# Valuation of Guarantees Set Relative to Cross-Currency Stochastic Rates of Return

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**Abstract**

We derive the pricing formulas for guarantees whose guaranteed minimum rates of return are set relative to cross-currency stochastic rates of return, "GCSRs" for short, via a cross-currency framework. GCSRs are often embedded in contracts which include life and pension insurance policies, guaranteed investment contracts and index-linked bonds, etc. The valuation of such guarantees has not been investigated in previous literature regarding guarantees. Our research finds that valuing GCSRs via a single-currency framework which is adopted in previous research on guarantees causes a significant underestimation of GCSRs under both maturity and multi-period guarantee. The underestimation of multi-period guarantee is much more significant than that of maturity guarantee. As a result, the pricing formulas derived in our research are more suitable, tractable and feasible for practice than those in previous relevant literature.

Keywords：Stochastic, Rate of Return Guarantee, Cross-currency, Interest rate, LIBOR Market Model

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## 1 Introduction

A common way to reduce the financial risk in financial contracts is to embed the policies with minimum rate of return guarantees to bind the return from below. Such contracts include life and pension insurance policies, guaranteed investment contracts (GICs), cf. e.g., Walker (1992), and index-linked bonds, etc. Because these guarantees are often embedded in policies issued by insurance companies, investment banks or government, it is important that the issuers know the value of the policies they are selling. Since these guarantees may be surprisingly expensive, this may cause the issuers to charge too small premiums to put their financial stability at risk. Besides, there are requirements that insurance companies explicitly inform the customers about the economic value of the embedded guarantees in some countries. As a result, further analysis for pricing rate of return guarantees correctly is important and warranted.

There are a variety of guarantee designs in financial contracts embedded with guaranteed rate of return in practice. One class of these guarantees is so-called absolute guarantees, i.e., guarantees where the minimum rate of return is set to be deterministic. The other is so called relative guarantees in the literature (Lindset, 2004), i.e., guarantees where the minimum guaranteed rates of return are linked to a stochastic rate of return on an asset such as an index, a reference portfolio, a specific asset traded in financial markets, etc. The occurrence of relative guarantees is due to the problem of absolute guarantees. Granting a deterministic guaranteed rate results in a problem which is unable to attract contract participants by a low guaranteed rate. On the other hand, contract issuers bear financial burdens to attract contract participants with a high guaranteed rate.

Previous research on valuing guarantees for life insurance products or pension funds has focused on absolute guarantees, which provide participants to receive a constant or predetermined minimum rate of return. The existing literature which analyzed absolute guarantees under the assumption of deterministic interest rate included Brennan and Schwartz (1976), Boyle and Schwartz (1977), Boyle and Hardy (1997), and Grosen and Jorgensen (1997, 2000). Persson and Aase (1997) and Hansen and Miltersen (2002) employed the Vasicek (1977) interest rate model. Miltersen and Persson (1999), Lindset (2003), and Bakken, Lindset and Olson (2006) adopted the Heath-Jarrow-Morton framework (HJM, 1992).

Despite the popularity of relative rate of return guarantees, especially those issued in

Latin America, the research on them is significantly less in number than absolute guarantees. Only few articles were written on the relative rate of return guarantees. Ekern and Persson (1996) investigated unit-linked life insurance contracts with different types of relative guarantees. Pennacchi (1999) valued both the absolute and the relative guarantee provided for Chilean and Uruguayan pension plans by using a contingent claim analysis. Both papers assume that interest rate is deterministic. However, Lindset (2004) analyzed several kinds of minimum guaranteed rates of return within the HJM framework. The guaranteed rate of return examined in these three papers is set relative to the rates of return on equity-market assets. Besides, Yang, Yueh and Tang (2008) studied rate of return guarantees for pension funds linked relative to a return measured by market realized δ-year spot rates with the HJM framework. However, one problem of their result is available only for a special case with a limited guarantee period under multi-period guarantee. Hsieh and Chen (2010) analyzed guarantees whose guaranteed rate of return is set relative to a stochastic interest rate under the LIBOR market model (LMM). They derived general pricing formulas for guarantees with an arbitrary guarantee period and solved the limitations of Yang et al. (2008).

In practice, it is common for that the underlying asset which provides the rate of return for the contract and the guaranteed rates of return are denominated in cross-currency, (GCSR denotes a minimum rate of return guarantee whose guaranteed rate of return is set relative to a cross-currency stochastic rate of return, hereafter.). This situation can be always observed in unit-linked products. However, the all previous literature regarding guarantees assumed that both the underlying assets and the guaranteed rate in contracts are denominated in a single-currency. This assumption is not consistent with the real economic environment and leads to that those pricing formulas are not suitable for valuing GCSRs since the "quanto-effect" which has been discussed in the finance literature is not considered. Amin and Jarrow (1991), Schlogl (2002), Musiela and Rutkowski (2005), and Wu and Chen (2007) show that the "quanto-effect" affects the pricing results and should be considered in valuing cross-currency financial products. [1]

[1] Examples of such contracts can be observed in pension plans and unit-linked life insurance contracts. The countries which provide pension plans with a stochastic guaranteed rate include Chile, Colombia, Peru and Argentina (see e.g., Pennacchi, 1999; Lindset, 2004). Ekern and Persson (1996) analyze a number of unit-linked contracts with stochastic guaranteed rates. Exhibit 5 in Appendix C shows the statistics regarding the unit-linked products provided by the European Insurance and Reinsurance Federation (CEA). From the statistics, the European life insurance market in 2010 was characterized by a significant rise and percentage in the share of unit-linked contracts in total life premium.

This research attempts to derive the general pricing formulas for GCSRs embedded in financial contracts. The guaranteed minimum rate of return is set relative to a cross-currency stochastic rate of return. This issue is important and has not been investigated in previous research on guarantees.

Our article has several contributions to the literature on relative guarantees, particularly in the presence of an open cross-currency economic environment and stochastic interest rates.

First, we derive the general pricing formulas of GCSRs. Our pricing formulas consider the "quanto-effect" and hence are consistent with the real economic environment. The pricing formulas of GCSRs in this research will be more general and suitable for pricing guarantees in a real cross-currency environment. If the model setting degenerates to the single-currency case, the pricing formulas of GCSRs become the pricing formulas of the single-currency guaranteed contract.

Second, our research finds that valuing GCSRs via a framework which is used in previous research regarding guarantees and does not consider the effect of exchange rate (we call this framework a single-currency framework hereafter) causes a significant underestimation of GCSRs. The underestimation may lead issuers to charge too small premiums to suffer financial distress.[2] The underestimation can be avoided by using our pricing formulas.

Third, the derived formulas can be applied to value GCSRs under both multi-period and maturity guarantees with an arbitrary guarantee period. The lack of general formulas which can be applied to an arbitrary guarantee period due to utilizing other interest rate models is solved by adopting LMM to describe the behavior of interest rate. Rate-of-return guarantees are embedded in these two fundamentally different types. The contract period of multi-period guarantees is divided into several subperiods. A binding guarantee is specified for each subperiod. Many life insurance contracts and guaranteed investment contracts (GIC) sold by investment banks, cf. e.g., Walker (1992), are examples of multi-period guarantees. In contrast, maturity guarantees are binding only at contract expiration. There are some extra bonuses of adopting the LMM. One is that the quotes of interest rates are consistent with market conventions and thus make the pricing formulas more tractable and feasible for practitioners. The other is that the problems exhibited in the other interest rate models, such as the Vasicek model, the Cox, Ingersoll

[2] More details about the results of numerical analysis are represented in Subsection 5.2.

and Ross (CIR) model, and the HJM model, are overcome.[3]

Finally, using our pricing formulas to value GCSRs will be more efficient than adopting time-consuming simulation, especially for those insurance and pension policies with a long duration.

This article is structured as follows. Section 2 describes the financial plans embedded with guarantees and the structure of each guarantee. Section 3 represents the economic environment and the dynamics of assets for pricing. In Section 4, the pricing formulas of each guarantee are derived. Section 5 represents the examination of accuracy of the pricing formulas via Monte Carlo simulation and shows some numerical analysis. In Section 6, the results of this paper are concluded with a brief summary.

## 2 Financial Plans Embedded with Guarantees and Guarantee Structure

We describe financial plans (such as insurance or pension policies) embedded with GCSRs under maturity and multi-period guarantees and represent each type of guarantees. In addition, the guaranteed rates of return of plans are set relative to cross-currency stochastic rates of return.

Assume that $T_0, T_1, ..., T_N \in [0, \tau]$ with $0 \le t \le T_0 \le T_1 ... \le T_N \le \tau$. In accordance with practice, we define $\delta = T_i - T_{i-1},\ i = 1, 2, ..., N$ and $T_0 - t \le \delta$. An investor contributes a notional principle to the financial plan in each period. We list the notations with "d" for domestic and "f" for foreign as follows.

### 2.1 Financial Plans Embedded with GCSRs under Maturity Guarantees (First-Type Guarantees)

A participator of financial plans contributes principals to the plan at time $T_0, T_1, ..., T_{N-1}$. At maturity, the participator receives the terminal payout of a financial plan embedded with GCSRs under maturity guarantees $FP_I(T_N)$, ie:

$$FP_I(T_N) = \sum_{n=1}^{N} \left\{ P_{d,n} \cdot \max\left[ \prod_{i=n-1}^{N-1} \frac{S_f^*(T_{i+1})}{S_f^*(T_i)} \quad , \quad \prod_{i=n-1}^{N-1} \left(1 + \delta L_d^{\delta}(T_i, T_i)\right) \right] \right\} \tag{2.1.1}$$

where

$P_{d,n}$ = *the principal which the investor contributes to the plan at time* $T_n$ *denominated in units of domestic currency.*

[3] For the purpose of brevity, these problems are specified in Section 3.

