$ $Z_{H3} = \begin{bmatrix} 0 & Z_{V3}^T \end{bmatrix}^T, Z_{H4} = diag(Z_4, Z_{V4}), \Sigma_{H2} = \Sigma_{V2},$ satisfying (25), (27) and the following matrix inequalities

$$\begin{bmatrix} \Delta_{11} & -A_S^T Q_S B_{S1} + P_S B_{S1} - C_{S2}^T Z_4^T D_{S12}^T \\ * & -B_{S1}^T Q_S B_{S1} - He\{-\alpha I + D_{S12} Z_4 D_{S21}\} \end{bmatrix} < 0 \quad (32)$$

where

$\Delta_{11} = -A_S^T Q_S A_S + He\{P_S A_S\} + \varpi_l^2 Q_S$

and

$$\begin{bmatrix} A_{SA}^T P_{SA} + P_{SA}A_{SA} & P_{SA}B_{SA1} - C_{SA2}^T Z_4^T D_{SA12}^T \\ * & -He\{D_{SA12} Z_4 D_{SA21}\} \end{bmatrix} < 0 \quad (33)$$

Then there exists a solution to Problem 2. The decentralized static output feedback gain K_D is determined by

$$K_D = D_M^{-1} D_N \quad (34)$$

where $\begin{bmatrix} D_M & D_N \end{bmatrix} = Z_4$.

Remark 2. Dual results of Theorems 1, 2, 3, and Corollary 1 can be derived in the same manner for the case $m_1 \geq p_2$ by applying the property of norm preserving under matrix transpose (e.g., the largest singular value). The assumptions differ and this extends the results of this work.

Remark 3. It is straightforward to extend this work to the other finite frequency ranges and discrete-time cases.

IV. SIMULATION

In this section, a numerical example is given that establishes the efficacy of the proposed methods.

Example 1: This example is a continuous-time unstable system adopted from Robert A. Paz (1993) [8]. The plant data is given as follows.

$$A=\begin{bmatrix}0&1&4&-4&1\\-3&-1&1&2&1\\0&1&-1&-1&0\\2&1&-1&0&1\\-1&2&1&-2&-2\end{bmatrix}, B_1=\begin{bmatrix}1&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\\0&0&0&0&0\\1&0&0&0&0\end{bmatrix}, B_2=\begin{bmatrix}0&0\\4&0\\0&0\\0&0\\0&2\end{bmatrix},$$

$$C_1=\begin{bmatrix}1&0&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}, C_2=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}, D_{21}=\begin{bmatrix}0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix} \quad (35)$$

$$D_{12}=\begin{bmatrix}0&0&0&1&0\\0&0&0&0&1\end{bmatrix}^T, D_{11}=0_{5\times5}, D_{22}=0_{4\times2}.$$

The results of [8] give the centralized and decentralized dynamic controller with H_∞ performance 1.91211 and 1.9122, respectively. For demonstrating the effectiveness of the proposed methods, which emphasizes alternatively on closed-loop system performance in a restricted frequency range, the frequency range under consideration is given by $|\omega|\le 1$. Let the notations γ_∞ and γ_{IW} represent the smallest values that were obtained from solving the synthesis conditions (i.e., Theorem 2 and Theorem 3) and the analysis condition (Lemma 1(ii)) for the resulting closed-loop systems, respectively.

Applying the proposed methods (i.e., Theorem 2 and Theorem 3) with $\lambda=1$ yields the results of the decentralized dynamic and static output feedback controllers as shown in Table 1. As can be seen, both perform better than that of [8] in the prescribed frequency range. The parameters of the decentralized controllers are provided as follows:

$$\hat{A}=\begin{bmatrix}-8.6925&1.8793&-1.0119&0&0\\-3.9095&-1.0863&-0.7814&0&0\\-8.0877&6.4241&-2.1383&0&0\\0&0&0&-19.1633&92.4829\\0&0&0&-9.0561&-33.1936\end{bmatrix}$$

