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Abstract

This paper investigates a potential methodology of the control law design for relative trajectories about a Keplerian near-circular orbit with applications to the formation flight of spacecraft. We first consider a spacecraft formation about a nominal Keplerian orbit, whose dynamics is usually described by the Tschauner-Hempel Equation (T-H Equation). Briefly reviewing the results from the T-H Equation, we analytically prove the applicability of the "local time approximation", which has been shown perform well in controlling the formation about the halo orbits, to the T-H Equation. With the guidance of local time approximation, we propose potential design methods of control law both in the time domain and in the true-anomaly domain. However, the cost seems too large for a micro/nano satellite. We come up with a new methodology that derives spacecraft relative dynamics with full solutions in the two-body problem. This result can be used for lowering down the fuel usage for relative orbit maintenance. Numerical simulations are also presented to verify our results.

Keywords：formation flight, orbit control, local time approximation, inteferometric imagine

1. Introduction

Formation flight of spacecraft is recently of great interest among scientists, especially with application to interferometric imaging of space or the Earth. Interferometric imagine is performed by measuring the mutual intensity (the two point correlation [1-4]) that results from the collection and subsequent interference or two electric field measurements of a target made at two different observation point, with which a 2-D or 3-D mapping can be generated. Several space missions using interferometry for scientific observation have been carried out or under planning all over the world since 1978, such as TOPSAT by NASA. However, those missions were accomplished by a single spacecraft. A more efficient way to perform interferometric imagine is the usage of spacecraft formation. Results of this study will contribute to possible application of formation flight techniques about a Low Earth circular orbit.

The current state of art in investigating formation flight around the Earth is to solve the linearized dynamics, such as the Clohessy-Wiltshire Equation (C-W Equation) [5] or the Tschauner-Hempel Equation (T-H Equation) [6], dealing with the relative motion about a circular or elliptic orbit, respectively. Many researchers have been devoted themselves into formation investigation in different areas, as shown in Ref. [7 – 13]. However, those scientists investigated formation problems around an Earth orbit usually approached by looking for natural periodic solutions. No methodology regarding control law design was proposed so far. The linearized model described by the T-H Equation is time varying. According, we would like to apply the "local time approximation", developed for the control of relative motion about a halo orbit by Hsiao and Scheeres, to guide out thinking, and develop a feasible control law in the spacecraft formation about a Keplerian orbit.

In this paper we try to investigate the feasibility of applying local time approximation to the spacecraft formation problem about a Keplerian orbit, and develop control law to stabilize the unstable trajectories. To apply the local time approximation, we first need to show that our problem satisfies certain assumptions detailed in Ref. [11, 12]. Having proved this, we then investigate possible control laws, one of which will generate a trajectory of the "scaled" nominal orbit.

In detail, we first consider a spacecraft formation about a nominal Keplerian orbit, whose dynamics described by the T-H Equation. Then the applicability of the "local time approximation is analytically proved, and utilized to guide our thinking of control law design. We also propose potential control law investigated both in the time domain and in the true-anomaly domain. By designing the control law in the true-anomaly domain, we not only stabilize the unstable relative trajectory, but also "re-construct" the "scaled" nominal orbit for our formation of spacecraft. Numerical simulations are also presented to verify our results. Then we subtract two adjacent orbits in the two-body problem to get the relative motion, and simplify the results after several approximations. We can reduce the cost of controller by choosing initial condition. The results will later contribute to the formation flight of spacecraft about the Earth orbit with applications to interferometric imagining.

2. Dynamics Model

2.1 General Equations of Motion

A spacecraft formation problem is usually formulated as the problem of relative motion with a chief spacecraft on the nominal trajectory and a deputy spacecraft in the vicinity. Therefore, the linearization technique is usually applied to derive the dynamics of the relative motion of the deputy.

Consider a body-fixed frame attached on the chief spacecraft where *x* points from Earth to the spacecraft, *y* points along direction of the motion of the chief spacecraft, and *z* completes the triad, as shown in Fig. 1. Assume that the nominal trajectory is only governed by the Earth, meaning that the chief spacecraft moves in a Keplerian orbit. Given the eccentricity *e*, the true anomaly *f*, and the position vector of the chief spacecraft **R** at any time instant, we can write down the dynamics of the deputy spacecraft as [13]:

$$\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = 2n_{lc}\sqrt{1+e\sin f}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} + n_{lc}^2\begin{bmatrix} 3+e\cos f & -2e\sin f & 0 \\ 2e\sin f & e\cos f & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{1}$$

where $n_{lc} = \sqrt{\mu / R^3}$ is the *local mean motion* and $\delta\mathbf{r} = (x, y, z)$ denotes the relative position of the deputy spacecraft. By setting $e = 0$, Eq.(1) degenerates to the Clohessy-Wiltshire Equation, which describes the relative motion about a circular orbit. Generally speaking, Eq. (1) does not have a closed-form solution, except for $e = 0$.