$S_f^*(\eta)$ = $S_f(\eta)\cdot X(\eta)$ ( $S_f^*(\eta)$ *is used for the simplicity of presentation, hereafter* ).

$S_f(\eta)$ = *the underlying foreign asset price at time* $\eta$ *denominated in units of foreign currency* ( $S_{f,\eta}^* = S_f^*(\eta)$ ).

$X(\eta)$ = *the exchange rate at time* $\eta$ *expressed as the domestic currency value of one unit of foreign currency.*

$\frac{S_f^*(T_{i+1})}{S_f^*(T_i)}$ = *the actual rate of return on the underlying in period* $(T_{i+1} - T_i) = \delta$.

$L_d^\delta(T_i, T_i)$ = *the domestic* $T_i$*-matured LIBOR rates with a compounding period* $\delta$.

The maturity guarantee is binding only at the contract expiration. The plan provides the investor a minimum interest rate guarantee on the principal paid into the contract. Note that the guaranteed rate of first-type guarantees is set relative to a cross-currency stochastic LIBOR interest rate. The underlying asset and the guaranteed rate are denominated in cross currency, which is common in unit-linked contracts. However, there is no research on the relative guarantee to deal with this issue.

The payout to the participator in $T_N$, $FP_I(T_N)$, can be written as the value of a pure financial plan without guarantees, $U(T_N)$, plus the value of the first-type guarantee, $G_I(T_N)$, such that

$$FP_I(T_N) = \underbrace{\sum_{n=1}^{N}\left\{P_{d,n}\prod_{i=n-1}^{N-1}\frac{S_f^*(T_{i+1})}{S_f^*(T_i)}\right\}}_{U(T_N)} + \underbrace{\sum_{n=1}^{N}\left\{P_{d,n}\max\left(\prod_{i=n-1}^{N-1}\left(1+\delta L_d^\delta(T_i,T_i)\right) - \prod_{i=n-1}^{N-1}\frac{S_f^*(T_{i+1})}{S_f^*(T_i)}\ ,\ 0\right)\right\}}_{G_I(T_N)} \tag{2.1.2}$$

$$U_I(T_N) = \sum_{n=1}^{N}\left\{P_{d,n}\prod_{i=n-1}^{N-1}\frac{S_f^*(T_{i+1})}{S_f^*(T_i)}\right\} \tag{2.1.3}$$

$$G_I(T_N) = \sum_{n=1}^{N}\left\{P_{d,n}\underbrace{\max\left(\prod_{i=n-1}^{N-1}\left(1+\delta L_d^\delta(T_i,T_i)\right) - \prod_{i=n-1}^{N-1}\frac{S_f^*(T_{i+1})}{S_f^*(T_i)}\ ,\ 0\right)}_{\Upsilon_I^{(n)}(T_N)}\right\} \tag{2.1.4}$$

$$\Upsilon_I^{(n)}(T_N) = \max\left(\prod_{i=n-1}^{N-1}\left(1+\delta L_d^\delta(T_i,T_i)\right) - \prod_{i=n-1}^{N-1}\frac{S_f^*(T_{i+1})}{S_f^*(T_i)}\ ,\ 0\right) \tag{2.1.5}$$

$\Upsilon_I^{(n)}(T_N)$ is defined as the time $T_N$ value of the first-type guarantee for one dollar contributed in the n-th period. Actually, the cash flow of $\Upsilon_I^{(n)}(T_N)$ is a type of options. The amount of principal only serves as a scalar of the actual payout. As a result, $G_I(T_N)$ is the sum of the option value of each period and the potential costs for issuers at the expiration date.

## 2.2 Financial Plans Embedded with GCSRs under Multi-period Guarantees (Second-Type Guarantees)

A financial plan embedded with GCSRs under multi-period guarantees provides the terminal payout, $FP_{II}(T_N)$, to the participator at maturity, ie:

$$FP_{II}(T_N)=\sum_{n=1}^{N}\left\{P_{d,n}\prod_{i=n-1}^{N-1}\max\left(\frac{S_f^*(T_{i+1})}{S_f^*(T_i)}\ ,\ \left(1+\delta L_d^{\delta}(T_i,T_i)\right)\right)\right\} \tag{2.1.6}$$

The contract period is divided into several subperiods for multi-period guarantees. The contract specifies a binding guarantee for each subperiod.

The payout to the participator in $T_N$, $FP_{II}(T_N)$, can be also written as the value of a pure financial plan without guarantees, $U(T_N)$, plus the value of the second-type guarantee, $G_{II}(T_N)$, such that

$$\begin{aligned}FP_{II}(T_N)=&\underbrace{\sum_{n=1}^{N}\left\{P_{d,n}\prod_{i=n-1}^{N-1}\frac{S_f^*(T_{i+1})}{S_f^*(T_i)}\right\}}_{U(T_N)}\\&+\underbrace{\sum_{n=1}^{N}\left\{P_{d,n}\left[\left(\prod_{i=n-1}^{N-1}\max\left(\frac{S_f^*(T_{i+1})}{S_f^*(T_i)},\left(1+\delta L_d^{\delta}(T_i,T_i)\right)\right)-\prod_{i=n-1}^{N-1}\frac{S_f^*(T_{i+1})}{S_f^*(T_i)}\right)\right]\right\}}_{G_{II}(T_N)}\end{aligned} \tag{2.1.7}$$

$$G_{II}(T_N)=\sum_{n=1}^{N}\left\{P_{d,n}\underbrace{\left[\left(\prod_{i=n-1}^{N-1}\max\left(\frac{S_f^*(T_{i+1})}{S_f^*(T_i)},\left(1+\delta L_d^{\delta}(T_i,T_i)\right)\right)-\prod_{i=n-1}^{N-1}\frac{S_f^*(T_{i+1})}{S_f^*(T_i)}\right)\right]}_{\Upsilon_{II}^{(n)}(T_N)}\right\} \tag{2.1.8}$$

$$\Upsilon_{II}^{(n)}(T_N)=\left[\left(\prod_{i=n-1}^{N-1}\max\left(\frac{S_f^*(T_{i+1})}{S_f^*(T_i)},\left(1+\delta L_d^{\delta}(T_i,T_i)\right)\right)-\prod_{i=n-1}^{N-1}\frac{S_f^*(T_{i+1})}{S_f^*(T_i)}\right)\right] \tag{2.1.9}$$

where $\Upsilon_{II}^{(n)}(T_N)$ is defined as the time $T_N$ value of the second-type guarantee for one dollar contributed in the n-th period.

The major difference between the first-type and the second-type guarantee is that a maturity guarantee is binding only at the contract expiration, while the contract period of a multi-period guarantee is divided into several subperiods, where a binding guarantee is specified for each subperiod.

## 3 Economic Model

From the payoff structure of GCSRs, the pricing model should include the dynamics of the foreign equity-type asset, the exchange rate and the domestic interest rates. These dynamics will be adopted to develop the arbitrage-free pricing formulas of GCSRs. Assume that trading takes place on a continuous basis in the time interval $[0,\tau]$, for some fixed horizon $0<\tau<\infty$. The uncertainty is described by the filtered probability space $\left(\Omega, F, Q, \{F_t\}_{t\in[0,\tau]}\right)$. The filtration $\{F_t\}_{t\in[0,\tau]}$ is the $Q$-augmentation of the filtration generated by independent standard Brownian motions $W(t)=\left(W_1(t),W_2(t),...,W_D(t)\right)$. $Q$ represents the spot martingale probability measure. The filtration $\{F_t\}_{t\in[0,\tau]}$ denotes the flow of information accruing to all the agents in the economy. The dynamics of assets under the martingale measure Q are given as follows (see Harrison and Kreps (1979), Amin and Jarrow (1991), Schlogl (2002), Musiela and Rutkowski (2005) and Wu and Chen (2007) for more details).

### The Dynamics of the Exchange Rate

The dynamics of the spot exchange rate $X(t)$ is assumed to have a lognormal volatility structure and its stochastic process under the martingale measure Q is given by

$$\frac{dX(t)}{X(t)}=\left[r_d(t)-r_f(t)\right]dt+\sigma_X(t)\cdot dW(t), \tag{3.1.1}$$

where $\sigma_X(t)$ is a deterministic volatility vector function of an exchange rate satisfying the standard regularity conditions and $r_k(t), k\in\{d,f\}$ is the *k*th country's risk-free short rate at time *t*.

### The Dynamics of the Foreign Equity-Type Asset

The dynamics of the foreign reference equity-type asset $S_f(t)$ under the martingale measure Q is given by

$$\frac{dS_f(t)}{S_f(t)} = \left[r_f(t) - \sigma_X(t)\cdot\sigma_{Sf}(t)\right]dt + \sigma_{Sf}(t)\cdot dW(t), \quad (3.1.2)$$

where $\sigma_{Sf}(t)$ is a deterministic volatility vector function satisfying the standard regularity conditions.

**The Dynamics of Interest Rates: LIBOR Market Model (LMM)**

We adopt the LMM to describe the behavior of domestic interest rate. The LMM has been developed by Brace, Gatarek and Musiela (1997, BGM). The processes of domestic LIBOR rates under the LMM are briefly expressed as follow. [4] The notations are given below:

$f_d(t,T)$ = the domestic forward interest rate contracted at time $t$ for instantaneous borrowing and lending at time $T$ with $0 \le t \le T \le \tau$.