$$\hat{B}=\begin{bmatrix}-0.0030&0.0069&0&0\\-0.0008&0.0025&0&0\\0.0044&-0.0324&0&0\\0&0&-0.0032&0.0001\\0&0&0.0003&-0.0008\end{bmatrix} \quad (36)$$

$$\hat{C}=\begin{bmatrix}368.9782&607.3013&222.1798&0&0\\0&0&0&4124.7937&-15560.4061\end{bmatrix}$$

$$\hat{D}=\begin{bmatrix}0.0989&-1.4193&0&0\\0&0&-0.6549&-1.1659\end{bmatrix}$$

$$K_D=\begin{bmatrix}-54.9601&-516.5782&0&0\\0&0&-1.3556&-1.2255\end{bmatrix} \quad (37)$$

Next, we restrict ourselves to finding a strictly proper decentralized dynamic controller of the same structure. This can easily be done by setting D_N (i.e, part of Z_4) to be null; see (31). In such a case, the direct-through part of the closed-loop system becomes null. Hence the achievable performance level measured by Lemma 1(ii) [3,4] and Lemma 1(iii) (this work), denoted by γ_{IW} and γ_{PGW}, respectively, should be the same. Indeed, the numerical result verifies this expectation; both take the value 1.5719. Besides, $\gamma_\infty = 2.384$ is higher than that in Table 1. This is reasonable because the design parameter λ and the controller structure are kept the same, and a smaller class of controllers is considered for the design.

Table 1 Comparative result of the decentralized dynamic and static controllers

	dynamic	static
α	0.0039	0.0979
γ_∞	1.776	1.573
γ_{IW}	1.5115	1.3965
order of V	10	15
Closed- loop poles	$-1.1534, -4.6386,$ $-9.8735, -1.8150,$ $-26.1119\pm 27.5413i,$ $-3.1912\pm 4.0167i,$ $-0.0982\pm 3.0453i,$	-2.0672×10^3 $-0.0562\pm 3.2612i$ $-3.5594, -1.8844$

V. APPENDIX

The proofs of conditions (iii) of Lemmas 1, 2, 3, Corollary 1, and Theorem 3 are omitted due to space limitation.

Proof of Lemma 4

Proof: Consider the following equivalent mathematical conditions: For any square matrix W and any real value α we have

$$\begin{aligned}&[W-\alpha I]^*[W-\alpha I]\ge 0 \Leftrightarrow W^*W \ge \alpha\left[W^*+W\right]-\alpha^2 I\\ &\Leftrightarrow W^*W\ge \alpha\left[(W-\alpha I)^*+(W-\alpha I)\right]+\alpha^2 I\end{aligned} \quad (A1)$$

By the hypothesis, we have $W^*W>\alpha^2 I$, which in turn implies that W^{-1} exists and $\bar{\sigma}(W^{-1})<\alpha^{-1}$. □

Proof of Lemma 5

Proof: By [1, Problem 12.6] Assumption (ii) implies that there exists a right coprime factorization (rcf) $P_{12}=N_{12}M_{22}^{-1}$ with N_{12} being inner. On the other hand, Assumption (i) implies that (A,B_2) is stabilizable. Thus, again by [1, Problem 12.6], Assumptions (i),(ii),(iii) imply that there exists a matrix F which realizes the desired rcf for P_{12} :

$$\begin{bmatrix}M_{22}\\N_{12}\end{bmatrix} := \left[\begin{array}{c|c}A+B_2F & B_2R^{-1/2}\\ \hline F & R^{-1/2}\\ C_1+D_{12}F & D_{12}R^{-1/2}\end{array}\right]\in RH_\infty \quad (A2)$$

where $R = D_{12}^T D_{12} > 0$. By the state-space realization for a doubly coprime factorization of a transfer function, see e.g., [1, Theorem 5.6] or [7]. It is readily verified that

$$\begin{bmatrix} \bar{M}_{22} \\ \bar{N}_{12} \\ \bar{N}_{22} \end{bmatrix} := \left[\begin{array}{c|c} A+B_2F & B_2 \\ \hline F & I \\ C_1+D_{12}F & D_{12} \\ C_2+D_{22}F & D_{22} \end{array}\right] \in RH_\infty \tag{A3}$$