2.2 The Tschauner-Hempel Equation

Instead of time, the mathematics model in the Tschauner-Hempel Equation uses the true anomaly *f* as the independent variable, and therefore implicit in time. Provided the deputy excursion $\delta\bar{\mathbf{r}} = (\bar{x}, \bar{y}, \bar{z})$, the T-H Equation are:

$$\begin{aligned} &\bar{x}'' - 2\bar{y}' - \frac{3\bar{x}}{(1+e\cos f)} = 0 \\ &\bar{y}'' + 2\bar{x}' = 0 \\ &\bar{z}'' + \bar{z} = 0 \end{aligned} \tag{2}$$

where $(')$ and $('')$ denotes the derivatives with respect to *f*. Actually, Eq. (1) and Eq. (2) can be transformed to each other through the transformation: $\delta\mathbf{r} = R\delta\bar{\mathbf{r}}$, where $R = \|\mathbf{R}\|$ is the distance of the between the chief spacecraft to the center of the planet. The symmetry in the force potential term makes the T-H Equation easier to analyze, and thus, Eq. (2) has a general analytic solution [6, 7, 8]:

$$\bar{x}(f) = c_1\cos f(1+e\cos f) + c_2\sin f(1+e\cos f) + \frac{2c_3}{\eta^2}\left[1 - \frac{3e}{2\eta^3}\sin f(1+e\cos f)M\right] \tag{3}$$

$$\bar{y}(f) = -c_1\sin f(2+e\cos f) + c_2\cos f(2+e\cos f) - \frac{3c_3}{\eta^5}(1+e\cos f)^2 M + c_4 \tag{4}$$

$$\bar{z}(f) = c_5\cos f + c_6\sin f \tag{5}$$

where c_1 to c_6 are coefficients determined by initial conditions, $\eta = \sqrt{1-e^2}$, and $M = nt = E - e\sin E$ is the Mean anomaly.

3. Local Time Approximation

3.1 Local Time Approximation

The "local time approximation" was first proposed by Hsiao and Scheeres [11, 12] to solve the control problem of spacecraft formation about an unstable halo orbit. The spirit of the local time approximation is to treat a time varying system as locally time invariant, and the control law is design based on the local time-invariant system. This algorithm has been shown valid and robust in controlling the formation about an unstable halo orbit, and in designing the orbit of the constellation.

Consider a linear, time varying, periodic dynamics, formulated by

$$\dot{\mathbf{x}} = A(t)\mathbf{x} \tag{6}$$

where $\mathbf{x}(t)$ is the states and $A(t)$ is periodic with the period T, i.e.,

$$A(t+T) = A(t)$$

According to the linear theory [14], the solution to the Eq. (6) can be written as $\mathbf{x}(t) = \Phi(t,t_0)\mathbf{x}(t_0)$, where $\Phi(t,t_0)$ is the state transition matrix (STM). Given the system periodic, by application of Floquet theorem, the analysis of motion on the order of an orbit period or longer can be written as the product of two matrices:

$$\Phi(t,t_0) = P(t-t_0)\exp((t-t_0)D) \tag{7}$$

where P is a periodic matrix of period T and D is a constant matrix which has, as its eigenvalues, the characteristic exponents of the periodic orbit over one orbital period.

For every small time interval δt satisfying $0 < \delta t < \Delta t << T$, the local STM can be approximated as

$$\Phi(t_i + \delta t, t_i) \approx I + A(t_i)\delta t + \cdots \tag{9}$$

provided $\|A(t_i)\| >> \|A'(t_i)\delta t\|$, where $(')$ denotes the derivative of $A(t)$ with respect to *t*. Equation (9) implies that the relative motion of the spacecraft over a short time span centered at t_i can be understood by analyzing the eigenvalues and eigenvectors of the matrix $A(t_i)$ [11, 12] if the variation of the dynamics matrix with time is small enough within the short time interval.

