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韋伯分配下之最佳逐步分群設限試驗計畫

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中文摘要

本研究將討論產品壽命為韋伯分配時，逐步分群設限下之壽命試驗設計問題。我們將探討此設限方案下之壽命分配參數的估計問題，並推導出其漸進分配之變異數共變異數矩陣。本研究將利用漸進分配的結果並採用D型最佳化的標準，來建立在試驗預算限制條件下之最佳的逐步分群設限試驗之設計。此外，本研究也將提出的方法應用到一例子上，並做有關敏感度分析。

關鍵詞： 分群資料；最大概度法；逐步型一設限。

Abstract

This study will discuss a life test under type I progressive group-censoring. A Weibull failure time distribution is considered. We will derive the maximum likelihood estimators of the model parameters and compute the Fisher information and the asymptotic variance-covariance matrix of the maximum likelihood estimators. We then use these values to investigate the optimal progressive group-censoring plans. D-optimality criterion will be considered to establish the optimal progressive group-censoring plans under a pre-determined budget of experiment. The method will be applied to a numerical example and the sensitivity analysis will be investigated.

Keywords: Grouped data; Maximum likelihood method; Progressive type I censoring.

1 Introduction

Censored sampling arises in a life test whenever the experimenter does not observe the lifetimes of all test units. The most common censoring schemes are type I censoring and type II censoring. These two censoring schemes have been studied rather extensively by a number of authors including Mann *et al.* (1974), Lawless (1982), and Meeker and Escobar (1998). One important characteristic of these two censoring schemes is that they do not allow for units to be removed from the test at the points other than the final termination point. However, if an experimenter desires to remove surviving units at points other than the final termination point of the life test, these two traditional censoring schemes will not be of use to the experimenter. The allowance of removing surviving units from the test before final termination point is desirable, as in the case of studies of wear, in which the study of the actual aging process requires units to be fully disassembled at different stages in the experiment. In addition, when a compromise between reduced time of experimentation and the observation of at least some extreme lifetimes is sought, such allowance is also desirable. These reasons lead us into the area of progressive censoring. Cohen (1963) also mentioned that one of the primary goals of progressive censoring is to save some live units for other tests, which is particularly useful when the units being tested are very expensive. Statistical inferences on the parameters of failure time distributions under progressive censoring have been studied by several authors such as Mann (1971), Gibbons and Vance (1983), Wong (1993), and Balakrishnan and Aggarwala (2000).

Note that in progressive censoring, the number of removals are all pre-determined. However, in some practical situations, these numbers may occur at random. Yuen and Tse (1996) indicated that, for example, in some reliability experiments, an experimenter may decide that it is inappropriate or too dangerous to carry on the testing on some of the tested units even though these units have not failed. In these cases, the pattern of removal is random.

In practice, it is often impossible continuously to observe the testing process, even with censoring. The test units might be able to be inspected intermittently. That is, we can only record whether a test unit fails in an interval instead of measuring failure time exactly. Hence, this type of inspection is called group (or interval) censoring. In the literature, group censored data have been studied by many researchers such as Cheng and Chen (1988), Chen and Mi (1996), Aggarwala (2001), and Tse *et al.* (2002).

In this study, we will focus on the designing problem of progressively type-I group-censored life test with Weibull failure time distribution. There are some important questions about how to design an appropriate life test that would result in the optimal estimation of the life parameters. Important questions include how to determine the number of units that should be tested, the number of inspections, and the length of the inspection intervals. One practical problem arising from designing a life test is the budget of experiment. The size of budget always affects the decisions of number units to test, number of inspections and length of inspection intervals and hence, affects the precision of estimating the failure time distribution. In this study, we are going to integrate these factors and the restricted cost of experiment to construct a mathematical model. Then we will use the method of nonlinear mixed integer programming to find the optimal solution of the total number of units to test, the times at which to measure the units, and the termination time of a experiment. Thus, we can set up an optimal life test by using the D-optimality criterion. We will apply the proposed method to a numerical example and discuss its sensitivity analysis.