and hence

$$\begin{bmatrix} M_{22} \\ N_{12} \\ N_{22} \end{bmatrix} := \begin{bmatrix} \bar{M}_{22}R^{-1/2} \\ \bar{N}_{12}R^{-1/2} \\ \bar{N}_{22}R^{-1/2} \end{bmatrix} = \left[\begin{array}{c|c} A+B_2F & B_2R^{-1/2} \\ \hline F & R^{-1/2} \\ C_1+D_{12}F & D_{12}R^{-1/2} \\ C_2+D_{22}F & D_{22}R^{-1/2} \end{array}\right] \in RH_\infty \tag{A4}$$

are rcf of the transfer function

$$\left[\begin{array}{c|c} A & B_2 \\ \hline C_1 & D_{12} \\ C_2 & D_{22} \end{array}\right] := \begin{bmatrix} P_{12} \\ P_{22} \end{bmatrix}. \tag{A5}$$

Furthermore, N_{12} is inner. □

Proof of Theorem 1

Proof: Under Assumptions (i),(ii),(iii), there exists a right coprime factorization $[P_{12}^T \quad P_{22}^T]^T = [N_{12}^T \quad N_{22}^T]^T M_{22}^{-1}$ for $[P_{12}^T \quad P_{22}^T]^T$ with N_{12} being inner. Next, we shall first show that assuming $K = \tilde{M}_K^{-1}\tilde{N}_K$ and letting $S_{cl} = \tilde{M}_K M_{22} - \tilde{N}_K N_{22}$, we have $T_{zw} = P_{11} + N_{12}S_{cl}^{-1}\tilde{N}_K P_{21}$. Indeed, substituting the alluded coprime factorizations of $[P_{12}^T \quad P_{22}^T]^T$ and K into T_{ZW}, yields

$$\begin{aligned} T_{zw} &= P_{11} + P_{12}\left(I - KP_{22}\right)^{-1}KP_{21} \\ &= P_{11} + N_{12}M_{22}^{-1}\left(I - \tilde{M}_K^{-1}\tilde{N}_K N_{22}M_{22}^{-1}\right)^{-1}\tilde{M}_K^{-1}\tilde{N}_K P_{21} \\ &= P_{11} + N_{12}\left(\tilde{M}_K M_{22} - \tilde{N}_K N_{22}\right)^{-1}\tilde{N}_K P_{21} \\ &= P_{11} + N_{12}S_{cl}^{-1}\tilde{N}_K P_{21}. \end{aligned} \tag{A6}$$

Inspired by [2] and by Lemma 5, since N_{12} is inner and $p_1 > m_2$, there exists a CIF of N_{12} such that $U := [N_{12} \quad N_{12}^{\perp}]$ is square and is inner. Because of the norm preserving property of inner functions, we have for any $\omega \in \mathbb{R}$

$$\begin{aligned} \bar{\sigma}\left(T_{zw}(j\omega)\right) &= \bar{\sigma}\left(\left(P_{11} + \begin{bmatrix} N_{12} & N_{12}^{\perp} \end{bmatrix}\begin{bmatrix} S_{cl}^{-1}\tilde{N}_K P_{21} \\ 0 \end{bmatrix}\right)(j\omega)\right) \\ &= \bar{\sigma}\left(\left(U^{\sim}\left(P_{11} + U\begin{bmatrix} S_{cl}^{-1}\tilde{N}_K P_{21} \\ 0 \end{bmatrix}\right)\right)(j\omega)\right) \\ &= \bar{\sigma}\left(\left(\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} + \begin{bmatrix} S_{cl}^{-1}\tilde{N}_K P_{21} \\ 0 \end{bmatrix}\right)(j\omega)\right) \\ &= \bar{\sigma}\left(\begin{bmatrix} S_{cl}^{-1} & 0 \\ 0 & V^{-1} \end{bmatrix}\begin{bmatrix} S_{cl}R_1 + \tilde{N}_K P_{21} \\ VR_2 \end{bmatrix}(j\omega)\right) \end{aligned} \tag{A7}$$

