行政院國家科學委員會專題研究計畫成果報告

線性矩陣不等式解的連續性及其涵義

Continuity property of the solutions of linear matrix inequalities (LMIs) and its implications

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一、 中英文摘要

本計畫的目的是要研究線性矩陣不等式解的連續性, 以及探討它在控制理論上的涵義。特別是我們將討論它在以下兩個議題潛在的應用: (1)參數極度緩慢變化的系統的 L_2 gain 近似原非時變系統中最大的 L_2 gain; (2) 一離散、非時變、穩定系統具有多個極度緩慢變化的不確定參數和一無結構化(unstructured)動態機制(dynamics)時, 其強健穩定性的充分且必要條件。

Abstract

The objectives of this research are to investigate continuity property of the solutions of LMIs and to explore its implications on control theory. In particular, we will consider its potential use in the following two issues: (1) the L_2 gain of an arbitrarily slowly varying parametrically system approximately equals to the maximal L_2 gain of its frozen-parameter systems; and (2) necessary and sufficient condition for uniformly robust stability of a discrete-time, linear time-invariant stable system against several arbitrarily slowly varying parameters and unstructured dynamics.

二、 計畫緣由與目的

A wide variety of problems arising from system theory and controller synthesis can be cast as a few standard convex or quasi-convex optimization problems involving linear matrix inequalities (LMIs) [1]. The algorithms developed by the interior point methods [2] to solve such constrained optimization problems has been proven to be very efficient in practice, which significantly increases the designers' ability to check many solvability conditions presented in the existing issues. Because of its broad range of applications and the advantages of formulating the solutions amenable for computers, LMI technique has grown to be a popular approach in analysis and design of control systems. However, concerning the fundamental issues of existence, uniqueness, and continuity of solutions, in contrast to a large amount of references for matrix equations, e.g., Riccati equations [3-6] (and the references therein), it appears to be rare for LMIs [5-10]. Thus the objectives of this research are to investigate these questions, in particular, continuity property of solutions of LMIs and its implications on control theory.

In the relevant literature, Oustry et al [7] provided sufficient conditions under which the optimal solution of a semi-definite program (SDP)(consisting of minimizing a linear objective over LMI constraints) is unique and continuous on problem data. They demonstrated that the result could be used to regularize ill-conditioned SDPs. That implies continuity of the solutions with respect to problem data is advantageous to obtain robust numerical solution. In [8-10], the (one-parameter) continuity property helps establish the equivalence of two well known robust stability conditions derived independently from μ theory and absolute stability theory, and therefore helps clarify certain close relation between the two theories. In [5,6] Stoorvogel et al showed semi-stabilizing solutions of H_2/H_∞ control problems might have discontinuities as a function of the system parameters. This implies that sub-optimal H_2/H_∞ control for certain systems might be discontinuous if the control gains were implemented from the semi-stabilizing solutions. In engineering sense, having robust numerical solutions in the design procedure and continuous control (insensitive to plant data) are of fundamental importance to a good controller design. In view of the above results, the importance of continuity property of LMI solutions is explicit.

Concerning implications of the continuity property, this research will focus on exploring new results on robust performance and robust stability of linear time-invariant systems with certain structured time varying mixed uncertainties. In [12] Shamma showed that constant D scaling condition is necessary and sufficient for uniformly robust stability against fast varying, structured dynamic uncertainty. In [8] Poolla and Tikku gave necessary and sufficient condition for robust performance against arbitrarily slowly varying, structured dynamic uncertainty (which is amount to a frequency-dependent D scaling condition). Recently, Paganini [13] provided necessary and sufficient conditions for uniformly robust stability against certain types of mixed uncertainties (including both parametric and dynamic uncertainties). In this research, we will generalize partial results in these issues.

三、研究方法與成果

This research proceeds with investigating the following three issues:

1. Continuity property of the solutions of strict LMI.
2. The L_2 gain of an arbitrarily slowly varying parametrically system approximately equals to the maximal L_2 gain of its frozen-parameter systems (i.e., systems with the parameters kept constant for all time).
3. Necessary and sufficient condition for uniformly robust stability against arbitrarily slowly varying parameters and arbitrarily slowly (or fast) varying dynamics.

Throughout, the LMI we consider refers to strict LMI [1], which has the form

$$L(x_1,\dots,x_l,M_0,\dots,M_l) := M_0+\sum_{i=1}^{l}x_iM_i+\left(M_0+\sum_{i=1}^{l}x_iM_i\right)^* > 0$$

where $x\in C^l$ is the variable and $M_i\in C^{n\times n}, i=0,\dots,l$ are given data. The symbol * denotes complex conjugate transpose and the inequality symbol indicates positive definiteness. Let

$F_p := \{F:[-1,1]^m \to C^{n\times n} : F \text{ is continuous } \}$.

The first result below establishes continuity property of solutions of the parameterized LMI

$$L(x_1,\dots,x_l,F_o(p),\dots,F_l(p))>0 \quad\dots\dots(1)$$

when $F_i\in F_p$. It states that under mild assumptions that the problem data of a strict LMI are continuously dependent on a few real parameters p which belong to a compact set and that for each value of the compact set the corresponding LMI has a solution, there exits a continuous function of p which serves as a solution to the entire parameterized family of LMI.

Theorem 1: Let $F_i\in F_p, i=1,\dots,l.$ The following conditions are equivalent.

(a) For every $(p_1,\dots,p_m)\in[-1,1]^m$, there exist real numbers $x_i, i=1,\dots l,$ such that (1) holds.