3.2 Applicability to the Keplerian Formation

To verify the applicability of the local time approximation to the dynamics of spacecraft formation about a Keplerian orbit, we have to examine the time variation of $A(t_i)$ over a short time interval. The complexity of Eq. (1), however, makes the analytical analysis impossible. To do the verification, we turn to the T-H Equation, given by:

$$\delta\bar{\mathbf{r}}'' - 2J\delta\bar{\mathbf{r}}' - V\delta\bar{\mathbf{r}} = 0 \tag{10}$$

where $\delta\bar{\mathbf{r}} = (\bar{x}, \bar{y}, \bar{z})$, and

$$J = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} \dfrac{3}{1+e\cos f} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Accordingly, the dynamics matrix can be expressed as

$$A(f) = \begin{bmatrix} 0 & I \\ V & 2J \end{bmatrix}_{6\times 6}$$

where $A(f)$ is periodic with period $f = 2\pi$. Then, Eq. (10) can be written as a standard form:

$$\bar{\mathbf{x}}' = A(f)\bar{\mathbf{x}} \tag{11}$$

where $(')$ denotes the derivatives with respect to *f*. We can find that the Eq. (11) is similar to

Eq. (6). Therefore, we can view the variable t in Eq. (6) as a free parameter, and apply the same analysis in the preceding section to Eq. (11) simply by letting $t = f$. Moreover, all norms are equal since $A(f)$ is finite dimension, we select the $\infty-$norm of a matrix in practical computation. Given $A(f)$ in Eq. (11) and $0 < e < 1$, the norm of $A(f_i)$ is computed as

$$\left\|A(f_i)\right\|_\infty = 2 + \frac{3}{1+e\cos f_i} \tag{12}$$

On the other hand, we should compare the above result with $\left\|A'(f_i)\delta f_i\right\|_\infty$. Given A(f) from Eq. (10), the derivative of $A(f)$ with respect to f is given by

$$A'(f) = \begin{bmatrix} 0 & 0 \\ V' & 0 \end{bmatrix}$$

where

$$V' = \begin{bmatrix} \frac{3e\sin f}{(1+e\cos f)^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thereofore,

$$\left\|A'(f_i)\delta f_i\right\|_\infty = \frac{3e\left|\sin f_i \delta f_i\right|}{(1+e\cos f_i)^2} \tag{13}$$

Given $\left|\delta f_i\right| << 1$, we conclude $\left|\sin f_i \delta f_i\right| << 1$. Let $\left|\sin f_i \delta f_i\right| = \varepsilon$. Then

$$\left\|A'(f_i)\delta f_i\right\|_\infty = \frac{3e\varepsilon}{(1+e\cos f_i)^2}$$

Evaluating $\left\|A(f_i)\right\| - \left\|A'(f_i)\delta f\right\|$ analytically, we have

$$\left\|A(f_i)\right\|_\infty - \left\|A'(f_i)\delta f\right\|_\infty = 2 + \frac{3(1+e\cos f_i - e\varepsilon)}{(1+e\cos f_i)^2} \tag{14}$$

As long as $e << 1$, we conclude that $1+e(\cos f_i - \varepsilon) = 1 \pm \bar{\varepsilon}$, where $\bar{\varepsilon} << 1$. Thus, Eq. (14) can be appromated as

$$\left\|A(f_i)\right\|_\infty - \left\|A'(f_i)\delta f\right\|_\infty = 2 + \frac{3(1 \pm \bar{\varepsilon})}{(1+e\cos f_i)^2} >> 0 \tag{15}$$

As a result, $\left\|A(f_i)\right\| >> \left\|A'(f_i)\delta f\right\|$, implying that the local time approximation is applicable to our system given the system variable *f*.

In practical issue, however, we usually deal with a dynamical system under time domain, in which the system variable is the time t. Hence, we have to consider the influence of variance in *t* to the variance in *f*. Recall that $df/dt = n_{lc}\sqrt{1+e\cos f}$, as given in Eq. (1). n_{lc} is the local mean motion and usually has the order of 10^{-3} to 10^{-5}, from the low Earth orbit to the geosynchronous orbit. Letting $\delta f = n_{lc}\sqrt{1+e\cos f}\,\delta t$, we conclude that $\left|\delta f\right| << 1$ as long as $\left|\delta t\right| << 1$. As a result, the algorithm of local time approximation is concluded applicable to the Keplerian orbit.

4. Control Law Design

4.1 Control law Design in Time Domain

Having proved that the local time approximation is applicable to the Keplerian orbit with small eccentricity, the algorithm shown in Ref. [10] can thus be utilized to guide the thought of control law design..