$P_d(t,T)$ = $\exp\left(-\int_t^T f_d(t,u)\,du\right)$, the time $t$ price of a domestic zero coupon bond (ZCB) paying one dollar at time $T$.

$r_d(t)$ = $f_d(t,t)$, the domestic risk-free short rate at time $t$.

$\beta_d(t)$ = $\exp\left[\int_0^t r_d(u)\,du\right]$, the domestic money market account at time $t$ with an initial value $\beta_d(0)=1$.

For some $\delta > 0,\ T \in [0,\tau]$, define the forward LIBOR rate process $\{L_d^\delta(t,T); 0 \le t \le T\}$ as given by

$$1+\delta L_d^\delta(t,T) = P_d(t,T)/P_d(t,T+\delta) = \exp\left(\int_T^{T+\delta} f_d(t,u)\,du\right)$$

The dynamics of the LIBOR rates, the ZCB price and the reference investment portfolio under the spot martingale measure Q are given as follows:

$$dL_d^\delta(t,T) = L_d^\delta(t,T)\gamma(t,T)\cdot\sigma_{Pd}(t,T+\delta) + L_d^\delta(t,T)\gamma_d(t,T)\cdot dW(t) \quad (3.1.3)$$

$$dP_d(t,T)/P_d(t,T) = r_d(t)dt - \sigma_{Pd}(t,T)\cdot dW(t) \quad (3.1.4)$$

where $\gamma_d(\cdot,T):[0,T]\to R^D$ is a deterministic, bounded and piecewise continuous volatility function and $\sigma_{Pd}(t,T)$ is defined as (3.1.5).

[4] A further description regarding the LMM can be found in advanced textbooks in finance, see, e.g. Svoboda (2004) and Musiela and Rutkowski (2005).

The bond volatility $\sigma_{Pd}(t,T)$ must be specified to fit the arbitrage-free condition in HJM and is given as follows:

$$\sigma_{Pd}(t,T)=\begin{cases}\displaystyle\sum_{k=1}^{\left[\delta^{-1}(T-t)\right]}\frac{\delta L_d^{\delta}(t,T-k\delta)}{1+L_d^{\delta}(t,T-k\delta)}\gamma_d(t,T-k\delta) & t\in[0,T-\delta]\ \&\ T-\delta>0\\ 0 & \textit{otherwise.}\end{cases}\tag{3.1.5}$$

where $\left[\delta^{-1}(T-t)\right]$ denotes the greatest integer that is less than $\delta^{-1}(T-t)$.

According to the bond volatility process (3.1.5), $\{\sigma_P(t,T+\delta)\}_{t\in[0,T+\delta]}$ is stochastic rather than deterministic. To solve equation (3.1.3) for the distribution of $L_d^{\delta}(T,T)$, BGM (1997) approximated $\sigma_{Pd}(t,T)$ by $\overline{\sigma}_{Pd}(t,T)$ at any fixed initial time *s*, and given by

$$\overline{\sigma}_{Pd}(t,T)=\begin{cases}\displaystyle\sum_{k=1}^{\left[\delta^{-1}(T-t)\right]}\frac{\delta L_d^{\delta}(s,T-k\delta)}{1+\delta L_d^{\delta}(s,T-k\delta)}\gamma_d(t,T-k\delta) & t\in[0,T-\delta]\ \&\ T-\delta>0,\\ 0 & \textit{otherwise}\end{cases}\tag{3.1.6}$$

where $0\le s\le t\le T\le\tau$. Hence, the calendar time of the process $\{L_d^{\delta}(t,T)\}_{t\in[s,T]}$ in (3.1.6) is frozen at its initial time $s$ and the process $\{\overline{\sigma}_{Pd}(t,T)\}_{t\in[s,T]}$ becomes deterministic. By substituting $\overline{\sigma}_{Pd}(t,T+\delta)$ for $\sigma_{Pd}(t,T+\delta)$ into the drift term of (3.1.3), the drift and the volatility terms become deterministic, so we can solve (3.1.3) and find the approximate distribution of $L_d^{\delta}(T,T)$ to be lognormal. This Wiener chaos order 0 approximation used in (3.1.6) is first utilized by BGM (1997) for pricing interest rate swaptions, developed further in Brace, Dun and Barton (1998) and formalized by Brace and Womersley (2000). It also appeared in Schlogl (2002) and Wu and Chen (2007).

There are some extra bonuses of adopting the LMM. One is that the quotes of interest rates are consistent with market conventions and thus make the pricing formulas more tractable and feasible for practitioners. The other is that the problems exhibited in the other interest rate models, such as the Vasicek model, the Cox, Ingersoll and Ross (CIR) model, and the HJM model, are overcome. These problems include: (a) the instantaneous short rate or the instantaneous forward rate is abstract, market-unobservable and continuously compounded. So it is complicated and difficult to recover model parameters from market-observed data; (b) the pricing formulas of extensively traded interest rate derivatives, such as caps, floors, swaptions, etc., based on the short rate models or the Gaussian HJM model are not consistent with market practice. This leads to some difficulties in parameter calibration; (c) as examined in Rogers (1996), the rates under Gaussian term structure models can become negative with a positive probability, which may cause pricing errors.

# 4 Valuation of Guarantees

In this section, two variants of the guarantees are priced based on the model framework above and by adopting martingale pricing method.[5]

## 4.1 Valuation of the First-Type Guarantee (Maturity Guarantee)

The pricing formulas of the first-type guarantee with the final payoff as specified in equation (2.1.4) and (2.1.5) are given as follows, and the proof is provided in Appendix A.

$$G_I(t) = \sum_{n=1}^{N} P_{d,n} \Upsilon_I^{(n)}(t) \tag{4.1.1}$$

$$\Upsilon_I^{(n)}(t) = \mathrm{H}_{1,t}^{(n)} N\left(d_1^{(n)}\right) - \mathrm{H}_{2,t}^{(n)} N\left(d_2^{(n)}\right) \tag{4.1.2}$$

*where*

$$\mathrm{H}_{1,t}^{(n)} = P_d(t, T_{n-1}) \cdot \exp\left(\Lambda_{(n)}\right), \quad \mathrm{H}_{2,t}^{(n)} = P_d(t, T_{n-1}),$$

$$P_d(t, T_{n-1}) = \left[1 + (T_0 - t) L_d(t, T_0)\right]^{-1} \left[\prod_{k=0}^{n-2} \left(1 + \delta L_d^{\delta}(t, T_k)\right)\right]^{-1},$$

$$\Lambda_{(n)} = \sum_{i=n-1}^{N-1} \int_t^{T_i} \lambda_B^i(u) \cdot \left(\lambda_B^i(u) - \lambda_A^i(u)\right) du$$
$$+ \sum_{i=n-1}^{(N-1)-1} \sum_{j=i+1}^{N-1} \int_t^{T_i} \left(\lambda_A^i(u) - \lambda_B^i(u)\right) \cdot \left(\lambda_A^j(u) - \lambda_B^j(u)\right) du,$$

$$\lambda_A^i(t) = \left[-\overline{\sigma}_{P_d}(t, T_i) + \overline{\sigma}_{P_d}(t, T_N)\right], \quad \lambda_B^i(t) = \left[-\overline{\sigma}_{P_d}(t, T_{i+1}) + \overline{\sigma}_{P_d}(t, T_N)\right],$$

$$d_1^{(n)} = \frac{\Lambda_{(n)} + \frac{1}{2} V_{(n)}^2}{V_{(n)}}, \quad d_2^{(n)} = d_1^{(n)} - V_{(n)}, \quad V_{(n)} = \sqrt{V_{(n)}^2},$$

[5] Details regarding the martingale pricing method can be seen in advanced textbooks in finance, see, e.g. Shreve (2004) and Musiela and Rutkowski (2005).

$N(\cdot)=$ the cumulative normal probability,

$$V_{(n)}^2=Var\left(X_{(n)}\right)+Var\left(Y_{(n)}\right)-2Cov\left(X_{(n)},Y_{(n)}\right),$$

$$Var\left(X_{(n)}\right)=\sum_{i=n-1}^{N-1}\int_t^{T_i}\left\|\lambda_A^i(u)-\lambda_B^i(u)\right\|^2du+2\sum_{i=n-1}^{(N-1)-1}\sum_{j=i+1}^{N-1}\int_t^{T_i}\left(\lambda_A^i(u)-\lambda_B^i(u)\right)\cdot\left(\lambda_A^j(u)-\lambda_B^j(u)\right)du,$$

$$Var\left(Y_{(n)}\right)=\int_t^{T_{n-1}}\left\|\lambda_C^{n-1}(u)\right\|^2du+\int_{T_{n-1}}^{T_N}\left\|\lambda_E(u)\right\|^2du,$$

$$\lambda_C^{n-1}(t)=\left[-\overline{\sigma}_{P_d}(t,T_{n-1})+\overline{\sigma}_{P_d}(t,T_N)\right],\qquad \lambda_D(t)=\left[\sigma_{Sf}(t)+\sigma_X(t)+\overline{\sigma}_{P_d}(t,T_N)\right],$$

$$Cov\left(X_{(n)},Y_{(n)}\right)=\sum_{i=n-1}^{N-1}\int_t^{T_{n-1}}\left(\lambda_A^i(u)-\lambda_B^i(u)\right)\cdot\lambda_C^{n-1}(u)du+\sum_{i=n-1}^{N-1}\int_{T_{n-1}}^{T_i}\left(\lambda_A^i(u)-\lambda_B^i(u)\right)\cdot\lambda_E(u)du$$

$\overline{\sigma}_{P_d}(t,\cdot)$ is defined as (3.1.6). $L_d(t,T_0)$ is the simply-compounded spot interest rate prevailing at time t for the maturity $T_0$ and $P_d(t,T_0)=\left[1+(T_0-t)L_d(t,T_0)\right]^{-1}$.

