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在偏態分配下的一個新的 R 管制圖

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Abstract

This paper proposes a heuristic method based on adjusted weighted standard deviation for constructing *R* chart for skewed process distributions. The asymmetric control limits of the chart are established with no assumption to the process distribution. If the process distribution is symmetric, these control limits are equivalent to those of Shewhart *R* chart. The proposed control limits are compared with weighted variance *R* chart and skewness correction *R* chart by Monte Carlo simulation. When the process distribution is Weibull or gamma, simulation results show that the proposed *R* chart performs better than both weighted variance and skewness correction *R* charts as the skewness and the sample size increase. For the case where the process distribution is exponential with known mean, the Type I risk and Type II risk of the proposed *R* chart are closer to those of the exact *R* chart than those of the weighted variance and skewness correction *R* charts.

Keywords: *R* chart; skewed distribution; skewness correction; weighted standard deviation; weighted variance.

1. Introduction

The conventional Shewhart-type control charts are powerful tools in statistical process control. They are widely accepted and applied in industry. They can be constructed by assuming that the process distribution is normal or approximately normal. However, there are many cases in which the process distribution is skewed, and such that the normality assumption is not valid. For example, the measurements from hole-drilling processes in printed circuit boards, coating processes, chemical processes such as hot-dip galvanizing processes, and semiconductor processes often follow skewed distributions. (see *e.g.*, Gunter (1989), Pyzdek (1995) and Bittanti *et al*. (1998)).

If the process distribution is skewed, the false alarm rate grows larger as the skewness increases because of the discrepancy between the variability pattern of the process distribution and the normality assumption. Three approaches have been suggested to deal with skewness of the process distribution:

(1) Disregard the skewness of the distribution and use the Shewhart-type charts by increasing the sample size so that the sample mean becomes approximately normally distributed. But this is often expensive.

(2) Assume the process distribution is known and constructs exact control charts that give desired false alarm rates (see Ferrell (1958), Nelson (1979), Lucas (1985) and Vardeman and Ray (1985)). In some instances, it is possible to transform the nonnormal variable into the normal, and then use the Shewhart-type chart. Also, one can fit a theoretical frequency curve such as the Gram-Charlier or Pearson system, and obtain asymmetric control limits satisfying the desired probability. However, these approaches can be complicated, and most quality practitioners would prefer to use the standard approach if the effect of non-normality is not

serious.

(3) Make no assumption about the underlying distribution and use heuristic methods to obtain a control chart so that the false alarm rate stays as close to the desired level as possible. However, only a few heuristic methods are discussed in the literatures, such as Choobineh and Branting (1986), Choobineh and Ballard (1987), Bai and Choi (1995), Chang and Bai (2001, 2004), Chang *et al.* (2002), and Chan and Cui (2003).

2. Literature Review: The Weighted Standard Deviation Method

Chang and Bai (2001) proposed a WSD method to set up control limits of $\overline{X}$, CUSUM and EWMA charts for skewed distributions. This method is based on the idea that the standard deviation σ of quality characteristic X can be divided into two parts: the upper and lower deviations, σ_U^W and σ_L^W, which represent the degrees of dispersions of the upper and lower sides from the process mean μ, respectively. Assume that the probability density function (p.d.f.) of an asymmetric distribution is $f(x)$. Chang and Bai (2001) showed that the p.d.f. $f(x)$ can be approximated by two normal p.d.f.'s, $f_U(x)$ and $f_L(x)$. These two normal distributions have the same mean μ but different standard deviations $2\sigma_U^W$ and $2\sigma_L^W$. The upper side of $f(x)$ can be approximated by the upper side of $f_U(x)$, and the lower sides of $f(x)$ can be approximated by the lower side of $f_L(x)$. Moreover, Chang and Bai (2001) showed that the standard deviation of the original distribution can be decomposed into $\sigma = \sigma_U^W + \sigma_L^W$, where the σ_U^W and σ_L^W can be expressed as

$$\sigma_U^W = P\sigma, \text{ and } \quad \sigma_L^W = (1-P)\sigma, \tag{1}$$

where $P = P(X \le \mu)$. If the underlying distribution is skewed to the right, then $P > \frac{1}{2}$ and $\sigma_U^W > \sigma_L^W$; otherwise, $P < \frac{1}{2}$ and $\sigma_U^W < \sigma_L^W$.

Suppose that n observations are taken from a distribution with p.d.f. $f(x)$ and these observations are divided into two groups by μ. Then, the observations from a half-side are copied into the other side. The average numbers of observations of upper and lower parts become $2n(1-P)$ and $2nP$, respectively.