The rest of the paper is organized as follows: Section 2 describes the model and some necessary assumptions. We use the maximum likelihood method to obtain the point and interval estimators of the parameters. Section 3 proposes a procedure to determine the number of units to test, the number of inspections and the length of inspection intervals. Section 4 applies the proposed procedure to a numerical example, and Section 5 studies the sensitivity analysis of the proposed procedure. Some conclusions are in Section 6.

2 Model and Parameter Estimation

Let random variable X have a Weibull distribution. The probability density function and cumulative distribution function are

$$f(x) = \frac{\nu}{\theta}\left(\frac{x}{\theta}\right)^{\nu-1} e^{-\left(\frac{x}{\theta}\right)^{\nu}}, \quad x > 0,$$

and

$$F(x) = 1 - e^{-\left(\frac{x}{\theta}\right)^{\nu}}, \quad x > 0, \tag{1}$$

respectively, where $\theta > 0$ and $\nu > 0$ are parameters.

Suppose that n independent units are simultaneously placed on a life test at time 0, and run until time τ_1, at which point the number of failed units n_1 are counted and r_1 surviving units are removed from the test; starting from time τ_1, the $n - n_1 - r_1$ non-removed surviving units are run until time τ_2, at which point the number of failures n_2 are counted and r_2 surviving units are removed from the test, and so on. At time τ_k, the number of failed units n_k are counted and the remaining surviving $r_k = n - \sum_{i=1}^{k} n_i - \sum_{j=1}^{k-1} r_j$ units are all removed, thereby terminating the test. This scheme is called progressive type-I group-censoring.

Suppose a progressively type-I group-censored sample is collected, beginning with a random sample of n units with an exponential lifetime distribution. Let n_i be the number of units known to have failed in the interval $(\tau_{i-1}, \tau_i]$ and let r_i be the number of surviving units being withdrawn from the test at time τ_i, for $i = 1, 2, \ldots, k$, where $\tau_0 = 0$. The values $r_1, r_2, \ldots, r_k$ are specified as percentages of the remaining live units $p_1, p_2, \ldots, p_k$ (with $p_k = 1$). That is, $r_i = (m_i - n_i)p_i$, for $i = 1, 2, \ldots, k$, where $m_i = n - \sum_{j=1}^{i-1} n_j - \sum_{j=1}^{i-1} r_j$ is the number of non-removed surviving units at the beginning of the ith stage. Then, we have the fact that

$$n_i | n_{i-1}, \ldots, n_1, r_{i-1}, \ldots, r_1 \sim binomial(m_i, q_i),$$

where $q_i = \frac{F(\tau_i)-F(\tau_{i-1})}{1-F(\tau_{i-1})} = 1 - e^{-\left[\left(\frac{\tau_i}{\theta}\right)^\nu - \left(\frac{\tau_{i-1}}{\theta}\right)^\nu\right]}$ is the probability that a unit survives at time τ_{i-1} and will fail before time τ_i, for $i = 1, 2, \ldots, k$. The function $F(\cdot)$ is defined in (1).

The likelihood function is

$$\begin{aligned} L(\theta, \nu) &\propto \prod_{i=1}^{k} \left[\frac{F(\tau_i) - F(\tau_{i-1})}{1 - F(\tau_{i-1})}\right]^{n_i} \left[1 - \frac{F(\tau_i) - F(\tau_{i-1})}{1 - F(\tau_{i-1})}\right]^{m_i - n_i} \\ &= \prod_{i=1}^{k} \left[1 - e^{-\left(\frac{\tau_i}{\theta}\right)^\nu + \left(\frac{\tau_{i-1}}{\theta}\right)^\nu}\right]^{n_i} \left[e^{-\left(\frac{\tau_i}{\theta}\right)^\nu + \left(\frac{\tau_{i-1}}{\theta}\right)^\nu}\right]^{m_i - n_i} \end{aligned}$$