By observation pricing equation, the effect of exchange rate is considered and characterized by $\sigma_X(t)$ in $\lambda_D(t)$, and the inappropriate estimation due to using the pricing formula in previous literature is avoided . The extra bonus of adopting the LMM model is that all the parameters in (4.1.1) and (4.1.2) can be easily obtained from market quotes, thus making the pricing formula more tractable and feasible for practitioners.

**4.2 Valuation of the Second-Type Guarantee (Multi-Period Guarantee)**

The pricing formulas of the second-type guarantees with the final payoff as specified in equation (2.1.8) and (2.1.9) are derived as follows and the proof is provided in Appendix B.

$$G_{II}(t)=\sum_{n=1}^{N}P_{d,n}\Upsilon_{II}^{(n)}(t) \tag{4.1.3}$$

$$\Upsilon_{II}^{(n)}(t)=P_d(t,T_{n-1})\left[\prod_{i=n-1}^{N-1}2N\left(d_3^i\right)-1\right] \tag{4.1.4}$$

*where*

$$d_3^i=\frac{V_i^2}{2},\quad V_i^2=\int_{T_i}^{T_{i+1}}\left\|\lambda_D^i(u)\right\|^2du,\quad \lambda_D^i(t)=\left[\sigma_{S_f}(t)+\sigma_X(t)+\overline{\sigma}_{P_d}(t,T_{i+1})\right].$$

Again, the effect of exchange rate is reflected by $\sigma_X(t)$ in $\lambda_D^i(t)$. Moreover, our formulas can be applied to any arbitrary guarantee period $\delta$, which solves the limitation of previous literature.

## 5 Numerical Analysis

### 5.1 Examination by Monte Carlo simulation and Numerical Analysis

Some practical examples are given to examine the accuracy of the pricing formulas derived in the previous section by comparing the results with Monte Carlo simulation. Based on actual 2-year market data,[6] two types of guarantees with different guarantee periods ($\delta$=1 year and $\delta$=0.5 year) are priced at the date, 2011/12/31, and the results are listed in Exhibit 1 and 2. The simulation is based on 50,000 sample paths. The FTSE index is used to replace the underlying foreign asset for the numerical purpose. The LIBOR rates in US are used to be domestic interest rates. The exchange rate is expressed as the US dollar value of one unit of pound. To ease the comparison and analysis, the principal which the investor contributes to the plan at each period is assumed to be \$1 in the case of $\delta$=1 and \$0.5 in the case of $\delta$=0.5.

Several notable points are yielded by observing the numerical results. First, the pricing formulas have been shown to be accurate and robust in comparison with Monte Carlo simulation for the recent market data. Second, Exhibit 2 shows that our formulas can be applied for arbitrary values of $\delta$ (other than $\delta$=1). The formula of Yang et al. (2008) is available only for the special case where the interest rate guarantee is linked to the one-year spot rate, i.e. $\delta$=1.

Third, the second-type guarantee is more expensive than the first-type guarantee in both cases of $\delta$=1 and $\delta$=0.5. With a longer maturity date, the cost difference is getting more and more significant. Because the effect of higher guaranteed rates in some periods can be alleviated by lower guaranteed rates in other periods for the first-type guarantee. Such alleviation does not work for the second-type guarantee.

Finally, using the derived formulas is more efficient than adopting time-consuming simulation for those guarantees with long duration.

[6] All the market data are drawn and computed from the DataStream database and are available upon request from the authors.

Exhibit 1. The Price of Two Types of Guarantees for $\delta$=1 Year

| | First-Type Guarantee $G_1(t)$ | | | Second-Type Guarantee $G_2(t)$ | | |
|---|---|---|---|---|---|---|
| Maturity Date $T_N$ | CS (A) | MC (B) | PD (C) | CS (D) | MC (E) | PD (F) |
| 5 | 0.7582 | 0.7581 | 0.0203% | 1.4078 | 1.4081 | 0.0242% |
| 10 | 2.4081 | 2.4076 | 0.0188% | 7.7672 | 7.7637 | 0.0449% |
| 15 | 4.6337 | 4.6327 | 0.0232% | 22.6605 | 22.6725 | 0.0531% |
| 20 | 7.1947 | 7.1950 | 0.0042% | 52.6054 | 52.6039 | 0.0029% |
| 25 | 9.9181 | 9.9214 | 0.0333% | 109.4984 | 109.5933 | 0.0866% |
| 30 | 12.7123 | 12.6997 | 0.0993% | 214.9352 | 215.0796 | 0.0672% |

*CS and MC represent, respectively, the results of the formula and Monte Carlo simulations.*
*PD denotes the percentage difference which is equal to* ｜CS-MC｜÷ [(CS+MC)÷2].

Exhibit 2. The Prices of Two Types of Guarantees for $\delta$=0.5 Year

| | First-Type Guarantee $G_1(t)$ | | | Second-Type Guarantee $G_2(t)$ | | |
|---|---|---|---|---|---|---|
| Maturity Date $T_N$ | CS (A) | MC (B) | PD (C) | CS (D) | MC (E) | PD (F) |
| 5 | 0.8664 | 0.8663 | 0.0177% | 2.6219 | 2.6216 | 0.0128% |
| 10 | 2.6042 | 2.6055 | 0.0491% | 15.5430 | 15.5451 | 0.0136% |
| 15 | 4.9059 | 4.9076 | 0.0340% | 52.4808 | 52.4566 | 0.0462% |
| 20 | 7.5231 | 7.5237 | 0.0068% | 146.4593 | 146.5345 | 0.0513% |
| 25 | 10.2892 | 10.2845 | 0.0459% | 377.0310 | 376.7209 | 0.0823% |
| 30 | 13.1166 | 13.1249 | 0.0632% | 935.5293 | 934.7146 | 0.0871% |

### 5.2 Comparison: Cross-Currency vs. Single-Currency

In this subsection, we show what the results are if pricing GCSRs is completed within a single-currency framework under which the effect of exchange rate is not considered. Such framework is adopted in previous research regarding guarantees. The results are compared with those which are valued via a cross-currency framework aforementioned.

In the case of $\delta$=1 year, exhibit 3 shows that the prices of guarantees are underestimated about 11.23%~20.21% for first-type guarantee and about 22.67%~44.96% for second-type guarantee. In the case of $\delta$=0.5 year, the prices of guarantees are undervalued about 12.48%~22.07% for first-type guarantee and about 27.20%~62.23% for second-type guarantee as listed in exhibit 4. The percentage of underestimation is getting larger with a shorter maturity for first-type guarantee but with a longer maturity for second-type guarantee. Moreover, the percentage of underestimation for second-type guarantee is much bigger than that for first-type guarantee. From the above analysis, we know that issuers may charge too small premiums to put their financial stability at risk if a single-currency framework is used to price GCSRs.

Exhibit 3. The Percentage of Underestimation of Two Types of Guarantees for $\delta$=1 Year

| | First-Type Guarantee $G_1(t)$ | | | Second-Type Guarantee $G_2(t)$ | | |
|---|---|---|---|---|---|---|
| Maturity Date $T_N$ | CS (A) | CSNO (B) | PU (C) | CS (D) | CSNO (E) | PU (F) |
| 5 | 0.7582 | 0.6050 | -20.2109% | 1.4078 | 1.0886 | -22.6686% |
| 10 | 2.4081 | 1.9832 | -17.6422% | 7.7672 | 5.7121 | -26.4589% |
| 15 | 4.6337 | 3.9309 | -15.1671% | 22.6605 | 15.7048 | -30.6953% |
| 20 | 7.1947 | 6.2283 | -13.4330% | 52.6054 | 34.0477 | -35.2772% |
| 25 | 9.9181 | 8.7087 | -12.1933% | 109.4984 | 65.6163 | -40.0756% |
| 30 | 12.7123 | 11.2843 | -11.2333% | 214.9352 | 118.3089 | -44.9560% |

*CS represents the results of the formula under a cross-currency framework.*

*CSNO represents the results of the formula under a single-currency framework.*

*PU denotes the percentage of underestimation which is equal to* (CSNO-CS)÷CS.

Exhibit 4. The Percentage of Underestimation of Two Types of Guarantees for $\delta$=0.5 Year

| | First-Type Guarantee $G_1(t)$ | | | Second-Type Guarantee $G_2(t)$ | | |
|---|---|---|---|---|---|---|
| Maturity Date $T_N$ | CS (A) | CSNO (B) | PU (C) | CS (D) | CSNO (E) | PU (F) |
| 5 | 0.8664 | 0.6752 | -22.0681% | 2.6219 | 1.9088 | -27.1968% |
| 10 | 2.6042 | 2.1037 | -19.2178% | 15.5430 | 10.3166 | -33.6258% |
| 15 | 4.9059 | 4.0893 | -16.6454% | 52.4808 | 31.0874 | -40.7643% |
| 20 | 7.5231 | 6.4075 | -14.8291% | 146.4593 | 75.8836 | -48.1879% |
| 25 | 10.2892 | 8.8989 | -13.5119% | 377.0310 | 167.9415 | -55.4568% |
| 30 | 13.1166 | 11.4798 | -12.4789% | 935.5293 | 353.3952 | -62.2251% |

## 5 Conclusions

Two different types of GCSRs have been developed via a risk-neutral valuation method. The guaranteed rates of return embedded in financial plans are set relative to a cross-currency stochastic rate of return. The derived pricing formulas reflect the effect of exchange rate and are more consistent with market practice than those given in the previous researches. The formulas of GCSRs under maturity and multi-period guarantees can be applied to any arbitrary guarantee period $\delta$. Pricing GCSRs with the derived formulas can be executed more efficiently than time-consuming simulation, especially for those plans with a long duration. Thus, our pricing formulas of GCSRs are more suitable, tractable and feasible for practical implementation. In addition, the underestimation of GCSRs due to utilizing a single-currency can be avoided by using our formulas.