Let $d_2^* = \dfrac{E(R_n)}{\sigma}$ be the chart constant for a skewed distribution corresponding to d_2 for a normal distribution, where $R_n = X_{(n)} - X_{(1)}$ is the sample range with sample size n, and $X_{(r)}$ denotes the rth order statistic in the sample, $1 \le r \le n$. Let $R_{2n(1-P)}^U = 2(X_{(n)} - \mu)$ be the sample range of copied $2n(1-P)$ observations from the upper part of the normal p.d.f. $f_U(x)$, and $R_{2nP}^{L} = 2(\mu - X_{(1)})$ be the sample range of copied $2nP$ observations from the lower part of the normal p.d.f. $f_L(x)$. Chang and Bai (2001) suggested that d_2^* can be approximated by

$$d_2^{WSD} = Pd_2(2n(1-P)) + (1-P)d_2(2nP).$$

Let X_{ij} denote the *j*th observation taken from the *i*th sample, and let *m* and *n* be the number of samples and the number of observations in a sample, respectively. If *P* is unknown, it can be estimated by

$$\hat{P} = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n} I(\overline{\overline{X}} - X_{ij})}{m \times n}, \tag{2}$$

where $\overline{\overline{X}} = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}}{m \times n}$ is the grand mean of the pre-samples, and *I*(*x*)=1 if $x \geq 0$ or *I*(*x*)=0 otherwise. We can approximate the standard deviation of the underlying distribution by $\frac{\overline{R}}{d_2^{WSD}}$, and then the control limits of WSD $\overline{X}$ chart can be expressed as

$$UCL_{WSD} = \overline{\overline{X}} + 3\frac{\overline{R}}{d_2^{WSD}\sqrt{n}}2\hat{P} \quad \text{and} \quad LCL_{WSD} = \overline{\overline{X}} - 3\frac{\overline{R}}{d_2^{WSD}\sqrt{n}}2(1-\hat{P}).$$

3. Adjusted Weighted Standard Deviation *R* Chart

Let $d_3^* = \sqrt{Var\left(\frac{R_n}{\sigma}\right)}$ be the chart constant for a skewed distribution corresponding to d_3 for a normal distribution. In the WSD method, $2\sigma_U^W$ and $2\sigma_L^W$ are used instead of σ for the upper and lower control limits, respectively. Using equation (1), the control limits of WSD *R* chart can be obtained as

$$\begin{aligned} UCL_{WSD} &= (d_2^* + 3d_3^*)(2\sigma_U^W) = 2(d_2^* + 3d_3^*)P\sigma, \\ LCL_{WSD} &= \left[(d_2^* - 3d_3^*)(2\sigma_L^W)\right]^+ = \left[2(d_2^* - 3d_3^*)(1-P)\sigma\right]^+, \end{aligned} \tag{3}$$

where $[a]^+$ denotes $\max(0, a)$. In accordance with the WSD method, it can be shown that

$$d_3^{*2} = E\left(\frac{R_{2n(1-P)}^U}{2\sigma}\right)^2 + E\left(\frac{R_{2nP}^L}{2\sigma}\right)^2 + 2E\left(\frac{R_{2n(1-P)}^U}{2\sigma}\frac{R_{2nP}^L}{2\sigma}\right) - d_2^{*2}. \tag{4}$$

The expectation of cross-product term, $E\left(\frac{R_{2n(1-P)}^U}{2\sigma}\frac{R_{2nP}^L}{2\sigma}\right)$, in equation (4) is hard to be derived exactly if the process distribution is not specified, so Chang and Bai (2001) did not address the

WSD *R* chart. Analytically, it can be shown as follows:

$$2E\left(\frac{R^U_{2n(1-P)}}{2\sigma}\frac{R^L_{2nP}}{2\sigma}\right) \le E\left(\frac{R^U_{2n(1-P)}}{2\sigma}\right)^2 + E\left(\frac{R^L_{2nP)}}{2\sigma}\right)^2 . \tag{5}$$

Substituting equation (1) and inequality (5) into equation (4) yields an upper bound of d_3^*, that is,

$$d_3^{*2} \le 2P^2\left(Var\left(\frac{R^U_{2n(1-P)}}{2\sigma^W_U}\right)+\left(E\left(\frac{R^U_{2n(1-P)}}{2\sigma^W_U}\right)\right)^2\right)+2(1-P)^2\left(Var\left(\frac{R^L_{2nP}}{2\sigma^W_L}\right)+\left(E\left(\frac{R^L_{2nP}}{2\sigma^W_L}\right)\right)^2\right)-d_2^{*2}.$$

Using the WSD method, $Var\left(\frac{R^U_{2n(1-P)}}{2\sigma^W_U}\right)$, $Var\left(\frac{R^L_{2nP}}{2\sigma^W_L}\right)$, $E\left(\frac{R^U_{2n(1-P)}}{2\sigma^W_U}\right)$, $E\left(\frac{R^L_{2nP}}{2\sigma^W_L}\right)$ and d_2^* can be estimated by $d_3(2n(1-P))$, $d_3(2nP)$, $d_2(2n(1-P))$, $d_2(2nP)$ and d_2^{WSD}, respectively. An upper bound of the WSD estimator of d_3^*, say d_3^{WSD}, can be defined by

$$d_3^{WSD} = \sqrt{2P^2 d_3^2(2n(1-P)) + 2(1-P)^2 d_3^2(2nP) + \delta}\,,$$

where $\delta = (Pd_2(2n(1-P)) - (1-P)d_2(2nP))^2$ provided that the probability of $P = P(X \le \mu)$ is specified. If P is unknown, it can be estimated by equation (2).