where $[R_1^T \quad R_2^T]^T := U^{\sim}P_{11}$, and V is a real-rational proper transfer function. By Lemma 4, conditions (ii),(iii) of Theorem 1 imply that

$$\bar{\sigma}\left(S_{cl}^{-1}(j\omega)\right) < \alpha^{-1}, \bar{\sigma}\left(V^{-1}(j\omega)\right) < \alpha^{-1} \tag{A8}$$

and hence

$$\bar{\sigma}\left(diag\left(S_{cl}^{-1}(j\omega), V^{-1}(j\omega)\right)\right) < \alpha^{-1} \ \ \forall \omega \in \Omega \tag{A9}$$

This together with condition (i) of Theorem 1 imply that $\bar{\sigma}(T_{zw}(j\omega)) < \gamma \ \forall \omega \in \Omega$. Finally, closed-loop stability follows from condition (iv) of Theorem 1. This completes the proof. □

Proof of Theorem 2

Proof: The proof proceeds by converting the conditions of Theorem 1 into matrix inequalities one by one. First, we apply Lemma 1(iii) to condition (i) of Theorem 1 with $(A,B,C,D) = (A_{Hcl}, B_{Hcl}, C_{Hcl}, D_{Hcl})$ where

$$\left[\begin{array}{c|c} A_{Hcl} & B_{Hcl} \\ \hline C_{Hcl} & D_{Hcl} \end{array}\right] := \left[\begin{array}{cc|c} A_H & 0 & B_{H1} \\ B_{KH}C_{H2} & A_{KH} & B_{KH}D_{H21} \\ \hline D_{H12}D_{KH}C_{H2} & D_{H12}C_{KH} & D_{H11} + D_{H12}D_{KH}D_{H21} \end{array}\right] \tag{A10}$$

which is a realization of $F_l(P_H, K_H)$, and the instrumental variables are in the following form:

$$\begin{aligned} P &= \begin{bmatrix} P_{h11} & P_{h12} \\ * & P_{h22} \end{bmatrix}, \ Q = \begin{bmatrix} Q_{h11} & Q_{h12} \\ * & Q_{h22} \end{bmatrix}, \\ G &= \begin{bmatrix} R_H & X_{H12} \\ X_{H21} & X_{H22} \end{bmatrix}, \ W = \begin{bmatrix} \bar{R}_H & \lambda X_{H12} \\ M_{H21} & M_{H22} \end{bmatrix} \end{aligned} \tag{A11}$$

where λ is a real scalar to be determined. Define notations

$$\begin{aligned} G^{-1} &= \begin{bmatrix} S_H & Y_{H12} \\ Y_{H21} & Y_{H22} \end{bmatrix}, W^{-1} = \begin{bmatrix} \bar{S}_H & \bar{Y}_{H12} \\ N_{H21} & N_{H22} \end{bmatrix}, T_{H1} = \begin{bmatrix} S_H & Y_{H12} \\ I & 0 \end{bmatrix}, \\ T_{H2} &= \begin{bmatrix} I & 0 \\ R_H & X_{H12} \end{bmatrix}, T_{H3} = \begin{bmatrix} \bar{S}_H & \bar{Y}_{H12} \\ I & 0 \end{bmatrix}, T_{H4} = \begin{bmatrix} I & 0 \\ \bar{R}_H & \lambda X_{H12} \end{bmatrix} \end{aligned} \tag{A12}$$