## Appendix A: Proof of Equation (4.1.1) and (4.1.2)

A lemma is first presented and then employed to price (2.1.4) and (2.1.5).

**Lemma 1**: Given that X and Y are normal random variables with mean zero and variances $\sigma^2(\cdot)$, the following identity holds:

$$E\left[K_1\exp\left(X-\tfrac{1}{2}\sigma_x^2\right)-K_2\exp\left(Y-\tfrac{1}{2}\sigma_y^2\right)\right]^+ = K_1N(d)-K_2N(d-V)$$

where $(Z)^+=\max(Z,0)$, $d=\left[\ln\left(\frac{K_1}{K_2}\right)+\frac{1}{2}V^2\right]\Big/V$ and $V^2$ is the variance of $X-Y$.

***Proof:*** See Amin and Jarrow (1991, p324) for details.

## Proof of Equation (4.1.1) and (4.1.2)

By applying the martingale pricing method,[7] the market value of the first-type guarantees at time *t*, $0\le t\le T_0\le T_1\le ...\le T_N$, is derived as follows:

$$G_I(t)=E^Q\left\{e^{-\int_t^{T_N}r_u du}G_I(T_N)\middle|F_t\right\},\quad \text{(where } E^*\{\cdot|F_t\}=E_t^*\{\cdot\},\ r(t)=r_t.\text{)} \tag{A.1}$$

Substituting $G_I(T_N)$ as shown in (3.1.2) into (A.1), we know

$$(\text{A.1})=E_t^Q\left\{e^{-\int_t^{T_N}r_u du}\left[\sum_{n=1}^{N}P_{d,n}\Upsilon_I^{(n)}(T_N)\right]\right\}=\sum_{n=1}^{N}P_{d,n}\Upsilon_I^{(n)}(t) \tag{A.2}$$

where $E^Q(\cdot)$ denotes the expectation under the domestic martingale measure $Q$ and $\Upsilon_t^{(n)}(I)$ is derived as follows.

$$\Upsilon_I^{(n)}(t)=E_t^Q\left\{e^{-\int_t^{T_N}r_u du}\Upsilon_I^{(n)}(T_N)\right\}=P_d(t,T_N)E_t^{T_N}\left\{\Upsilon_I^{(n)}(T_N)\right\} \tag{A.3}$$

$E^{T_N}(\cdot)$ denotes the expectation under the forward martingale measure $Q^{T_N}$ (with respect to the numeraire $P_d(t,T_N)$ ) defined by the Radon-Nikodym derivative $dQ^{T_N}/dQ=\dfrac{P_d(T_N,T_N)/P_d(t,T_N)}{\beta_d(T_N)/\beta_d(t)}$.[8]

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[7] Details regarding the martingale pricing method can be seen in advanced textbooks in finance, see, e.g. Shreve (2004) and Musiela and Rutkowski (2005).

[8] See Shreve (2004) and Musiela and Rutkowski (2005) for details on the changing-numeraire mechanism.

By inserting the definition of $\Upsilon_I^{(n)}(T_N)$ as shown in (3.1.4) into (A.3), (A.3) can be shown to be

$$(\text{A.3}) = P_d(t,T_N) E_t^{T_N} \left\{ \left[ \underbrace{\prod_{i=n-1}^{N-1} \frac{P_d(T_i,T_i)}{P_d(T_i,T_{i+1})}}_{(A-1)} - \underbrace{\frac{S_f^*(T_N)}{S_f^*(T_{n-1})}}_{(A-2)} \right]^+ \right\} \tag{A.4}$$

where $1+\delta L_d^{\delta}(T_i,T_i) = \dfrac{P_d(T_i,T_i)}{P_d(T_i,T_{i+1})}$, $(A-1) = \prod_{i=n-1}^{N-1} \dfrac{P_d(T_i,T_i)}{P_d(T_i,T_{i+1})}$ and

$$(A-2) = S_f^*(T_N)/S_f^*(T_{n-1}) = S_f(T_N)X(T_N)/S_f(T_{n-1})X(T_{n-1}).$$

We then solve (A-1) and (A-2), respectively.

The dynamics of $P_d(T_i,T_i)/P_d(T_i,T_{i+1}),\ i=n-1,\ldots,N-1$, and $S_f^*(T_N)/S_f^*(T_{n-1})$ are determined below.

$$\frac{P_d(T_i,T_i)}{P_d(T_i,T_{i+1})} = \frac{P_d(T_i,T_i)/P_d(T_i,T_N)}{P_d(T_i,T_{i+1})/P_d(T_i,T_N)} \equiv \frac{A^i(T_i)}{B^i(T_i)} \tag{A.5}$$

$$\begin{aligned}\frac{S_f^*(T_N)}{S_f^*(T_{n-1})} &= \frac{S_f(T_N)X(T_N)}{S_f(T_{n-1})X(T_{n-1})} \\ &= \frac{S_f(T_N)X(T_N)/P_d(T_N,T_N)}{S_f(T_{n-1})X(T_{n-1})/P_d(T_{n-1},T_N)} \frac{P_d(T_{n-1},T_{n-1})}{P_d(T_{n-1},T_N)} \equiv \frac{D(T_N)}{D(T_{n-1})} C^{n-1}(T_{n-1})\end{aligned} \tag{A.6}$$

We define each variable at time t as follows.

$$A^i(t) = P_d(t,T_i)/P_d(t,T_N), \quad i=n-1,\ldots,N-1 \tag{A.7}$$

$$B^i(t) = P_d(t,T_{i+1})/P_d(t,T_N), \quad i=n-1,\ldots,N-1 \tag{A.8}$$

$$C^{n-1}(t) = P_d(t,T_{n-1})/P_d(t,T_N), \tag{A.9}$$

$$D(t) = S_f(t)X(t)/P_d(t,T_N). \tag{A.10}$$

By employing (3.1.1)~(3.1.5) and Ito's Lemma and substituting $\overline{\sigma}_{Pd}(t,\cdot)$ defined in (3.1.6) for $\sigma_{Pd}(t,\cdot)$, the dynamics of (A.7)~(A.10) under the forward measure $Q^{T_N}$ can be obtained. Under the forward measure $Q^{T_N}$, the random variables defined from (A.7) to

(A.10) are martingales. Their dynamics can be written as follows.

$$\frac{dA^i(t)}{A^i(t)} = \left[ \underbrace{-\bar{\sigma}_{Pd}(t,T_i) + \bar{\sigma}_{Pd}(t,T_N)}_{\lambda_A^i(t)} \right] \cdot dW_t^{T_N} = \lambda_A^i(t) \cdot dW_t^{T_N} \tag{A.11}$$

$$\frac{dB^i(t)}{B^i(t)} = \left[ \underbrace{-\bar{\sigma}_{Pd}(t,T_{i+1}) + \bar{\sigma}_{Pd}(t,T_N)}_{\lambda_B^i(t)} \right] \cdot dW_t^{T_N} = \lambda_B^i(t) \cdot dW_t^{T_N} \tag{A.12}$$

$$\frac{dC^{n-1}(t)}{C^{n-1}(t)} = \left[ \underbrace{-\bar{\sigma}_{Pd}(t,T_{n-1}) + \bar{\sigma}_{Pd}(t,T_N)}_{\lambda_C^{n-1}(t)} \right] \cdot dW_t^{T_N} = \lambda_C^{n-1}(t) \cdot dW_t^{T_N} \tag{A.13}$$

$$\frac{dD(t)}{D(t)} = \left[ \underbrace{\sigma_{Sf}(t) + \sigma_X(t) + \bar{\sigma}_{Pd}(t,T_N)}_{\lambda_E(t)} \right] \cdot dW_t^{T_N} = \lambda_E(t) \cdot dW_t^{T_N} \tag{A.14}$$

(where $dW^{T_N}(t) = dW_t^{T_N}$ )

Solving the stochastic differential equations from (A.11) to (A.14), we obtain:

$$A^i(T_i) = A^i(t) e^{-\frac{1}{2}\int_t^{T_i} \|\lambda_A^i(u)\|^2 du + \int_t^{T_i} \lambda_A^i(u) \cdot dW_u^{T_N}}, \tag{A.15}$$

$$B^i(T_i) = B^i(t) e^{-\frac{1}{2}\int_t^{T_i} \|\lambda_B^i(u)\|^2 du + \int_t^{T_i} \lambda_B^i(u) \cdot dW_u^{T_N}}, \tag{A.16}$$

$$C^{n-1}(T_{n-1}) = \frac{P(t,T_{n-1})}{P(t,T_N)} e^{-\frac{1}{2}\int_t^{T_{n-1}} \|\lambda_C^{n-1}(u)\|^2 du + \int_t^{T_{n-1}} \lambda_C^{n-1}(u) \cdot dW_u^{T_N}}, \tag{A.17}$$

$$\frac{D(T_N)}{D(T_{n-1})} = e^{-\frac{1}{2}\int_{T_{n-1}}^{T_N} \|\lambda_D(u)\|^2 du + \int_{T_{n-1}}^{T_N} \lambda_D(u) \cdot dW_u^{T_N}}, \tag{A.18}$$

$$\left(1 + \delta L_d^\delta(T_i,T_i)\right) = \frac{P_d(t,T_i)}{P_d(t,T_{i+1})} e^{-\frac{1}{2}\int_t^{T_i} \left( \|\lambda_A^i(u)\|^2 - \|\lambda_B^i(u)\|^2 \right) du + \int_t^{T_i} \left[ \lambda_A^i(u) - \lambda_B^i(u) \right] \cdot dW_u^{T_N}} \tag{A.19}$$