Unfortunately, it can be shown that the d_3^{WSD} underestimates d_3^* for almost all specified distributions such as Weibull, lognormal and gamma with sample sizes *n*=3, 5, 7, 10. In practice, *R* chart is often used to monitor the process variability when sample size $n \le 10$. Tsai and Wu (2006) show the results of positively-skewed cases for Weibull, lognormal and gamma distributions. Those distributions are chosen because they represent a wide variety of shapes as parameters change.

Tsai and Wu (2006) show that the d_3^{WSD} is severely biased downward for all specified distributions and sample sizes and the bias becomes more serious as the skewness of the underlying distribution and sample size increase. However, d_3^{WSD} is simply computed. It is only functional of the values of *P*, $d_2(n)$ and $d_3(n)$, and does not refer to the original distribution. Actually, we can treat the δ in d_3^{WSD} as an adjusted factor of the WSD estimator of d_3^* to a skewed process distribution. The δ depends on the both skewness of the underlying distribution and the sample size, and it reduces to zero if the underlying distribution tends to be symmetric. But the δ seems not sensitive enough to decline the influence of the skewness of the underlying distribution. Hence, the δ must be modified so that the proposed estimator can work well. Assume that the δ can be modified to be a new quantity named δ^*, and the WSD estimator can be adjusted to be a new estimator

$$d_3^{AWSD} = \sqrt{2P^2 d_3^2(2n(1-P)) + 2(1-P)^2 d_3^2(2nP) + \delta^*}\ ,$$

so that d_3^{AWSD} is a good estimator of d_3^*; that is, the estimate of d_3^{AWSD} is closer to d_3^*. Let $D_n(P) = d_3^{*2} - (d_3^{WSD})^2$ denote the difference of d_3^{*2} and $(d_3^{WSD})^2$ with a random sample of size *n*. Then $D_n(P) \cong \delta^* - \delta$. Tsai and Wu (2006) plot the values of $D_n(P)$ versus the values of |*P*-0.5| for the sample sizes *n*=3, 5, 7, 10. They show that $D_n(P)$ is a curve function of |*P*-0.5| for all specified distributions, and the pattern depends on the sample size and skewness of the process distribution. Let D_{nW}, D_{nL}, and D_{nG} be the values of $D_n(P)$ computed from Weibull, lognormal and gamma distributions, respectively. Furthermore, let Y be the mean of D_{nW}, D_{nL}, and D_{nG}. Because these distributions represent a wide variety of shapes as parameters change, we approximate $D_n(P)$ by Y and then approximate δ^* by $\delta + Y$ for the situations of *P*=0.52(0.02)0.70 and *n*=3, 5, 7, 10 for all specified distributions. Accordingly, we can establish the following model.

$$Y = b_1|P-0.5| + b_2|P-0.5|^2 + b_3 n + \varepsilon\ , \tag{6}$$

where ε is an error term. Using the ordinary least squares estimation method, we can approximate δ^* by

$$\delta^* \cong \delta - 2.892|P-0.5| + 49.390|P-0.5|^2 + 0.021n. \tag{7}$$

Tsai and Wu (2006) show that the d_3^{AWSD} improves the performance of d_3^{WSD} significantly and get closer to d_3^* for all specified distributions.

In formulating a heuristic estimator of d_3^*, one may wish the proposed estimator can reduce to $d_3(n)$ for symmetric distribution which has the probability *P*=0.5. But the proposed estimator d_3^{AWSD} cannot meet the requirement due to the δ^* in equation (7) depends on the sample sizes. Actually, the d_3^{WSD} works well when the skewness of the underlying distribution is small for all specified sample sizes. Based on our numerical study, we suggest to take $d_3^{AWSD} = d_3^{WSD}$ if $|P-0.5| \le 0.04$ so that the proposed estimator d_3^{AWSD} can reduce to $d_3(n)$ for a symmetric distribution. If the parameters σ, d_2^* and d_3^* are unknown, they can be estimated by $\frac{\overline{R}}{d_2^{WSD}}$, d_2^{WSD} and d_3^{AWSD}, respectively. The control limits in equation (3) become

$$UCL_{AWSD} = 2\left(1+3\frac{d_3^{\mathrm{AWSD}}}{d_2^{WSD}}\right)P\overline{R}, \text{ and } LCL_{AWSD} = \left[2\left(1-3\frac{d_3^{\mathrm{AWSD}}}{d_2^{WSD}}\right)(1-P)\overline{R}\right]^+ . \tag{8}$$

An example is illustrated to the use of the proposed method, and a Monte Carlo simulation study is conducted to evaluate the performance of the proposed chart by Tsai and Wu (2006). Simulation study shows that the proposed chart works well.

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