Let $h_i = \left(\frac{\tau_i}{\theta}\right)^\nu - \left(\frac{\tau_{i-1}}{\theta}\right)^\nu$, $i = 1, 2, \ldots, k$. The log-likelihood function may then be written as

$$\log L(\theta, \nu) \propto \sum_{i=1}^{k} \left[n_i \log(1 - e^{-h_i}) - (m_i - n_i) h_i\right]. \tag{2}$$

and, hence the maximum likelihood estimators (MLEs) of θ and ν can be obtained by maximizing (2) directly. The likelihood equations for θ and ν are

$$\frac{\partial}{\partial \theta} \log L(\theta, \nu) = \sum_{i=1}^{k} \left[n_i \frac{e^{-h_i} \frac{\partial h_i}{\partial \theta}}{1 - e^{-h_i}} - (m_i - n_i) \frac{\partial h_i}{\partial \theta}\right]$$

and

$$\frac{\partial}{\partial \nu} \log L(\theta, \nu) = \sum_{i=1}^{k} \left[n_i \frac{e^{-h_i} \frac{\partial h_i}{\partial \nu}}{1 - e^{-h_i}} - (m_i - n_i) \frac{\partial h_i}{\partial \nu}\right]$$

where $\frac{\partial h_i}{\partial \theta} = -\frac{\nu}{\theta}\left[\left(\frac{\tau_i}{\theta}\right)^\nu - \left(\frac{\tau_{i-1}}{\theta}\right)^\nu\right]$ and $\frac{\partial h_i}{\partial \nu} = \left(\frac{\tau_i}{\theta}\right)^\nu \log\left(\frac{\tau_i}{\theta}\right) - \left(\frac{\tau_{i-1}}{\theta}\right)^\nu \log\left(\frac{\tau_{i-1}}{\theta}\right)$. The MLEs $\hat{\theta}$ and $\hat{\nu}$ can be found by solving the likelihood equations. Standard numerical methods such as Newton-Raphson can be used to obtain the solutions of the likelihood equations.

The asymptotic normal distribution for the MLEs can be derived in the usual way. From the log-likelihood function in (2), we have

$$\begin{aligned} \frac{\partial^2}{\partial \theta^2} \log L(\theta, \nu) = \sum_{i=1}^{k} \frac{1}{(1 - e^{-h_i})^2} \Bigg\{ n_i \left[-e^{-h_i} \left(\frac{\partial h_i}{\partial \theta}\right)^2 + (1 - e^{-h_i}) \frac{\partial^2 h_i}{\partial \theta^2}\right] \\ - m_i \frac{\partial^2 h_i}{\partial \theta^2} (1 - e^{-h_i})^2 \Bigg\}, \end{aligned} \tag{3}$$

$$\frac{\partial^2}{\partial\nu^2}\log L(\theta,\nu) = \sum_{i=1}^{k}\frac{1}{(1-e^{-h_i})^2}\left\{n_i\left[-e^{-h_i}\left(\frac{\partial h_i}{\partial\nu}\right)^2 + (1-e^{-h_i})\frac{\partial^2 h_i}{\partial\nu^2}\right] - m_i\frac{\partial^2 h_i}{\partial\nu^2}(1-e^{-h_i})^2\right\}, \tag{4}$$

and

$$\frac{\partial^2}{\partial\theta\partial\nu}\log L(\theta,\nu) = \sum_{i=1}^{k}\frac{1}{(1-e^{-h_i})^2}\left\{n_i\left[-e^{-h_i}\frac{\partial h_i}{\partial\theta}\frac{\partial h_i}{\partial\nu} + (1-e^{-h_i})\frac{\partial^2 h_i}{\partial\theta\partial\nu}\right] - m_i\frac{\partial^2 h_i}{\partial\theta\partial\nu}(1-e^{-h_i})^2\right\}, \tag{5}$$

where $\frac{\partial^2 h_i}{\partial\theta^2} = \frac{\nu(\nu+1)}{\theta^2}\left[\left(\frac{\tau_i}{\theta}\right)^\nu - \left(\frac{\tau_{i-1}}{\theta}\right)^\nu\right]$, $\frac{\partial^2 h_i}{\partial\nu^2} = \left(\frac{\tau_i}{\theta}\right)^\nu\left[\log\left(\frac{\tau_i}{\theta}\right)\right]^2 - \left(\frac{\tau_{i-1}}{\theta}\right)^\nu\left[\log\left(\frac{\tau_{i-1}}{\theta}\right)\right]^2$ and $\frac{\partial^2 h_i}{\partial\theta\partial\nu} = -\frac{1}{\theta}\left[\left(\frac{\tau_i}{\theta}\right)^\nu - \left(\frac{\tau_{i-1}}{\theta}\right)^\nu\right] - \frac{\nu}{\theta}\left[\left(\frac{\tau_i}{\theta}\right)^\nu\log\left(\frac{\tau_i}{\theta}\right) - \left(\frac{\tau_{i-1}}{\theta}\right)^\nu\log\left(\frac{\tau_{i-1}}{\theta}\right)\right]$. To obtain the Fisher's information, we need the expectations of (3), (4) and (5). To get this, let us compute the expectations of m_i, for $i = 1, 2, \ldots, k$. From the property of conditional expectation, we have