By using (A.19), (A-1) can be derived as follows.

$$\text{(A-1)} = \prod_{i=n-1}^{N-1} \frac{P_d(t,T_i)}{P_d(t,T_{i+1})} \cdot \exp\left\{ \begin{aligned} & -\frac{1}{2} \sum_{i=n-1}^{N-1} \left[ \int_t^{T_i} \left( \|\lambda_A^i(u)\|^2 - \|\lambda_B^i(u)\|^2 \right) du \right] \\ & \qquad + \sum_{i=n-1}^{N-1} \int_t^{T_i} \left[ \lambda_A^i(u) - \lambda_B^i(u) \right] \cdot dW_u^{T_N} \end{aligned} \right\} \tag{A.20}$$

Define

$$X_i = \int_t^{T_i} \left(\lambda_A^i(u) - \lambda_B^i(u)\right) \cdot dW_u^{T_N}, i = n-1, \ldots, N-1 \quad , \quad X_{(n)} = \sum_{i=n-1}^{N-1} X_i. \tag{A.21}$$

Then, we know that

$$Var(X_i) = \int_t^{T_i} \left\| \lambda_A^i(u) - \lambda_B^i(u) \right\|^2 du \tag{A.22}$$

$$Cov(X_i, X_j) = \int_t^{T_i} \left(\lambda_A^i(u) - \lambda_B^i(u)\right) \cdot \left(\lambda_A^j(u) - \lambda_B^j(u)\right) du, \qquad j = i+1 \tag{A.23}$$

$$Var\left(X_{(n)}\right) = \sum_{i=n-1}^{N-1} \int_t^{T_i} \left\| \lambda_A^i(u) - \lambda_B^i(u) \right\|^2 du + 2 \sum_{i=n-1}^{(N-1)-1} \sum_{j=i+1}^{N-1} \int_t^{T_i} \left(\lambda_A^i(u) - \lambda_B^i(u)\right) \cdot \left(\lambda_A^j(u) - \lambda_B^j(u)\right) du \tag{A.24}$$

By combing equations from (A.21) to (A.24) with (A.20), (A.20) can be written as (A.25).

$$(\text{A.20}) = \underbrace{\frac{P_d(t, T_{n-1})}{P_d(t, T_N)} \exp\left[\Lambda_{(n)}\right]}_{K_1^{(n)}} \cdot \exp\left[X_{(n)} - \frac{1}{2} Var\left(X_{(n)}\right)\right] \tag{A.25}$$

where

$$\frac{P_d(t, T_{n-1})}{P_d(t, T_N)} = \prod_{i=n-1}^{N-1} \frac{P_d(t, T_i)}{P_d(t, T_{i+1})} \tag{A.26}$$

$$\Lambda_{(n)} = \sum_{i=n-1}^{N-1} \left[ \int_t^{T_i} \gamma_B^i(u) \cdot \left(\gamma_B^i(u) - \gamma_A^i(u)\right) du \right] + \sum_{i=n-1}^{(N-1)-1} \sum_{j=i+1}^{N-1} \left[ \int_t^{T_i} \left(\gamma_A^i(u) - \gamma_B^i(u)\right) \cdot \left(\gamma_A^j(u) - \gamma_B^j(u)\right) du \right] \tag{A.27}$$

Next, (A-2) can be obtained by adopting (A.17) and (A.18) as below.

$$(\text{A-2}) = \frac{P_d(t, T_{n-1})}{P_d(t, T_N)} \exp\left\{ \begin{array}{l} -\frac{1}{2}\left[ \int_t^{T_{n-1}} \left\| \lambda_C^{n-1}(u) \right\|^2 du + \int_{T_{n-1}}^{T_N} \left\| \lambda_E(u) \right\|^2 du \right] \\ \qquad + \left[ \int_t^{T_{n-1}} \lambda_C^{n-1}(u) \cdot dW_u^{T_N} + \int_{T_{n-1}}^{T_N} \lambda_E(u) \cdot dW_u^{T_N} \right] \end{array} \right\} \tag{A.28}$$

Define

$$Y_1 = \int_t^{T_{n-1}} \lambda_C^{n-1}(u) \cdot dW_u^{T_N}, \quad Y_2 = \int_{T_{n-1}}^{T_N} \lambda_E(u) \cdot dW_u^{T_N}, \quad Y_{(n)} = \sum_{i=1}^{2} Y_i \tag{A.29}$$

As a result,

$$Var(Y_1) = \int_t^{T_{n-1}} \left\| \lambda_C^{n-1}(u) \right\|^2 du, \ Var(Y_2) = \int_{T_{n-1}}^{T_N} \left\| \lambda_E(u) \right\|^2 du \tag{A.30}$$

$$Var\left(Y_{(n)}\right)=\int_t^{T_{n-1}}\left\|\lambda_C^{n-1}(u)\right\|^2 du+\int_{T_{n-1}}^{T_N}\left\|\lambda_E(u)\right\|^2 du \tag{A.31}$$

Combing equations from (A.29) to (A.31) with (A.28), (A.28) can be written as (A.32).

$$(\text{A.28})=\underbrace{\frac{P_d\left(t,T_{n-1}\right)}{P_d\left(t,T_N\right)}}_{K_2^{(n)}}e^{Y_{(n)}-\frac{1}{2}Var\left(Y_{(n)}\right)} \tag{A.32}$$

By the results of (A1) and (A2) as shown in (A.25) and (A.32), (A.4) can be represented as

$$\Upsilon_t^{(n)}(I)=P_d\left(t,T_N\right)E_t^{T_N}\left\{\left[K_1^{(n)}e^{X_{(n)}-\frac{1}{2}Var\left(X_{(n)}\right)}-K_2^{(n)}e^{Y_{(n)}-\frac{1}{2}Var\left(Y_{(n)}\right)}\right]^+\right\} \tag{A.33}$$

Applying Lemma 1, we know

$$(\text{A.33})=P_d\left(t,T_N\right)\left\{\frac{P_d\left(t,T_{n-1}\right)}{P_d\left(t,T_N\right)}e^{\Lambda_{(n)}}N\left(d_1^{(n)}\right)-\frac{P_d\left(t,T_{n-1}\right)}{P_d\left(t,T_N\right)}N\left(d_2^{(n)}\right)\right\}=\mathrm{H}_{1,t}^{(n)}N\left(d_1^{(n)}\right)-\mathrm{H}_{2,t}^{(n)}N\left(d_2^{(n)}\right) \tag{A.34}$$

where

$$\mathrm{H}_{1,t}^{(n)}=P_d\left(t,T_{n-1}\right)\exp\left[\Lambda_{(n)}\right],\quad \mathrm{H}_{2,t}^{(n)}=P_d\left(t,T_{n-1}\right) \tag{A.35}$$

$$P_d\left(t,T_{n-1}\right)=\left[1+\left(T_0-t\right)L_d\left(t,T_0\right)\right]^{-1}\left[\prod_{k=0}^{n-2}\left(1+\delta L_d^{\delta}\left(t,T_k\right)\right)\right]^{-1} \tag{A.36}$$

$$d_1^{(n)}=\left[\Lambda_{(n)}+\tfrac{1}{2}V_{(n)}^2\right]\Big/V_{(n)},\quad d_2^{(n)}=d_1^{(n)}-V_{(n)},\quad V_{(n)}=\sqrt{V_{(n)}^2} \tag{A.37}$$

$$V_{(n)}^2=Var\left(X_{(n)}-Y_{(n)}\right)=Var\left(X_{(n)}\right)+Var\left(Y_{(n)}\right)-2Cov\left(X_{(n)},Y_{(n)}\right) \tag{A.38}$$

$Cov\left(X_{(n)},Y_{(n)}\right)$ can be derived as below.

$$Cov\left(X_{(n)},Y_{(n)}\right)=\sum_{i=n-1}^{N-1}\int_t^{T_{n-1}}\left(\lambda_A^i(u)-\lambda_B^i(u)\right)\cdot\lambda_C^{n-1}(u)du+\sum_{i=n-1}^{N-1}\int_{T_{n-1}}^{T_i}\left(\lambda_A^i(u)-\lambda_B^i(u)\right)\cdot\lambda_E(u)du \tag{A.39}$$

$P_d\left(t,T_{n-1}\right)$ can be derived as follows.

$$1=P_d\left(t,T_{n-1}\right)\frac{1}{P_d\left(t,T_0\right)}\left[\prod_{k=0}^{n-2}\frac{P_d\left(t,T_k\right)}{P_d\left(t,T_{k+1}\right)}\right] \tag{A.40}$$

According to (A.40), we can obtain (A.36) after rearrangement.

