1 Introduction

Traditionally, life test plans are developed based on both the produce and consumer risks. The produce risk denotes the probability that a good lot of items will be rejected. The consumer risk denotes the probability of accepting a lot of items with poor quality. Based on the producer and consumer risks, some life test plans were developed as follows: Epstein and Sobel [10] constructed life test plans for exponential distribution with type-II censoring. Epstein [9] proposed a hybrid life test plan which combines the type-I censoring and the type-II censoring. Jeong and Yum [12] developed life test plans for exponential distribution with type-I censoring. Kim and Yum [13] did comparisons of different designs of the exponential life test plan with intermittent inspections. Wu and Tsai [19] developed the design of truncated life test plan for Birnbaum-Saunders distribution. Tsai and Wu [16] proposed the design of truncated life test plan for generalized Rayleigh distribution.

The type-I and type-II censoring schemes, however, do not allow removing surviving items at the times other than the termination time of the life test. This allowance, however, may be desirable when a compromise between reducing test time and an expectation of some extreme lifetimes in life test can be sought. This reason motivates the quality personnel to adopt a progressively censoring scheme. A comprehensive reference regarding the progressive censoring, its applications and mathematical results can be found in Balakrishnan and Aggaarwala [3]. The parameter estimations of the exponential, lognormal and Weibull lifetime models with progressive type-I censoring have discussed by Cohen [7] and [8], and Wong [18], respectively. The problems of simulation, estimation, calculation of moments and the construction of a life test plan with progressive type-II censoring have been discussed by Aggarwala and Balakrishnan [2], Balakrishnan and Sandhu [4], Balakrishnan and Saw [5], Cacciari and Montanari [6], Tse and Yuen [15], Viverps and Balakrishnan [17] and Yuen and Tse [20], respectively.

It is easier to develop the statistical works with type-II censored data than using the type-I censored data. But the termination time of a life test under type-II censoring is randomly and cannot be predetermined. Moreover, the quality personnel may inspect the life test only at some specific times so that the exact failure times cannot be observed. These reasons encourage quality personnel to conduct a life test with the type-I interval censoring which only records the failure numbers in some predetermined time intervals. Based on the administrative convenience and a compromise between reducing test time and the allowance to remove surviving items during the life test, the study is thus motivated to develop life test plans under the progressively type-I interval censoring with intermittent inspections. Basically, the construction of the proposed life test plan is easy to practitioners.

2 Progressively Type-I Interval Censored Life Test Plans

2.1 Progressively Type-I Interval Censored test

Assume that the lifetimes of items $Y_1, Y_2, \cdots$ are independent and identically exponentially distributed with the following probability density function (p.d.f.) and cumulative density

function (c.d.f.), respectively

$$\begin{aligned} f(y|\eta) &= \frac{1}{\eta} exp\left(-\frac{y}{\eta}\right), \\ F(y|\eta) &= 1 - exp\left(-\frac{y}{\eta}\right),\ y > 0,\ \eta > 0, \end{aligned} \tag{1}$$

where η is the scale parameter and the mean lifetime of items. Assume that a lot of N items taken from (1) are submitted to an inspection of acceptance sampling. The lot is accepted if the mean lifetime η is longer than or equal to η_0, and rejected otherwise, where η_0 is a predetermined level. In the development of the life test, we can transform the lifetime data by letting $T_i = Y_i/\eta_0$, $i = 1, 2, \cdots, N$ so that the transformed data do not depend on the measuring scale. Let $\theta = \eta/\eta_0$, it can be shown that $T_1, T_2, \cdots, T_N$ are independent and identically exponentially distributed with parameter θ.

The progressively type-I interval censored test can be conducted as follows: Assume that $m(\geq 2)$ inspection times $0 < \tau_1 < \tau_2 < \cdots < \tau_m < \infty$ are predetermined and n items are drawn randomly from the lot and put on a life test at the initial time 0. At the time τ_1, X_1 failure items in the interval $(0, \tau_1]$ are recorded, and R_1 of the $n - X_1$ surviving items are selected randomly and removed. Continuing on the test, at the time τ_2, X_2 failure items in the interval $(\tau_1, \tau_2]$ are recorded, and R_2 of the $n - X_1 - X_2 - R_1$ surviving items are selected randomly and removed. Finally, at the censoring time τ_m, X_m failure items in the interval $(\tau_{m-1}, \tau_m]$ are recorded, and all surviving numbers $R_m = n - \sum_{j=1}^{m} X_j - \sum_{j=1}^{m-1} R_j$ are removed. Then the test is stopped.