$$E(n_i) = E(m_i)q_i, \quad i = 1, 2, \ldots, k.$$

Now, beginning with $E(m_1) = n$, $r_i = (m_i - n_i)p_i$ and $m_{i+1} = m_i - n_i - r_i$, we obtain, by induction,

$$E(m_i) = n\prod_{j=1}^{i-1}(1-q_j)(1-p_j), \quad i = 2, 3, \ldots, k.$$

Hence, the Fisher's information matrix is

$$\mathbf{I}(\theta,\nu) = \begin{bmatrix} \sum_{i=1}^{k} E(m_i)\frac{\left(\frac{\partial q_i}{\partial\theta}\right)^2}{q_i(1-q_i)} & \sum_{i=1}^{k} E(m_i)\frac{\left(\frac{\partial q_i}{\partial\theta}\right)\left(\frac{\partial q_i}{\partial\nu}\right)}{q_i(1-q_i)} \\ \sum_{i=1}^{k} E(m_i)\frac{\left(\frac{\partial q_i}{\partial\theta}\right)\left(\frac{\partial q_i}{\partial\nu}\right)}{q_i(1-q_i)} & \sum_{i=1}^{k} E(m_i)\frac{\left(\frac{\partial q_i}{\partial\nu}\right)^2}{q_i(1-q_i)} \end{bmatrix}$$

where $\frac{\partial q_i}{\partial\theta} = e^{-h_i}\frac{\partial h_i}{\partial\theta}$ and $\frac{\partial q_i}{\partial\nu} = e^{-h_i}\frac{\partial h_i}{\partial\nu}$. For a large sample size n, the MLEs $(\hat{\theta},\hat{\nu})'$ has an approximate bivariate normal distribution with mean vector $(\theta,\nu)'$ and variance-covariance matrix $\mathbf{I}^{-1}(\theta,\nu)$. Thus, the approximate confidence intervals for θ and ν can be easily established.

3 Planning of Life Test

To obtain the optimal estimation of life parameters, frequently asked questions include "How many units does the experimenter need to test?", "How long does the experimenter need to run the life test?" or "How many times does the experimenter need to inspect the units in the life test?" Simply put, more test units, more test time, and more number of inspections will generate more information, which improves the precision of estimates. However, the restricted cost of experiment does not allow us to do so. Therefore, the problem of obtaining the optimal estimation of life parameters under a restricted cost of experiment is an important issue to the reliability analyst.

There are a lot of decision variables that affect the cost of experiment and the performance of the estimation of life parameters. The most important three decision variables are: (1) the number of test units, (2) the number of inspections, and (3) the lengths of inspection intervals. In this study, we assume that the lengths of inspection intervals are all equal. Let n denote the number of units on test, k be the number of inspections and τ be the length of inspection interval. The cost of experiment consists of the following four parts.

1. Installation cost. This is the cost of installing all test units in the beginning of life experiment, say C_a. It does not depend on the number of test units.
2. Sample cost. This is the cost of test units. Let C_s be the cost of a test unit. Then the total sample cost is nC_s.
3. Inspection cost. The inspection cost includes the cost of using inspection equipment and material. It depends on the number of inspections. Let C_i denote the cost of one inspection. Then the total inspection cost is kC_i.
4. Operation cost. The cost consists of the salary of operators, utility, and depreciation of test equipment, etc. It is proportional to the testing time. Let C_o be the operation cost in the time interval between two inspections. Then the total operation cost is $k\tau C_o$.