Note that the time differences, $\left(T_0 - t\right)$ and $\delta = T_n - T_{n-1}$, are measured as year fraction between two dates. $L_d\left(t, T_0\right)$ is the simply-compounded spot interest rate prevailing at time t for the maturity $T_0$.

Therefore, equation (4.1.1) and (4.1.2) have been derived.

## Appendix B: Proof of Equation (4.1.3) and (4.1.4)

By applying the martingale pricing method, the market value of the second-type guarantees at time $t$, $0 \le t \le T_0 \le T_1 \le ... \le T_N$, is derived as follows:

$$G_{II}\left(t\right) = E^Q\left\{ e^{-\int_t^{T_N} r_u du} G_{II}\left(T_N\right) \middle| F_t \right\} \tag{B.1}$$

Substituting $G_{T_N}\left(II\right)$ as shown in (3.2.2) into (B.1), we know

$$(\text{B.1}) = E_t^Q\left\{ e^{-\int_t^{T_N} r_u du} \left[ \sum_{n=1}^{N} P_{d,n} \cdot \Upsilon_{II}^{(n)}\left(T_N\right) \right] \right\} = \sum_{n=1}^{N} P_{d,n} \cdot \Upsilon_{II}^{(n)}\left(t\right) \tag{B.2}$$

Define

$$M_{II}^{(n)}\left(T_N\right) = \prod_{i=n-1}^{N-1} \max\left[ \left(1 + \delta L_d^{\delta}\left(T_i, T_i\right)\right), \frac{S_f^*\left(T_{i+1}\right)}{S_f^*\left(T_i\right)} \right] \tag{B.3}$$

Hence,

$$\Upsilon_{II}^{(n)}\left(t\right) = M_{II}^{(n)}\left(T_N\right) - \frac{S_f^*\left(T_N\right)}{S_f^*\left(T_{n-1}\right)} \tag{B.4}$$

Therefore, $\Upsilon_{II}^{(n)}\left(t\right)$ can be derived as below.

$$\Upsilon_{II}^{(n)}\left(t\right) = E_t^Q\left\{ e^{-\int_t^{T_N} r_u du} \Upsilon_{II}^{(n)}\left(T_N\right) \right\} = E_t^Q\left\{ e^{-\int_t^{T_N} r_u du} M_{II}^{(n)}\left(T_N\right) \right\} - E_t^Q\left\{ e^{-\int_t^{T_N} r_u du} \frac{S_f^*\left(T_N\right)}{S_f^*\left(T_{n-1}\right)} \right\} \tag{B.5}$$

According to the definition of $M_{II}^{(n)}\left(T_N\right)$, we know

$$E_t^Q\left\{ e^{-\int_t^{T_N} r_u du} M_{II}^{(n)}\left(T_N\right) \right\} = E_t^Q\left\{ e^{-\int_t^{T_N} r_u du} \left[ \prod_{i=n-1}^{N-1} C_{T_{i+1}} \right] \right\} \tag{B.6}$$

where

$$C_{T_{i+1}} = \max\left[\left(1+\delta L_d^{\delta}(T_i,T_i)\right), \frac{S_f^*(T_{i+1})}{S_f^*(T_i)}\right]$$
$$= \left[\left(1+\delta L_d^{\delta}(T_i,T_i)\right), \frac{S_f^*(T_{i+1})}{S_f^*(T_i)}\right]^+ + \frac{S_f^*(T_{i+1})}{S_f^*(T_i)}, \qquad i=n-1,n,\ldots N-1 \tag{B.7}$$

By "The Law of Iterated Conditional Expectation" in Duffie (1988), (B.8) can be obtained as follows.

$$\text{(B.6)} = E_t^Q\left\{e^{-\int_t^{T_{n-1}} r_u du}\right\}\prod_{i=n-1}^{N-1}\underbrace{E_{T_i}^Q\left\{e^{-\int_{T_i}^{T_{i+1}} r_u du} C_{T_{i+1}}\right\}}_{(B-1)} = P_d(t,T)\prod_{i=n-1}^{N-1} 2N\left(d_3^i\right) \tag{B.8}$$

where

$$E_t^Q\left\{e^{-\int_t^{T_{n-1}} r_u du}\right\} = P_d(t,T_{n-1}), \tag{B.9}$$

$$E_{T_i}^Q\left\{e^{-\int_{T_i}^{T_{i+1}} r_u du} C_{T_{i+1}}\right\} = 2N\left(d_3^i\right), \tag{B.10}$$

$$d_3^i = \frac{\frac{1}{2}V_i^2}{V_i} = \frac{1}{2}V_i, \quad V_i = \sqrt{V_i^2}, \tag{B.11}$$

$$V_i^2 = \int_{T_i}^{T_{i+1}} \left\|\lambda_D^i(u)\right\|^2 du, \tag{B.12}$$

$$\lambda_E^i(t) = \left[\sigma_{Sf}(t) + \sigma_X(t) + \overline{\sigma}_{Pd}(t,T_{i+1})\right]. \tag{B.13}$$

We solve (B-1) as follows.

$$\text{(B-1)} = P_d(T_i,T_{i+1}) E_{T_i}^{T_{i+1}}\left\{C_{T_{i+1}}\right\} \tag{B.14}$$

where $E_\eta^{T_{i+1}}(\cdot)$ denotes the expectation under the forward martingale measure $Q^{T_{i+1}}$ defined by the Radon-Nikodym derivative $dQ^{T_{i+1}}/dQ = \dfrac{P_d(T_{i+1},T_{i+1})/P_d(T_i,T_{i+1})}{\beta_d(T_{i+1})/\beta_d(T_i)}$.

By using (B.7), $E_{T_i}^{T_{i+1}}\left\{C_{T_{i+1}}\right\}$ can be derived below.

$$E_{T_i}^{T_{i+1}}\left\{C_{T_{i+1}}\right\} = \underbrace{E_{T_i}^{T_{i+1}}\left\{\left[\left(1+\delta L_d^{\delta}(T_i,T_i)\right) - \frac{S_f^*(T_{i+1})}{S_f^*(T_i)}\right]^+\right\}}_{(B-1a)} + \underbrace{E_{T_i}^{T_{i+1}}\left\{\frac{S_f^*(T_{i+1})}{S_f^*(T_i)}\right\}}_{(B-1b)} \tag{B.15}$$

where $\left(1+\delta L_d^{\delta}\left(T_i,T_i\right)\right)=P_d\left(T_i,T_i\right)/P_d\left(T_i,T_{i+1}\right)$

We then solve (B-1a) and (B-1b), respectively.

$$\text{(B-1a)}=E_{T_i}^{T_{i+1}}\left[\left(\frac{P_d\left(T_i,T_i\right)}{P_d\left(T_i,T_{i+1}\right)}-\frac{S_f^*\left(T_{i+1}\right)}{S_f^*\left(T_i\right)}\right)\cdot I_\alpha\right],\alpha=\left\{\frac{P_d\left(T_i,T_i\right)}{P_d\left(T_i,T_{i+1}\right)}>\frac{S_f^*\left(T_{i+1}\right)}{S_f^*\left(T_i\right)}\right\} \quad \text{(B.16)}$$

$I_\alpha$ is an indicator function with $\begin{cases} 1, & \frac{P_d(T_i,T_i)}{P_d(T_i,T_{i+1})}>\frac{S_f^*(T_{i+1})}{S_f^*(T_i)}. \\ 0, & otherwise. \end{cases}$

The dynamics of $P_d\left(T_i,T_i\right)/P_d\left(T_i,T_{i+1}\right)$, and $S_f^*\left(T_{i+1}\right)/S_f^*\left(T_i\right)$ are determined below.

$$\frac{P_d\left(T_i,T_i\right)}{P_d\left(T_i,T_{i+1}\right)}\equiv C^i\left(T_i\right). \quad \text{(B.17)}$$

$$\frac{S_f^*\left(T_{i+1}\right)}{S_f^*\left(T_i\right)}=\frac{S_f\left(T_{i+1}\right)X\left(T_{i+1}\right)/P_d\left(T_{i+1},T_{i+1}\right)}{S_f\left(T_i\right)X\left(T_i\right)/P_d\left(T_i,T_{i+1}\right)}\frac{P_d\left(T_i,T_i\right)}{P_d\left(T_i,T_{i+1}\right)}\equiv\frac{D^i\left(T_{i+1}\right)}{D^i\left(T_i\right)}C^i\left(T_i\right). \quad \text{(B.18)}$$

We define each variable at time t as follows.

$$C^i\left(t\right)=P_d\left(t,T_i\right)/P_d\left(t,T_{i+1}\right) \quad \text{(B.19)}$$

$$D^i(t)=S_f\left(t\right)X\left(t\right)/P_d\left(t,T_{i+1}\right) \quad \text{(B.20)}$$

By employing (3.1.1)~(3.1.4) and Ito's Lemma and substituting $\overline{\sigma}_{Pd}\left(t,\cdot\right)$ defined in (3.1.6) for $\sigma_{Pd}\left(t,\cdot\right)$, the dynamics of (B.19) and (B.20) under the forward measure $Q^{T_{i+1}}$ can be obtained as given below. Under the forward measure $Q^{T_{i+1}}$, the random variables defined in (B.19) and (B.20) are martingales, and their dynamics can be written as follows.