Then it is easy to verify that $T_{H1}G = T_{H2}$ and $T_{H3}W = T_{H4}$. Performing congruence transformation $diag(T_{H1}, T_{H3}, I, I)$ to the resulting matrix inequality yields

$$\begin{bmatrix} \Xi_{11} & -T_{H1}PT_{H3}^T - T_{H1}T_{H4}^T + T_{H2}A_{Hcl}T_{H3}^T & -T_{H2}B_{Hcl} & 0 \\ * & \varpi_l^2 T_{H3}QT_{H3}^T - He\{T_{H4}A_{Hcl}T_{H3}^T\} & T_{H4}B_{Hcl} & T_{H3}C_{Hcl}^T \\ * & * & -\alpha\gamma I & -D_{Hcl}^T \\ * & * & * & -\alpha\gamma I \end{bmatrix} < 0 \tag{A13}$$

where

$$\Xi_{11} = -T_{H1}QT_{H1}^T + He\{T_{H2}T_{H1}^T\},$$

$$T_{H2}T_{H1}^T = \begin{bmatrix} S_H^T & I \\ R_H S_H^T + X_{H12}Y_{H12}^T & R_H \end{bmatrix}, \ T_{H1}T_{H4}^T = \begin{bmatrix} S_H & S_H\bar{R}_H^T + \lambda Y_{H12}X_{H12}^T \\ I & \bar{R}_H^T \end{bmatrix},$$

$$T_{H2}B_{Hcl} = \begin{bmatrix} B_{H1} \\ R_H B_{H1} + X_{H12}B_{KH}D_{H21} \end{bmatrix}, T_{H4}B_{Hcl} = \begin{bmatrix} B_{H1} \\ \bar{R}_H B_{H1} + \lambda X_{H12}B_{KH}D_{H21} \end{bmatrix},$$

$$T_{H2}A_{Hcl}T_{H3}^T = \begin{bmatrix} A_H\bar{S}_H^T & A_H \\ \Xi_{A21} & R_H A_H + X_{H12}B_{KH}C_{H2} \end{bmatrix},$$

$$T_{H4}A_{Hcl}T_{H3}^T = \begin{bmatrix} A_H\bar{S}_H^T & A_H \\ \Xi_{B21} & \bar{R}_H A_H + \lambda X_{H12}B_{KH}C_{H2} \end{bmatrix},$$

$$T_{H3}C_{Hcl}^T = \begin{bmatrix} \bar{S}_H C_{H2}^T D_{KH}^T D_{H12}^T + \bar{Y}_{H12}C_{KH}^T D_{H12}^T \\ C_{H2}^T D_{KH}^T D_{H12}^T \end{bmatrix}.$$

with

$$\begin{aligned} \Xi_{A21} &= \left(R_H A_H + X_{H12}B_{KH}C_{H2}\right)\bar{S}_H^T + X_{H12}A_{KH}\bar{Y}_{H12}^T, \\ \Xi_{B21} &= \left(\bar{R}_H A_H + \lambda X_{H12}B_{KH}C_{H2}\right)\bar{S}_H^T + \lambda X_{H12}A_{KH}\bar{Y}_{H12}^T. \end{aligned}$$

Without loss of generality we may assume S_H and $\bar{S}_H$ are invertible. Performing the congruence transformation $diag(S_H^{-1}, I, \bar{S}_H^{-1}, I, I, I)$ to (A13) and assuming $S_H^{-1}Y_{H12} = \bar{S}_H^{-1}\bar{Y}_{H12}$ yield

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & \Pi_{14} & -S_H^{-1}B_{H1} & 0 \\ * & \Pi_{22} & \Pi_{23} & \Pi_{24} & \Pi_{25} & 0 \\ * & * & \Pi_{33} & \Pi_{34} & \bar{S}_H^{-1}B_{H1} & \Pi_{36} \\ * & * & * & \Pi_{44} & \Pi_{45} & C_{H2}^T D_{KH}^T D_{H12}^T \\ * & * & * & * & -\alpha\gamma I & -D_{H11}^T - D_{H21}^T D_{KH}^T D_{H12}^T \\ * & * & * & * & * & -\alpha\gamma I \end{bmatrix} < 0 \quad \text{(A14)}$$