Therefore, the total cost of experiment is:

$$C_T = C_a + nC_s + kC_i + k\tau C_o.$$

Note that the asymptotic variance-covariance matrix of $\hat{\theta}$ and $\hat{\nu}$ is a function of n, k and τ. For simplicity, we can write $G(n,k,\tau) = |\mathbf{I}^{-1}(\theta,\nu)|$. Consider the D-optimality criterion. Then, the optimal design problem consists of finding n, k and τ that minimize $G(n,k,\tau)$. However, the determination of n, k and τ is restricted to the budget of experiment, say, C_r. Hence, the optimal design problem can be expressed as follows.

$$\begin{array}{rl} \text{minimize} & G(n,k,\tau) \\ \text{subject to} & C_a + nC_s + kC_i + k\tau C_o \le C_r \\ & k, n \in N,\ \ k \ge 1, \quad \text{and} \quad \tau > 0, \end{array} \tag{6}$$

where N is the set of positive integers. Since the objective function and constraint are both nonlinear functions of decision variables n, k, and τ, the method of nonlinear mixed integer programming can be used to find the optimal solution.

4 Numerical Example

We apply the proposed methods to a numerical example. Use the algorithm in Aggarwala (2001) to generate data with $n = 15$, $k = 5$, $\tau = 2$, $\theta = 4.8$ and $\nu = 1.5$. The pre-specified percentages of removals are $(p_1, p_2, p_3, p_4, p_5) = (0.25, 0.25, 0.4, 0.5, 1)$. The data are presented in Table 1.

Table 1: Progressively group censored sample

i	1	2	3	4	5
n_i	4	3	3	0	1
r_i	2	1	0	1	0
p_i	0.25	0.25	0.4	0.5	1

We obtain the maximum likelihood estimates of θ and ν to be $\hat{\theta} = 4.7969$ and $\hat{\nu} = 1.3789$, respectively. We use these estimates in the design of our new experiment. Assume that the percentages of removals are $p_1 = p_2 = \cdots = p_{k-1} = 0.25$ and $p_k = 1$. Suppose further that the values of cost parameters are as follows: $C_a = \$10$, $C_s = \$85/\text{unit}$, $C_i = \$3.25$, $C_o = \$3/\text{ten hours}$, and $C_r = \$6000$. Consider the objective function to be the determinant of asymptotic variance-covariance matrix of $\hat{\theta}$ and $\hat{\nu}$. Thus, the optimal design problem is:

$$\begin{array}{ll} \text{minimize} & |\mathbf{I}^{-1}(\theta, \nu)| \\ \text{subject to} & 10 + 85n + 3.25k + 3k\tau \leq 6000 \end{array}$$

We can obtain the optimal design as follows.

$$n^* = 69, \quad k^* = 11, \quad \tau^* = 2.7045.$$

5 Sensitivity Analysis

We now study sensitivity of the optimal solution to change in the values of the different parameters associated with the life experiment. These parameters can be divided into two parts: (1) the parameters in failure time distribution, *i.e.*, θ and ν; and (2) the parameters in the cost of experiment, *i.e.*, C_a, C_s, C_i, C_o, and C_r. We will study how the optimal solution is influenced by the estimates of distribution parameters and by the cost parameters of experiment in Sections 5.1 and 5.2, respectively.

5.1 The Effect of Estimated Distribution Parameters

As we mentioned in Section 3, in practice, the values of distribution parameters are usually unknown. We have to use prior information or data from a pilot test to get their estimates.

However, no one can guarantee that the estimates are exactly equal to the unknown parameters. Thus, in this subsection, we will discuss the influence of changing values of estimated parameters on the optimal solutions.

From Section 4, we obtain the optimal design $n^* = 69$, $k^* = 11$ and $\tau^* = 2.7045$ when the estimates of parameters $\hat{\theta} = 4.7969$ and $\hat{\nu} = 1.3789$, and the cost parameters of experiment $(C_a, C_s, C_i, C_o, C_r) = (10, 85, 3.25, 3, 6000)$. The 95% approximate confidence intervals for θ and ν are $(1.7123, 7.8814)$ and $(0.2634, 1.7365)$, respectively. Thus, we choose various values of θ and ν in their 95% approximate confidence intervals for sensitivity analysis. Set $(C_a, C_s, C_i, C_o, C_r) = (10, 85, 3.25, 3, 6000)$, the cost parameters used in Section 4. Now, the optimal solutions of n, k, and τ for $p = 0.25$ are given in Table 2. It shows that n is not sensitive but k and τ are sensitive to the changes in these values of parameter.