$$\frac{dC^i\left(t\right)}{C^i\left(t\right)}=\left[\underbrace{-\overline{\sigma}_{Pd}\left(t,T_i\right)+\overline{\sigma}_{Pd}\left(t,T_{i+1}\right)}_{\lambda_C^i(t)}\right]\cdot dW_t^{T_{i+1}}=\lambda_C^i\left(t\right)\cdot dW_t^{T_{i+1}} \quad \text{(B.21)}$$

$$\frac{dD^i(t)}{D^i(t)}=\left[\underbrace{\sigma_{Sf}\left(t\right)+\sigma_X\left(t\right)+\overline{\sigma}_{Pd}\left(t,T_{i+1}\right)}_{\lambda_D^i(t)}\right]\cdot dW_t^{T_{i+1}}=\lambda_D^i\left(t\right)\cdot dW_t^{T_{i+1}} \quad \text{(B.22)}$$

Solving the stochastic differential equations (B.21) and (B.22), we obtain:

$$\frac{D^i(T_{i+1})}{D^i(T_i)} = \exp\left(-\frac{1}{2}\int_{T_i}^{T_{i+1}} \left\|\lambda_D^i(u)\right\|^2 du + \int_{T_i}^{T_{i+1}} \lambda_D^i(u)\cdot dW_u^{T_{i+1}}\right), \tag{B.23}$$

$$\begin{aligned} \frac{S_f^*(T_{i+1})}{S_f^*(T_i)} &= \frac{S_f(T_{i+1})X(T_{i+1})}{S_f(T_i)X(T_i)} \\ &= \frac{P_d(T_i,T_i)}{P_d(T_i,T_{i+1})}\exp\left(-\frac{1}{2}\int_{T_i}^{T_{i+1}} \left\|\lambda_D^i(u)\right\|^2 du + \int_{T_i}^{T_{i+1}} \lambda_D^i(u)\cdot dW_u^{T_{i+1}}\right) \end{aligned}. \tag{B.24}$$

Hence,

$$\text{(B.16)} = \frac{P_d(T_i,T_i)}{P_d(T_i,T_{i+1})} P_r^{T_{i+1}}\left(\frac{P_d(T_i,T_i)}{P_d(T_i,T_{i+1})} > \frac{S_f^*(T_{i+1})}{S_f^*(T_i)}\right) - E_{T_i}^{T_{i+1}}\left(\frac{S_f^*(T_{i+1})}{S_f^*(T_i)}\cdot I_\alpha\right), \tag{B.25}$$

where $P_r^{T_{i+1}}(\cdot)$ denotes the probability under the forward martingale measure $Q^{T_{i+1}}$.

By inserting (B.24) into $P_r^{T_{i+1}}(\cdot)$, the probability can be obtained after rearrangement as follows:

$$P_r^{T_{i+1}}\left(\frac{P_d(T_i,T_i)}{P_d(T_i,T_{i+1})} > \frac{S_f^*(T_{i+1})}{S_f^*(T_i)}\right) = N\left(d_3^i\right), \tag{B.26}$$

where

$$d_3^i = V_i/2,\ \ V_i^2 = \int_{T_i}^{T_{i+1}} \left\|\lambda_D^i(u)\right\|^2 du,\ \ V_i = \sqrt{V_i^2}. \tag{B.27}$$

Using (B.24), we know

$$E_{T_i}^{T_{i+1}}\left(\frac{S_f^*(T_{i+1})}{S_f^*(T_i)}\cdot I_\alpha\right) = \frac{P_d(T_i,T_i)}{P_d(T_i,T_{i+1})} P_r^{R_i}\left(\frac{P_d(T_i,T_i)}{P_d(T_i,T_{i+1})} > \frac{S_f^*(T_{i+1})}{S_f^*(T_i)}\right), \tag{B.28}$$

where $P_r^{R_i}(\cdot)$ denotes the probability under the martingale measure $R_i$ which is defined by the Radon-Nikodym derivative

$$\frac{dR_i}{dQ^{T_{i+1}}} = \exp\left(-\frac{1}{2}\int_{T_i}^{T_{i+1}} \left\|\lambda_D^i(u)\right\|^2 du + \int_{T_i}^{T_{i+1}} \lambda_D^i(u)\cdot dW_u^{T_{i+1}}\right). \tag{B.29}$$

From the Radon-Nikodym derivative $dR_i/dQ^{T_{i+1}}$, we know that

$$dW_t^{T_{i+1}} = dW_t^{R_i} + \lambda_D^i(t)\,dt. \tag{B.30}$$

Under the measure $R_i$, we obtain (B.31) by substituting (B.30) into (B.24)

$$\frac{S_f^*(T_{i+1})}{S_f^*(T_i)}=\frac{P_d(T_i,T_i)}{P_d(T_i,T_{i+1})}\exp\left(\frac{1}{2}\int_{T_i}^{T_{i+1}}\left\|\lambda_D^i(u)\right\|^2du+\int_{T_i}^{T_{i+1}}\lambda_D^i(u)\cdot dW_u^{R_i}\right). \tag{B.31}$$

By inserting (B.31) into $P_r^{R_i}(\cdot)$, the probability can be obtained after rearrangement as follows:

$$P_r^{R_i}\left(\frac{P_d(T_i,T_i)}{P_d(T_i,T_{i+1})}>\frac{S_f^*(T_{i+1})}{S_f^*(T_i)}\right)=N\left(-d_3^i\right). \tag{B.32}$$

By combing (B.16), (B.25), (B.26), (B.28) with (B.32), (B-1a) can be obtained below.

$$\text{(B-1a)}=\frac{P_d(T_i,T_i)}{P_d(T_i,T_{i+1})}N\left(d_3^i\right)-\frac{P_d(T_i,T_i)}{P_d(T_i,T_{i+1})}N\left(-d_3^i\right). \tag{B.33}$$

From (B.24), we obtain

$$\text{(B-1b)}=E_{T_i}^{T_{i+1}}\left\{\frac{S_f^*(T_{i+1})}{S_f^*(T_i)}\right\}=\frac{P_d(T_i,T_i)}{P_d(T_i,T_{i+1})}. \tag{B.34}$$

Hence,

$$\text{(B.15)}=2\frac{P_d(T_i,T_i)}{P_d(T_i,T_{i+1})}N\left(d_3^i\right). \tag{B.35}$$

And we obtain (B-1) as shown in (B.36).

$$\text{(B-1)}=P_d(T_i,T_{i+1})E_{T_i}^{T_{i+1}}\left\{C_{T_{i+1}}\right\}=2N\left(d_3^i\right). \tag{B.36}$$

Besides, (B.37) can be obtained by using (A.28).

$$E_t^Q\left\{e^{-\int_t^{T_N}r_u du}\frac{S_f^*(T_N)}{S_f^*(T_{n-1})}\right\}=P_d(t,T_N)E_t^{T_N}\left\{\frac{S_f^*(T_N)}{S_f^*(T_{n-1})}\right\}=P_d(t,T_{n-1}) \tag{B.37}$$

Inserting (B.8) and (B.37) into (B.5), we derive the result as follows.

$$\psi_{II}^{(n)}(t)=P_d(t,T_{n-1})\prod_{i=n-1}^{N-1}2N\left(d_3^i\right)-P_d(t,T_{n-1})=P(t,T_{n-1})\left[\prod_{i=n-1}^{N-1}2N\left(d_3^i\right)-1\right] \tag{B.38}$$

Therefore, the proof of equation (4.1.3) and (4.1.4) is completed.

## Appendix C

Exhibit 5: Share of Unit-Linked Contracts in Total Life Premium

| Source: CEA | Life Premium (Euro million) | | †Share-UL(%) | |
|---|---|---|---|---|
| Country | 2009 | 2010 | 2009 | 2010 |
| Belgium | 18,404 | 19,141 | 9.04% | 10.70% |
| Bulgaria | 103 | 115 | 5.28% | 5.62% |
| Switzerland | 19,484 | 21,828 | 9.51% | 10.13% |
| Czech Republic | 2,044 | 2,601 | 39.11% | 46.80% |
| Germany | 81,371 | 87,165 | 13.91% | 13.48% |
| Denmark | 14,342 | 14,938 | 25.04% | 35.18% |
| Estonia | 133 | 182 | 45.25% | 61.52% |
| Spain | 29,074 | 27,297 | 14.77% | 17.44% |
| Finland | 2,847 | 4,570 | 56.76% | 56.04% |
| France | 137,923 | 143,837 | 13.02% | 13.39% |
| United Kingdom | 155,417 | 152,583 | 17.00% | 14.94% |
| Croatia | 339 | 337 | 6.59% | 6.79% |
| Hungary | 1,466 | 1,606 | 57.21% | 61.35% |
| Italy | 81,116 | 90,102 | 12.00% | 17.10% |
| Malta | 193 | 224 | 12.95% | 15.18% |
| Netherlands | 24,381 | 21,573 | 34.04% | 43.00% |
| Norway | 7,140 | 8,382 | 20.04% | 19.99% |
| Poland | 6,982 | 7,848 | 21.37% | 25.89% |
| Portugal | 9,876 | 12,103 | 29.22% | 22.04% |
| Sweden | 18,134 | 22,203 | 35.94% | 34.38% |
| Slovenia | 630 | 656 | 58.57% | 60.37% |
| Romania | 384 | 214 | n.a. | 41.69% |
| Cyprus | 353 | 375 | n.a. | n.a. |
| Latvia | 28 | n.a. | 13.50% | n.a. |
| Greece | 2,202 | n.a. | n.a. | n.a. |
| Others | 19,802 | 20,795 | n.a. | n.a. |
| CEA (Total) | 634,169 | 660,676 | 15.96% | 16.98% |

CEA: European Insurance and Reinsurance Federation.
† ”Share-UL” represents the share of unit-linked contracts in total life premium.
‡ ”n.a.” denotes ”not available”.

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