According to the local time approximation, during the time interval $t_i - \delta t \le t_i \le t_i + \delta t$, $|\delta t| << 1$, the system can be formulated as locally linear-time-invariant, given by Eq. (1) with $t = t_i$. In Ref. [10], a position-and-velocity feedback controller, given by

$$\mathcal{T}(t_i) = -2\varpi(t_i) J \delta\dot{\mathbf{r}} - V_{cr}(t_i)\delta\mathbf{r},$$

is designed to null out the Coriolis force and generate a non-Coriolis environment in the rotational system. The controller is applied in the following way

$$\dot{\mathbf{x}} = A(t_i)\mathbf{x} + \mathcal{T}(t_i)$$

such that the controlled equations of motion are then in the following form:

$$\delta\ddot{\mathbf{r}} - V_{des}(t_i)\delta\mathbf{r} = 0 \tag{17}$$

where the force potential matrix $V_{des}(t_i)$ contains the desired oscillating modes. Accordingly, the position feedback in the controller can be obtained as

$$V_{cr}(t_i) = V(t_i) - V_{des}(t_i) \tag{18}$$

By making $V_{des} = diag(-\omega_1^2,\ -\omega_2^2,\ -\omega_3^2)$, we can create a system with three oscillators and arbitrary trajectories [10]. A relative circular formation can be achieved by designing a controller with all the oscillating frequencies the same, i.e., $V_{des}(t_i) = -\varpi^2\mathbf{I}$, and with proper selection of initial conditions.

4.2 Control Law Design in T-H Equation

In addition to do the trajectory control in the time domain, an alternate approach is from the T-H Equation. The T-H Equation describes the relative motion about a Keplerian orbit, and there exist analytic solutions to the equation, as shown in Eqs. (3) – (5). However, we should notice that the solutions in Eqs. (3) – (5) are normalized about the nominal position. Thus, the full solutions must be multiplied by the nominal position, $R(t)$. Given the controller

$$\overline{\mathcal{T}}(f_i) = -2J\delta\overline{\mathbf{r}}' - V_{cr}(f_i)\delta\overline{\mathbf{r}}, \tag{19}$$

the closed-looped system can be written as $\overline{\mathbf{x}}' = \overline{A}(f_i)\overline{\mathbf{x}} + \overline{\mathcal{T}}(f_i)$ or $\delta\overline{\mathbf{r}}'' - V_{des}(f_i)\delta\overline{\mathbf{r}} = 0$, provided $V_{des}(t_i) = -\mathbf{I}$. The relative motion of the deputy spacecraft can be analytically solved as

$$\begin{aligned} \overline{x}(f) &= c_1\cos f + c_2\sin f & \qquad & x(f) = c_1R\cos f + c_2R\sin f \\ \overline{y}(f) &= c_3\cos f + c_4\sin f & \text{or} \qquad & y(f) = c_3R\cos f + c_4R\sin f \\ \overline{z}(f) &= c_5\cos f + c_6\sin f & \qquad & z(f) = c_5R\cos f + c_6R\sin f \end{aligned}$$

There are some interesting points in the above choice of control law. First of all, by properly selecting the initial conditions we can generate a formation about the y-z plane. Moreover, with the participation of $R(t)$ the nominal trajectory can be "re-constructed" in this formation, by selecting $c_4 = c_5 = 0$ and $c_3 = c_6 = y_0$.

5. FULL APPROACH

In this section we are going to discuss the so-called "full approach" that obtains a natural, periodic, relative trajectory from full solutions instead of integration from a linearized model. The costs of above controllers are too large. We can lower down the cost by choosing proper initial conditions.

5.1 Natural Periodic Solution

In the two body problem there exist natural periodic trajectories, the elliptical and circular orbits, and their periods are determined by the semi-major axes. Taking the geocentric inertial frame into account, we can view the uncontrolled formation as spacecraft orbiting in two neighboring elliptical orbits with same period. Under this situation, the relative motion of the deputy spacecraft to the chief spacecraft would naturally be periodic.

To be consistent with proceeding discussion, we would also like to investigate this algorithm in the rotational frame attached with the chief spacecraft. We first formulate the motion of chief spacecraft in the geocentric rotational frame whose rotation rate equals to that of chief spacecraft orbiting the Earth, meaning that $\mathbf{R}_{rc} = (R,\ 0,\ 0)$, where the subscript "*rc*" means rotational frame with the chief. Similarly, we can formulate the motion of the deputy spacecraft as $\mathbf{r}_{rd} = (r,\ 0,\ 0)$ in the geocentric rotational frame whose rotation rate equals to that of deputy spacecraft orbiting the Earth.