Table 2: Optimal values for fixed $c_a = 10$, $c_s = 85$, $c_i = 3.25$, $c_o = 3.0$ and $c_r = 6000$

θ	ν	n	k	τ	$G(n,k,\tau)$
1.7123	0.2634	69	14	1.8929	0.0072
	1.3789	70	7	0.8214	0.0011
	1.7365	70	7	0.8214	0.0010
4.7969	0.2634	68	12	4.7500	0.0584
	1.3789	69	11	2.7045	0.0091
	1.7365	69	12	2.3889	0.0081
7.8814	0.2634	68	10	5.9167	0.1620
	1.3789	69	8	4.1250	0.0244
	1.7365	69	8	4.1250	0.0219

5.2 The Effect of Cost Parameters

Changes in cost parameters of experiment can affect the determination of the optimal design. Let us consider the value of distribution parameter $\theta = 4.9326$ and the cost parameters of experiment $(C_a, C_s, C_i, C_o, C_r) = (10, 85, 3.25, 3, 6000)$. The solutions of the optimal design are shown in Section 4. Using the same value of distribution parameter, the sensitivity of each decision variables n, k, and τ to changes in the cost parameters of experiment is examined.

Table 3 shows that the number of test units n is insensitive to changes in C_a, C_i and C_o, and is highly sensitive to changes in C_s and C_r. In addition, n is an increasing function of C_r and a decreasing function of C_s. The number of inspections k is slightly sensitive to changes in C_a, and it is sensitive to changes in the cost parameters C_s, C_i, C_o, and C_r. The length of inspection interval τ is slightly sensitive to all cost parameters. It can be found that the termination time of the experiment $k\tau$ is a decreasing function of C_i.

Table 3: Optimal values for different costs values under $\theta = 4.7969$ and $\nu = 1.3789$

c_a	c_s	c_i	c_o	c_r	n	k	τ	$k\tau$	$G(n,k,\tau)$
10	85	3.25	3.0	4000	46	7	2.7262	19.0833	0.0204
				5000	58	5	2.9167	14.5833	0.0132
				6000	69	11	2.7045	29.7500	0.0091
				7000	81	10	2.4167	24.1667	0.0066
				8000	93	8	2.4583	19.6667	0.0050
10	85	3.25	1.0	6000	70	7	2.4643	17.2500	0.0088
			2.0		69	15	2.5417	38.1250	0.0090
			3.0		69	11	2.7045	29.7500	0.0091
			4.0		69	9	2.6597	23.9375	0.0090
			5.0		69	8	2.4750	19.8000	0.0091
10	85	0.75	3.0	6000	69	15	2.5278	37.9167	0.0090
		2.00			69	13	2.5385	33.0000	0.0090
		3.25			69	11	2.7045	29.7500	0.0091
		4.50			69	10	2.6667	26.6667	0.0090
		5.75			69	9	2.7130	24.4167	0.0091
10	55	3.25	3.0	6000	107	10	2.4167	24.1667	0.0038
	70				79	6	2.5278	15.1667	0.0070
	85				69	11	2.7045	29.7500	0.0091
	100				59	8	2.6667	21.3333	0.0124
	115				51	11	2.7045	29.7500	0.0166
6	85	3.25	3.0	6000	69	12	2.5000	30.0000	0.0090
8					69	12	2.4444	29.3333	0.0091
10					69	11	2.7045	29.7500	0.0091
12					69	11	2.6439	29.0833	0.0090
14					69	11	2.5833	28.4167	0.0090

6 Conclusion

Determining appropriate number of test units, number of inspections, and length of inspection interval under restricted budget of experiment is an important decision problem for experimenters when conducting a life test. We use the method of nonlinear mixed integer programming to set up the optimal design. Under this design, we obtain the optimal estimation of life parameters. Finally, the proposed method can lead to better designs for conducting life tests. It provides the most efficient use of one's resources and to achieve the precision that one can expect to have with such a design. This approach is intuitive, and can be useful to engineers.

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