In the life test, either the proposed values of $R_1, R_2, \cdots, R_{m-1}$, or probabilities of units removed $p_1, p_2, \cdots, p_{m-1}$ are determined in advance so that $p_m = 1$ and $R_i = [(n - \sum_{j=1}^{i} X_j - \sum_{j=1}^{i-1} R_j)p_i]$, $i = 1, 2, \cdots, m$, where $[s]$ is the largest integer smaller than or equal to s.

2.2 Ordinary Life Test Plans

Let $\boldsymbol{R} = (R_1, , R_2, \cdots, R_m)$ and $\boldsymbol{X} = (X_1, X_2, \cdots, X_m)$. Let $B(a, b)$ denote the binomial distribution with the number of individuals a and the probability of success b. Let $\nu_1 = n$, $\delta_1(\theta) = F(\tau_1|\theta)$, $\nu_i = n - \sum_{j=1}^{i-1}(X_j + R_j)$ and $\delta_i(\theta) = \frac{F(\tau_i|\theta) - F(\tau_{i-1}|\theta)}{1 - F(\tau_{i-1}|\theta)} = 1 - exp(-\frac{\tau_i - \tau_{i-1}}{\theta})$, $i = 2, 3, \cdots, m$. It can be shown that

$$\begin{aligned} &X_1 \sim B(\nu_1, \delta_1(\theta)) \\ &X_i|X_{i-1}, ..., X_1, R_{i-1}, ..., R_1 \ \sim B\left(\nu_i, \delta_i(\theta)\right). \end{aligned} \tag{2}$$

The statistical hypotheses regarding the acceptance sampling plan can be described as

$$H_0 : \eta = \eta_0 \text{ vs } H_1 : \eta = \eta_a (< \eta_0)$$

or equivalently

$$H_0 : \theta = 1 \text{ vs } H_1 : \theta = \theta_a (< 1)$$

The θ_a is so-called the discrimination ratio. According to the computation procedure of Aggarwala [1], the likelihood function can be found as

$$\begin{aligned} L(\theta) &\propto \prod_{i=1}^{m}[F(\tau_i|\theta) - F(\tau_{i-1}|\theta)]^{x_i}[1 - F(\tau_i|\theta)]^{R_i} \\ &= \prod_{i=1}^{m}\left[1 - exp(-\frac{\tau_i - \tau_{i-1}}{\theta})\right]^{x_i} exp(-\frac{x_i\tau_{i-1} + R_i\tau_i}{\theta}), \end{aligned}$$

where $F(\tau_0|\theta) = 0$. In particular, the life test is conducted with intermittent inspections, that is, m inspections are equally-spaced such that the length of the i-th time interval $\tau_i - \tau_{i-1} = \tau$ and $\delta_i(\theta) = \delta(\theta) = 1 - exp(-\tau/\theta)$, $i = 1, 2, \cdots, m$. If $X_1 = n$ then $\ln(L(\theta))$ is maximized when θ approaches to zero, otherwise, if $R_m = n$, then $\ln(L(\theta))$ is maximized when θ approaches to infinity, otherwise the maximum likelihood estimator of θ can be obtained as

$$\hat{\theta} = \frac{\tau}{\ln[1 + \sum_{i=1}^{m} x_i/(\sum_{i=2}^{m}(i-1)x_i + \sum_{i=1}^{m} iR_i)]}\ . \tag{3}$$

The producer risk is defined as the probability that a lot of items with mean lifetime $\theta = 1$ is rejected, and the consumer risk is defined as the probability that a lot of items with mean lifetime $\theta = \theta_a$ is accepted. Both risks can be represented, respectively as

$$\begin{aligned} P_R &= P(\hat{\theta} < c|\theta = 1) = \sum_{R_e} P(\boldsymbol{x}|\boldsymbol{R}, \theta = 1), \\ P_C &= P(\hat{\theta} \geq c|\theta = \theta_a) = 1 - \sum_{R_e} P(\boldsymbol{x}|\boldsymbol{R}, \theta = \theta_a), \end{aligned}$$

where c is a constant called the critical value, and R_e is a set of $\boldsymbol{x} = (x_1, x_2, \cdots, x_m)$ corresponding to the rejection region defined by

$$R_e = \{\boldsymbol{x}|\sum_{i=1}^{m}(x_i + R_i) = n, \hat{\theta} < c\},$$

and $P(\boldsymbol{x}|\boldsymbol{R}, \theta)$ is the joint probability of $\boldsymbol{X} = \boldsymbol{x}$. Based on the maximum likelihood estimate of $\hat{\theta}$, the ordinary life test plan (n, c) can be developed so that the producer and consumer risks as follows are satisfied:

$$P_R = \alpha \ \text{ and } \ 1 - P_C = 1 - \beta, \tag{4}$$

where α and β are two constants between 0 and 1. The joint probability of $\boldsymbol{x}$ can be presented as

$$\begin{aligned} & P(\boldsymbol{x}|\boldsymbol{R}, \theta) \\ = \ & P(x_1|\theta)P(x_2|x_1, R_1, \theta) \times \cdots \times P(x_m|x_1, x_2, \cdots, x_{m-1}, R_1, R_2, \cdots, R_{m-1}, \theta) \\ = \ & \prod_{i=1}^{m}\binom{\nu_i}{x_i}\delta_i^{x_i}(\theta)(1 - \delta_i(\theta))^{\nu_i - x_i} = \prod_{i=1}^{m}\binom{\nu_i}{x_i}\delta(\theta)^{\sum_{j=1}^{m} x_j}(1 - \delta(\theta))^{\sum_{j=1}^{m}(\nu_j - x_j)}. \end{aligned} \tag{5}$$

Since the discreteness of the sample size n, it may not be possible to get a pair of (n, c) such that the equations in (4) are satisfied simultaneously. Alternatively, one can develop the ordinary life test plan based on the smallest n and the corresponding c so that the following inequalities are satisfied.

$$P_R \leq \alpha \text{ and } 1 - P_C \geq 1 - \beta. \tag{6}$$

Assume that the removals are given by $R_i = [(n - \sum_{j=1}^{i} X_j - \sum_{j=1}^{i-1} R_j)p_i]$, $i = 1, 2, \cdots, m-1$, where $R_0 = 0$. A searching procedure with the following steps is suggested to establish the ordinary life test plan:

Step 1 Specify the values of m, τ, α, β, θ_a and $p_1, p_2, \cdots, p_m$.

Step 2 Let $n = 3$.

Step 3 Generate all possible combinations of $\boldsymbol{x}$ and $\boldsymbol{R}$ based on (2) and compute the maximum likelihood estimate of θ for each combination of $\boldsymbol{x}$ and $\boldsymbol{R}$, and denoted by $\hat{\theta}$.

Step 4 The sets of $\{\hat{\theta}, P(\boldsymbol{x}|\boldsymbol{R}, \theta = 1)\}$ and the $\{\hat{\theta}, P(\boldsymbol{x}|\boldsymbol{R}, \theta = \theta_a)\}$ for all possible combinations of $\boldsymbol{x}$ constitute the p.d.f.'s of $\hat{\theta}$ under $\theta = 1$ and $\theta = \theta_a$, respectively. Hence, the c.d.f.'s of $\hat{\theta}$ under $\theta = 1$ and $\theta = \theta_a$ can be determined by accumulating the joint probabilities of $P(\boldsymbol{x}|\boldsymbol{R}, \theta = 1)$ and $P(\boldsymbol{x}|\boldsymbol{R}, \theta = \theta_a)$, respectively.

Step 5 If there exist c's for which equations in (6) are satisfied, then select the largest one among those c's and stop. The desired ordinary life test plan is determined. Otherwise, go to Step 6.

Step 6 Let $n = n + 1$, go to Step 3.

2.3 Approximate Life Test Plans

The ordinary life test plans are exact, however, the method may fail due to a memory overflow error especially for large m and n. The computation is complicated and time consuming. For get over this difficulty, an approximate method is proposed to find the approximate life test plans. Based on asymptotic theory of maximum likelihood estimator, we can approximate the sampling distribution of $\hat{\theta}$ by a normal distribution with mean θ and variance $I^{-1}(\theta)$, where $I(\theta)$ is the Fisher information (see Hogg and Craig [11]). Since

$$\sum_{i=1}^{m} iR_i = \sum_{i=1}^{m-1} iR_i + mR_m = \sum_{i=1}^{m-1} iR_i + m\left(n - \sum_{i=1}^{m} x_i - \sum_{i=1}^{m-1} R_i\right),$$

the Fisher information can be found as

$$\begin{aligned} I(\theta) &= -E\left[\frac{\partial^2 \ln L(\theta)}{\partial \theta^2}\right] \\ &= \left[\frac{\tau(1-\delta(\theta))}{\theta^3 \delta(\theta)}\right]\left[\frac{\tau}{\theta\delta(\theta)} - 2\right]\sum_{i=1}^{m} E(X_i) - \frac{2\tau}{\theta^3}\sum_{i=1}^{m}(m-i+1)E(X_i) \\ &\quad + \frac{2n\tau}{\theta^3}\left[m - \sum_{i=1}^{m-1}(m-i)\frac{R_i}{n}\right]. \end{aligned} \tag{7}$$