Then we can transform the position vector into the geocentric inertial frame with rotation matrices, $[\mathrm{R}_1]$ and $[\mathrm{R}_2]$, such that

$$\mathbf{R}_I = [\mathrm{R}_1]\mathbf{R}_{rc},\ \ \mathbf{r}_I = [\mathrm{R}_2]\mathbf{r}_{rd}$$

The composite rotation matrices $[\mathrm{R}_1]$ and $[\mathrm{R}_2]$ can be computed in the following way:

$$[\mathrm{R}_k] = R_z(\Omega_k)R_x(i_k)R_z(\omega_k + f_k) \qquad k = 1,2$$

where Ω, i, ω, and f denote *the longitude of ascending node*, *the inclination*, *the argument of periapsis*, and *the true anomaly*, respectively, and the rotation matrices R_x and R_z are defined as:

$$R_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix},\ \ R_z(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then, the relative motion in the inertial frame is:

$$\delta\mathbf{r}_I = [\mathrm{R}_2]\mathbf{r}_{rd} - [\mathrm{R}_1]\mathbf{R}_{rc}$$

By pre-multiplying the inverse of $[\mathrm{R}_1]$, we can obtain the relative motion in the rotation frame centered at the chief spacecraft, as usually discussed:

$$\delta\mathbf{r} = [\mathrm{R}_1]^{-1}[\mathrm{R}_2]\begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix} \tag{20}$$

To do formation flight, the relative trajectory must be periodic, implying that the original two orbits must have the same periods. In two-body problem, this leads to two spacecraft staying in the orbits with identical semi-major axis. Moreover, for future use we define $\Delta\Omega = \Omega_1 - \Omega_2$, $\Delta i = i_1 - i_2$, $\Delta\omega = \omega_1 - \omega_2$ and $\Delta e = e_1 - e_2$. Here we also assume that $|\Delta\Omega| \ll 1$, $|\Delta i| \ll 1$, $|\Delta\omega| \ll 1$ and $|\Delta e| \ll 1$, leading to $\|\delta\mathbf{r}\| \ll 1$, so that we can further simplify our result. To distinguish the regular difference from the a small amount, we change the capital "Δ" to the small "δ" in the orbit elements in the following derivation. Equation (20) then can be simplified as:

$$\delta\mathbf{r} = r\begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} - \begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix} \tag{21}$$

where,

$$\Delta x = \cos(f_1 - f_2) - \delta\omega\sin(f_1 - f_2) + \delta\Omega[\cos i_2 \cos(\omega_1 + f_1)\sin(\omega_2 + f_2) - \cos i_1 \sin(\omega_1 + f_1)\cos(\omega_2 + f_2)]$$

$$\Delta y = -\sin(f_1 - f_2) - \delta\omega\cos(f_1 - f_2) + \delta\Omega[\cos i_2 \sin(\omega_1 + f_1)\sin(\omega_2 + f_2) - \cos i_1 \cos(\omega_1 + f_1)\cos(\omega_2 + f_2)]$$

$$\Delta z = -\delta i \sin(\omega_2 + f_2) + \delta\Omega \sin i_1 \cos(\omega_2 + f_2)$$

5.2 **Application to Formation about Circular Orbits**

Equation (21) gives an analytical expression of relative motion about an elliptical orbit with a small initial excursion. Without lost of generality, we let $\omega_1 = \pi/2$ to simplify the derivation. Also, consider a simpler case, $\delta\Omega = 0$ and $\delta\omega = 0$, and both orbits with the same epoch time, t_p. Equation (21) then becomes:

$$\delta\mathbf{r} = r\begin{bmatrix} \cos f_1 \cos f_2 + \sin f_1 \sin f_2 \\ -\sin f_1 \cos f_2 + \cos f_1 \sin f_2 \\ -\delta i \cos f_2 \end{bmatrix} - \begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix} \tag{22}$$

We can further simplify Eq. (22) by replacing r and f_2 with R, r, e_1, and e_2. First of all, in the two body problem r is given by $r = a(1-e_2^2)/(1+e_2\cos f_2)$. Moreover, we transform the above results to the eccentric anomaly. Hence,

$$r\cos f_2 = a(\cos E_2 - e_2) \quad , \quad r\sin f_2 = a\sqrt{1-e_2^2}\sin E_2$$