where

$$\Pi_{11} = -Q_{H11} + He\left\{S_H^{-T}\right\},$$
$$\Pi_{12} = -Q_{H12} + R_H^T + S_H^{-1}Y_{H12}X_{H12}^T + S_H^{-1},$$
$$\Pi_{22} = -Q_{H22} + He\left\{R_H^T\right\},$$
$$\Pi_{13} = -P_{H11} - \bar{S}_H^{-T} + S_H^{-1}A_H,$$
$$\Pi_{23} = -P_{H12}^T - \bar{S}_H^{-T} + R_H A_H + X_{H12}B_{KH}C_{H2} + X_{H12}A_{KH}Y_{H12}^T S_H^{-T},$$
$$\Pi_{33} = \varpi_l^2 Q_{H11} - He\left\{\bar{S}_H^{-1}A_H\right\},$$
$$\Pi_{14} = -P_{H12} - \bar{R}_H^T - \lambda S_H^{-1}Y_{H12}X_{H12}^T + S_H^{-1}A_H,$$
$$\Pi_{24} = -P_{H22} - \bar{R}_H^T + R_H A_H + X_{H12}B_{KH}C_{H2},$$
$$\Pi_{34} = \varpi_l^2 Q_{H12} - \bar{S}_H^{-1}A_H - A_H^T\bar{R}_H^T - \lambda C_{H2}^T B_{KH}^T X_{H12}^T - \lambda S_H^{-1}Y_{H12}A_{KH}^T X_{H12}^T,$$
$$\Pi_{44} = \varpi_l^2 Q_{H22} - He\left\{\bar{R}_H A_H + \lambda X_{H12}B_{KH}C_{H2}\right\},$$
$$\Pi_{25} = -R_H B_{H1} - X_{H12}B_{KH}D_{H21}$$
$$\Pi_{45} = \bar{R}_H B_{H1} + \lambda X_{H12}B_{KH}D_{H21}$$
$$\Pi_{36} = C_{H2}^T D_{KH}^T D_{H12}^T + S_H^{-1}Y_{H12}C_{KH}^T D_{H12}^T.$$

and

$$\begin{bmatrix} Q_{H11} & Q_{H12} \\ * & Q_{H22} \end{bmatrix} := \begin{bmatrix} I & S_H^{-1}Y_{H12} \\ I & 0 \end{bmatrix} \begin{bmatrix} Q_{h11} & Q_{h12} \\ * & Q_{h22} \end{bmatrix} \begin{bmatrix} I & I \\ Y_{H12}^T S_H^{-1} & 0 \end{bmatrix}^T$$
$$\begin{bmatrix} P_{H11} & P_{H12} \\ * & P_{H22} \end{bmatrix} := \begin{bmatrix} I & S_H^{-1}Y_{H12} \\ I & 0 \end{bmatrix} \begin{bmatrix} P_{h11} & P_{h12} \\ * & P_{h22} \end{bmatrix} \begin{bmatrix} I & I \\ Y_{H12}^T S_H^{-1} & 0 \end{bmatrix}^T$$

Next, applying the following change of variables to (A14):

$$\begin{aligned} &\Sigma_{H1} := S_H^{-1},\ \bar{\Sigma}_{H1} := \bar{S}_H^{-1}, \Sigma_{H2} := X_{H12}Y_{H12}^T S_H^{-T}, \\ &Z_{H1} := X_{H12}A_{KH}Y_{H12}^T S_H^{-T}, Z_{H2} := X_{H12}B_{KH}, \\ &Z_{H3} := C_{KH}Y_{H12}^T S_H^{-T}, Z_{H4} := D_{KH}, X_{H12} = \Sigma_{H2}\Sigma_{H1}^{-T}Y_{H12}^{-T}. \end{aligned} \quad \text{(A15)}$$

which becomes the condition (25) of Theorem 2.