Using induction, the mean of X_i can be determined as

$$\begin{aligned} E(X_1) &= n\delta(\theta) \\ E(X_i|X_{i-1},\cdots,X_1,R_{i-1},\cdots,R_1) &= \delta(\theta)E(\nu_i) \\ &= n\delta(\theta)\left[1-\sum_{j=1}^{i-1}\frac{R_j}{n}(1-\delta(\theta))^{i-j-1}\right],\ i=2,3,\cdots,m. \end{aligned}$$

In practical situations, the life test plans should be established before sampling, the exact values of R_i may not be determined in advance. For overcoming the difficulty, we approximate X_i by its mean value in R_i as follows:

$$\begin{aligned} &\frac{R_1}{n} \simeq \left(1-\frac{E(X_1)}{n}\right)p_1 = (1-\delta(\theta))p_1, \\ &\frac{R_2}{n} \simeq \left(1-\frac{E(X_1)}{n}-\frac{n-E(X_1)-R_1}{n}\delta(\theta)-\frac{R_1}{n}\right) \simeq p_2(1-\delta(\theta)^2(1-p_1)), \\ &\cdots \\ &\frac{R_i}{n} \simeq p_i(1-\delta(\theta))^i\prod_{h=1}^{i-1}(1-p_h),\ i=2,3,\cdots,m-1. \end{aligned} \tag{8}$$

Let

$$\phi_1(\theta)=\delta(\theta),\ \phi_i(\theta)=\delta(\theta)\left[1-\sum_{j=1}^{i-1}p_j(1-\delta(\theta))^{2i-j-1}\prod_{h=1}^{i-1}(1-p_h)\right],\ i=2,3,\cdots,m,$$

$$G_1(\theta)=\left[\frac{\tau(1-\delta(\theta))}{\theta^3\delta(\theta)}\right]\left[\frac{\tau}{\theta\delta(\theta)}-2\right]\sum_{i=1}^{m}\phi_i(\theta),\ G_2(\theta)=\frac{2\tau}{\theta^3}\sum_{i=1}^{m}(m-i+1)\phi_i(\theta);$$

and let

$$G_3(\theta)=\frac{2\tau}{\theta^3}[2-(1-\delta(\theta))p_1],$$

if $m=2$,

$$G_3(\theta)=\frac{2\tau}{\theta^3}\left[m-(m-1)(1-\delta(\theta))p_1-\sum_{i=2}^{m-1}(m-i)p_i(1-\delta(\theta))^i\prod_{j=1}^{i-1}(1-p_j)\right],$$

if $m\geq 3$; and let

$$G(\theta)=G_1(\theta)+G_2(\theta)+G_3(\theta).$$

Approximate R_i/n by (8) in (7), it can be shown that $I(\theta)\simeq nG(\theta)$. Solving the equations in (4) simultaneously, the required sample size can be determined by

$$n\simeq\left[\frac{z_{1-\beta}/\sqrt{G(\theta_a)}+z_{1-\alpha}/\sqrt{G(1)}}{1-\theta_a}\right]^2,$$

where z_ζ is the ζ-th quantile satisfying $\Phi(z_\zeta) = \zeta$ and $\Phi(\cdot)$ denotes the c.d.f. of the standard normal distribution. Then the critical value can be found by

$$c = 1 - \frac{z_{1-\alpha}}{\sqrt{n}\sqrt{G(1)}} \text{ or } c = \theta_a + \frac{z_{1-\beta}}{\sqrt{n}\sqrt{G(\theta_a)}}.$$

3 Conclusions

In this paper, the design of progressively type-I interval censored sampling plans with equally-spaced inspection times is developed based on the ordinary and approximate methods. Some of the proposed life test plans are tabulated and the use of the tables is illustrated by examples. The ordinary life test plans are exact, however, the computation is complicated and time consuming if the number of inspection and the sample size are large. Moreover, the search procedure may fail due to a memory overflow error on computer. If the computer facilities are excellent for practitioners, the ordinary life test plans are suggested. Otherwise, the approximate life test plans are suggested to replace the ordinary life test plans.

The sampling plans for progressively type-I interval censoring with unequally-spaced inspection times can be established in a similar way as the proposed approach. Compared with the design of equally-spaced inspection times, the design of unequally-spaced inspection times is troublesome for practitioners, moreover, the computation is complicated. Based on the administrative convenience, we do not suggest practitioners to construct the life test plans under progressively type-I interval censoring with unequally-spaced inspection times. Extending the proposed study to the case of two-parameter lifetime models, for example, Weibull, lognormal lifetime models and so forth may be a fruitful area of future research.

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