Let $E_2 = E_1 + \delta E$, and we can show that $|\delta E| \ll 1$ if $|\Delta e| \ll 1$ (See Appendix II). As a result,

$$\cos E_2 \approx \cos E_1 - \sin E_1 \delta E, \quad \sin E_2 \approx \sin E_1 + \cos E_1 \delta E$$

where, δE can be replaced with the following relation (See Appendix II):

$$\delta E = \frac{(e_2 - e_1)\sin E_1}{1 - e_2 \cos E_1} \tag{23}$$

Plugging the approximation into Eqs. (20)-(22) yields,

$$x = R\{\frac{(e_1 - e_2)(1 + e_1\cos f_1)\cos f_1}{1 - e_1^2} + \frac{[\sqrt{(1-e_2^2)(1-e_1^2)} - (1 - e_1 e_2)]\sin^2 f_1}{1 + e_1\cos f_1 - e_2\cos f_1 - e_1 e_2}\} \tag{26}$$

$$y = R\sin f_1\{\frac{(e_1 - e_2)(1 + e_1\cos f_1)}{1 - e_1^2} + \frac{[\sqrt{(1-e_2^2)(1-e_1^2)} - (1 - e_1 e_2)]\cos f_1 - (e_1 - e_2)}{1 + e_1\cos f_1 - e_2\cos f_1 - e_1 e_2}\} \tag{27}$$

$$z = R\delta i\{\cos f_1 \frac{(e_1 - e_2)(1 + e_1\cos f_1)}{1 - e_1^2} + \frac{(e_1 - e_2)\sin^2 f_1}{1 + e_1\cos f_1 - e_2\cos f_1 - e_1 e_2}\} \tag{28}$$

Since the nominal trajectory is circular, leading to $e_1 = 0$, and eventually $\sqrt{(1-e_2^2)}$ is approximated by

$$\sqrt{(1-e_2^2)} = 1 - \frac{1}{2}e_2^2 + \ldots$$

We notice that $R\delta i$ is actually the initial excursion in the z direction, z_0, in the C-W Equation. By letting $-Re_2 = x_0$, then Eqs. (26) to (28) are approximated as:

$$x = x_0[\cos f_1 + \frac{\frac{1}{2}e_2\sin^2 f_1}{1 - e_2\cos f_1}] \tag{29}$$

$$y = -x_0[2\sin f_1 + \frac{\frac{1}{2}e_2\cos f_1 \sin f_1}{1 - e_2\cos f_1}] \tag{30}$$

$$z = z_0[\cos f_1 - e_2 + \frac{e_2\sin^2 f_1}{1 - e_2\cos f_1}] \tag{31}$$

The first term of our result agree with the solution from the C-W Equation. This proves the validity of our derivation. In addition, we obtain a higher order term through this procedure so that we can predict the trajectory more precisely. Another advantage of our methodology is that we don't have to solve a differential equation as the C-W Equation did. This simplify the computation and improve accuracy.

5. Numerical Simulation

Figures 2 – 4 give some numerical simulation about the algorithm derived previously. The original excursions in both cases are about 50 m, leading to the spread of the whole formation being 100 m. We can see that the formation trajectory is similar to the original nominal trajectory, but with scaled sizes. Costs of formation of a single spacecraft are also presented in Figs. 4 and 7, and from the plots we can see that they are expensive for small satellites.

As shown in Fig. 5, the true trajectory with initial conditions from the C-W Equation drifts away although they remain circular in the linear version. On the contrary, the true trajectory with initial conditions from the full approach remain circular even under nonlinear circumstance. Figure 6 gives the error analysis of the two methodologies. We define the error between true trajectory and predicted trajectory as $\varepsilon = \left\| \delta\mathbf{r}_t - \delta\mathbf{r}_p \right\|$, where $\delta\mathbf{r}_t$ denotes the true trajectory and $\delta\mathbf{r}_p$ denotes the predicted trajectory by the C-W Equation or the full approach methodology. We can see that the largest error generated in our prediction is only half of that generated in the C-W Equation. The two examples are simulated with initial 70 km excursion in the $y-z$ plane for interferometric image purpose.

6. Conclusions

The control law design for relative trajectories about a Keplerian near-circular orbit with applications to the formation flight of spacecraft is investigated in this paper. Starting from results of the Tschauner-Hempel Equation (T-H Equation), we analytically prove the applicability of the "local time approximation", which has been shown perform well in controlling the formation about the halo orbits, to the T-H Equation. We then come up with two potential design methods of control law, one in the time domain and the other in the true-anomaly domain. The control law will result in a three dimensional oscillator so that a general trajectory can be generated. By designing the control law in the true-anomaly domain, we not only stabilize the unstable relative trajectory, but also "re-construct" the "scaled" nominal orbit for our formation of spacecraft. Numerical simulations are also presented to verify our results. We also conclude that the cost is reasonable for the formation flight of regular satellites from the numerical results.