Once the condition (25) is satisfied, we need to show that there do exist instrumental variables that justify the corresponding analysis condition (Lemma 1). To this end, by the change of variables previously defined, we have

$$P = \begin{bmatrix} P_{h11} & P_{h12} \\ * & P_{h22} \end{bmatrix} = \begin{bmatrix} I & \Sigma_{H1}Y_{H12} \\ I & 0 \end{bmatrix}^{-1} \begin{bmatrix} P_{H11} & P_{H12} \\ * & P_{H22} \end{bmatrix} \begin{bmatrix} I & \Sigma_{H1}Y_{H12} \\ I & 0 \end{bmatrix}^{-T}$$
$$Q = \begin{bmatrix} Q_{h11} & Q_{h12} \\ * & Q_{h22} \end{bmatrix} = \begin{bmatrix} I & \Sigma_{H1}Y_{H12} \\ I & 0 \end{bmatrix}^{-1} \begin{bmatrix} Q_{H11} & Q_{H12} \\ * & Q_{H22} \end{bmatrix} \begin{bmatrix} I & \Sigma_{H1}Y_{H12} \\ I & 0 \end{bmatrix}^{-T} \quad \text{(A16)}$$
$$X_{H12} = \Sigma_{H2}\Sigma_{H1}^{-T}Y_{H12}^{-T}.$$

Choose Y_{H12} to be any nonsingular matrix, we immediately recover P,Q and X_{H12}. It follows from the identity $G^{-1}G = I$ that we can recover $X_{H21} = Y_{H12}^{-1}(I - \Sigma_{H1}^{-1}R_H)$, and $X_{H22} = -Y_{H12}^{-1}\Sigma_{H1}^{-1}X_{H12}$. Now we have recovered G. For recovering $W, \bar{Y}_{H12}$ is determined by the constraint $S_H^{-1}Y_{H12} = \bar{S}_H^{-1}\bar{Y}_{H12}$. The rest part of the recovering procedure is similar to that of G. Finally we can recover (A_{KH},B_{KH},C_{KH},D_{KH}) as follows

$$\begin{bmatrix} A_{KH} & B_{KH} \\ C_{KH} & D_{KH} \end{bmatrix} = \begin{bmatrix} X_{H12}^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} Z_{H1}\Sigma_{H2}^{-T} & Z_{H2} \\ Z_{H3}\Sigma_{H2}^{-T} & Z_{H4} \end{bmatrix} \begin{bmatrix} X_{H12} & 0 \\ 0 & I \end{bmatrix} \quad \text{(A17)}$$

Indeed, this is the particular form as described in (21). Furthermore,

$$\left[\begin{array}{cc|c} \tilde{M}_K & \tilde{N}_K & 0 \\ \hline 0 & 0 & V \end{array}\right] = C_{KH}\left(sI - A_{KH}\right)^{-1}B_{KH} + D_{KH} \quad \text{(A18)}$$
$$= Z_{H3}\Sigma_{H2}^{-1}\left(sI - Z_{H1}\Sigma_{H2}^{-1}\right)^{-1}Z_{H2} + Z_{H4}$$

With the constraints imposed on Z_{Hi} $(i = 1,2,3,4)$ and Σ_{H2} as mentioned in Theorem 2, we obtain

$$\begin{bmatrix} \tilde{M}_K & \tilde{N}_K \end{bmatrix} = Z_3\Sigma_2^{-1}\left(sI - Z_1\Sigma_2^{-1}\right)^{-1}Z_2 + Z_4 \quad \text{(A19)}$$

After a tedious algebraic manipulation, the controller formula (29)-(31) does lead to controllers of the prescribed structure. The rest of the conditions of Theorem 2 can be derived in the same manner. This part is omitted due to space limitation. □

VI. CONCLUSIONS

The main contribution of this work lies on extending the current results of centralized H-infinity control in finite frequency domain to the subject of decentralized control. A unified approach to the finite frequency H-infinity control problems via decentralized dynamic and static output feedback for continuous-time linear systems has been presented. Under some mild assumptions, frequency-domain solvability conditions were derived and state-space solutions in terms of LMIs were provided.
A comprehensive numerical example has established the efficacy of the proposed controller design methods and it numerically confirmed the equivalence between the proposed analysis condition (i.e., Lemma 1(iii)) and that of the well-known GKYP lemma (i.e., Lemma 1(ii)) for the case of strictly proper systems. Extensions to the other (semi)finite frequency ranges and discrete-time cases can be easily carried out in the same manner.

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