(b) There exist continuous functions $x_i : [-1,1]^m \to R, i = 1,...,l$, such that (1) holds for all $(p_1,..., p_m) \in [-1,1]^m$.

The proof can be shown by induction on the number of the parameters. Following the same approach used in [8,10] and the concept of uniform continuity of continuous functions on a compact set we can inductively partially extend Proposition 2.1 in [10] to multiple parameters case. With this result in hand, we are ready to solve the following two problems.

1. The L_2 gain of an arbitrarily slowly varying parametrically system approximately equals to the maximal L_2 gain of its frozen-parameter systems: Suppose that we have a discrete-time linear parametrically state space model without direct through part. The realization are assumed to be continuously dependent on some parameters (possibly time-varying) which take on values from a compact set, and has L_2 gain less than certain value for all constant values of the set. By the above continuity result the difference of the LMI solutions at adjacent time instances can be made as small as possible as long as the parameters vary slow enough. Thus the LMI version strict Bounded Real lemma for discrete linear time-varying systems can be invoked to prove the claim.

2. Necessary and sufficient conditions for uniformly robust stability against arbitrarily slowly varying parameters and arbitrarily slowly (or fast) varying dynamics:

$\|M\|_{\mu R} := \sup_{\theta \in [0,2\pi]} \mu_R(M(e^{j\theta}))$, (i.e., the real structured singular value of M, see, e.g., [14]),

$$\Delta_r(\nu_p) := \{ \text{block-diag}(\delta_1 I_{n_1},...,\delta_l I_{n_l}) : \delta_i \text{ is linear time-varying, parametric}, \|\delta_i\| \le 1, \|z^{-1}\delta_i - \delta_i z^{-1}\| \le \nu_p \}$$

$$\Delta_c(\nu_c) := \{ \Delta : \Delta \text{ is a } k \times k \text{ linear time-varying dynamics}, \|\Delta\| \le 1, \|z^{-1}\Delta - \Delta z^{-1}\| \le \nu_c \}$$

$$\blacktriangle := \{\text{block\_diag}(\Delta_p, \Delta_d) : \Delta_p \in \Delta_r(\nu_p), \Delta_d \in \Delta_c(\nu_c)\}$$

$RH_\infty :=$ the set of square transfer matrices (with appropriate dimension determined from the context) whose entries are proper, real-rational functions with no poles outside the open unit disk.

Consider the perturbed system as shown in Figure 1, where M represents a discrete-time, stable linear time-invariant system, and $\Delta_p \in \blacktriangle$. Analogous to the Main Loop Theorem for constant matrix case, e.g., [11-12], uniformly robust stability of the $M-\Delta$ loop is equivalent to uniformly robust stability of the $M_{11}-\Delta_p$ loop for all $\Delta_p \in \Delta_r(\nu_p)$ and uniformly robust stability of the $F(M,\Delta_p)$-Δ_d loop for all $\Delta_p \in \Delta_r(\nu_p)$ and $\Delta_d \in \Delta_c(\nu_c)$, where $F(M,\Delta_p) = M_{22} + M_{21}\Delta_p(I - M_{11}\Delta_p)^{-1}M_{12}$. and M_{ij}, $i = 1,2$, $j = 1,2$ are sub-matrices of M with appropriate dimensions.

Consider the following statements:

S1 : $\|M_{11}\|_{\mu R} < 1$ and there exists $\nu_p > 0$ such that $\sup_{\Delta_p \in \Delta_r(\nu_p)} \|F(M,\Delta_p)\| < 1$.

S2 : M is uniformly robustly stable against mixed uncertainties in $\Delta \in \blacktriangle$ for some $\nu_p > 0$ and $\nu_c > 0$.

S3 : $\|M_{11}\|_{\mu R} < 1$ and $\sup_{\Delta_p \in \Delta_r(0)} \|F(M,\Delta_p)\| < 1$.

Note that S3 is equivalent to the mixed μ condition $\|M\|_\mu < 1$ (with several repeated real blocks and one full complex block) which is a necessary and sufficient condition for uniformly robust stability against linear time-invariant mixed uncertainty. Furthermore, it is readily checked that S1 implies S2 (via the Main Loop Theorem analogue) and that S2 implies S3. If we can show that S3 implies S1, then the three statements are all equivalent and thus give necessary and sufficient conditions for uniformly robust stability against the mixed uncertainties. The result follows directly from the previous result about the L_2 gain. Thus we have the following theorem.

Theorem 2: Let $M \in RH_\infty$ be strictly proper. Then M is uniformly robustly stable against arbitrarily slowly varying mixed uncertainty $\Delta \in$ ▲ if and only if $\|M\|_\mu < 1$.

Remark: It's easy to check (from a slight modification of statement S2) that Theorem 2 is also true when there is no constraint on the rate-of-variation of the dynamic uncertainty.

四、結論與討論

(a) 嚴格線性矩陣不等式在不同問題參數(problem data)的解(不唯一)經適當選取,可以內插成一依問題參數而變的連續函數而且滿足原線性矩陣不等式。(b)參數極度緩慢變化系統的 L_2 gain 近似原非時變系統中最大的 L_2 gain。(c)一線性非時變穩定系統能容忍多個極度緩慢變化的參數和一無結構化(unstructured) 極度緩慢(或快速)變化動態機制(dynamics)而不失穩定性的充分且必要條件是一個 mixed μ condition。結論(c)中之動態機制若為結構化,此問題仍有待研究。

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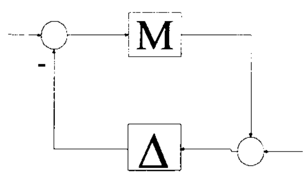

Figure 1