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8.Appendix I

Given $\delta\mathbf{r} = R\delta\bar{\mathbf{r}}$, we have

$$\delta\dot{\mathbf{r}} = \dot{R}\delta\bar{\mathbf{r}} + R\delta\dot{\bar{\mathbf{r}}} \tag{32}$$

However, we can rewrite Eq. (32) as

$$\delta\dot{\mathbf{r}} = \dot{f}\left(R'\delta\bar{\mathbf{r}} + R\delta\bar{\mathbf{r}}'\right) \tag{33}$$

We can explicitly derive $\dot{f}$ and R' as

$$\dot{f} = \frac{h}{R^2} \quad , \quad R' = \frac{d}{df}\left(\frac{p}{1+e\cos f}\right) = \left(\frac{p}{1+e\cos f}\right)\left(\frac{e\sin f}{1+e\cos f}\right) = RF$$

where $F = e\sin f/(1+e\cos f)$

Therefore, we can establish the transformation matrix T, given by

$$T = \begin{bmatrix} R\mathbf{I} & \mathbf{0} \\ \dot{f}RF\mathbf{I} & \dot{f}R\mathbf{I} \end{bmatrix} \tag{34}$$

Similarly, we can obtain T' by taking the derivative of T with respect to f:

$$T' = \begin{bmatrix} RF\mathbf{I} & \mathbf{0} \\ \dot{f}R\bar{F}\mathbf{I} & -\dot{f}RF\mathbf{I} \end{bmatrix} \tag{35}$$

where $\bar{F} = e\cos f/(1+e\cos f)$.

Appendix II

To investigate the the change in eccentric anomaly due to the change in eccentricity, we first look into the Kepler's equation:

$$M_e = n(t - t_p) = E - e\sin E$$

There is no closed-form solution for eccentric anomaly E in Kepler's equation. However, this equation can be solved with a power series of *e* [15]:

$$E = M_e + \sum_{n=1}^{\infty} a_n e^n \tag{36}$$

Equation (34) is named "Lagrange Series" and converges when $e < 0.662743419$. To simplify the derivation but still keep the accuracy, we take the first three terms of the series into consideration, given by,

$$E = M_e + e\sin M_e + \frac{e^2}{2}\sin 2M_e + \frac{e^3}{8}(3\sin 3M_e - \sin M_e)$$

In the preceding section we have selected two adjacent orbits with the same t_p. Thus, at any instant *t* the two spacecraft would have the same M_e. Define $\delta E = E_2 - E_1$ and we obtain

$$\delta E = (e_2 - e_1)\sin M_e + \frac{e_2^2 - e_1^2}{2}\sin 2M_e + \frac{e_2^3 - e_1^3}{8}(3\sin 3M_e - \sin M_e) \tag{37}$$

Since the magnitude of the sinusoidal function is no larger than 1, $|\delta E|$ is of the same order as $e_2 - e_1$. As a result, we conclude $|\delta E| \ll 1$ if $|e_2 - e_1| \ll 1$, which has been assumed previously.

With a very small δE, we can re-express E_2 in terms of E_1 and δE, and approximate $\sin E_2$ by Taylor series expansion:

$$\sin E_2 = \sin(E_1 + \delta E) = \sin E_1 + \cos E_1 \delta E + O(\delta E^2) + \ldots \tag{38}$$

Moreover, the equality of mean motion between two spacecraft gives us another relation in E_1, E_2, e_1, e_2:

$$M_e = E_1 - e_1\sin E_1 = E_2 - e_2\sin E_2 \tag{39}$$

Plugging Eq. (36) into Eq. (37) and doing algebraic manipulations, we otain

$$\delta E = \frac{(e_2 - e_1)\sin E_1}{(1 - e_2\cos E_1)} \tag{40}$$

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9. Chart arrangement

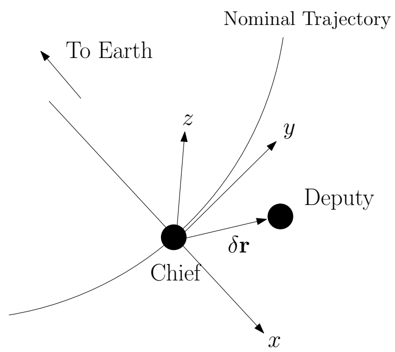
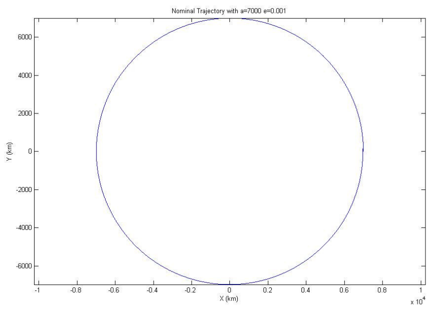


Fig.1 A cartoon showing how the coordinates are defined, and the relative positions between the chief and deputy spacecraft.

a)



b)

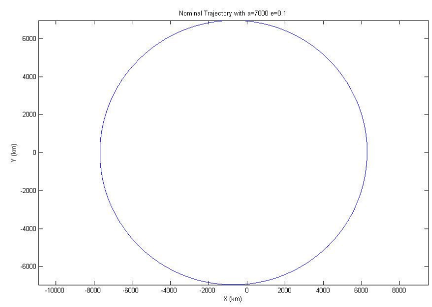


Fig.2 A nominal orbit with semi-major axis of 7000 km, and a) eccentricity of 0.001 b) eccentricity of 0.1 is used to simulate the formation flight.

a)

b)

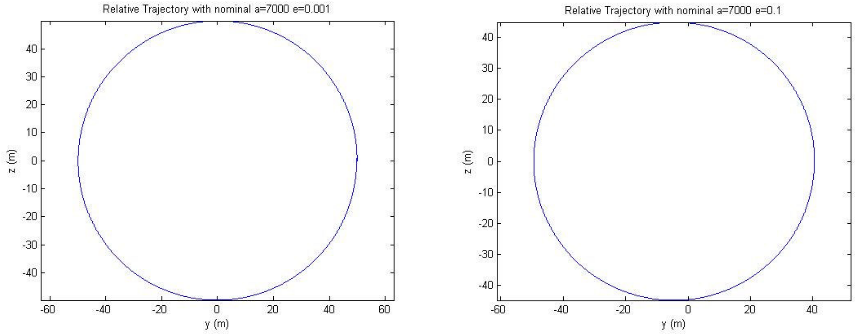


Fig.3 A spacecraft formation with initial excursion of 50 m from the nominal is simulated. The nominal orbit is shown in Fig. 2 (a) and (b), respectively, and the formation is simulated for one orbit period.

a) b)

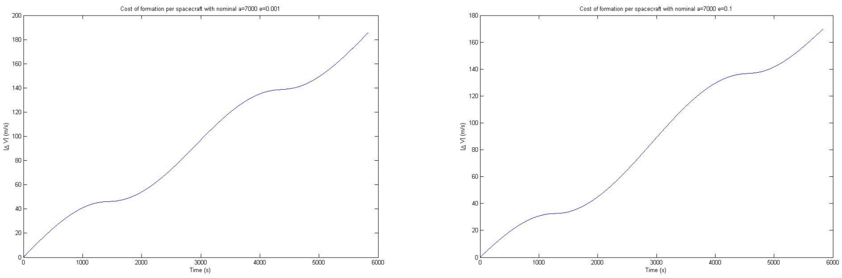


Fig.4 The cost of formation with initial excursion of 50 m from the nominal is simulated. The nominal orbit is shown in Fig. 2 (a) and (b), respectively, and the formation is simulated for one orbit period.

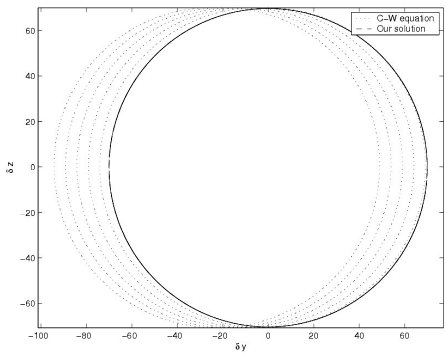


Fig.5 The real trajectories, integrated with original nonlinear model, are shown to compare the validity of the C-W Equation with that of the full

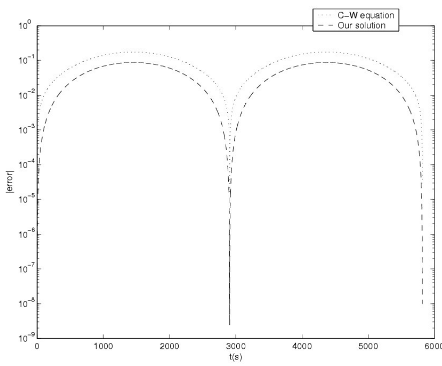


Fig6 The error between predicted trajectories and the real trajectories, integrated with original nonlinear model, are shown to compare the validity of the C-W Equation with that of the full approach. The simulation is run for one period.