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逐步設限下之貝氏統計推論

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中文摘要

在壽命分配參數的估計問題中，經常可自過去的實驗或經驗中知道參數的一些訊息，而貝氏方法即提供了結合過去訊息和現有資料來進行統計推論。本研究將考慮一種多用途的逐步型 II 設限方案，探討在此設限方案下壽命分配參數的估計問題。我們將推導分配參數的貝氏估計量和可靠區間，並同時推導出可靠度函數的貝氏估計量及可靠區間。此外，我們也將推導出未來觀察值的貝氏預測估計量及其最高事後密度區間。對於所提出的方法，我們將進行一些模擬上的比較，並透過例子來闡述我們所得到的結果。

關鍵詞： 貝氏估計量；最高事後密度區間；預測區間；逐步設限；可靠度函數。

Abstract

It is often the case that some information is available on the parameter of failure time distribution from previous experiments or analyses of failure time data. The Bayesian approach provides the methodology for incorporation of previous information with the current data. In this study, given a progressively type II censored sample from a Rayleigh distribution, Bayesian estimators and credible intervals will be obtained for the parameter and reliability function. We will also derive the Bayes predictive estimator and highest posterior density prediction interval for future observations. A numerical example will be presented for illustration, and some simulation study and comparisons will be performed.

Keywords: Bayes estimator; Highest posterior density interval; Prediction interval; Progressive censoring; Reliability function.

1 Introduction

The Rayleigh distribution is a special case of the Weibull distribution and has wide applications, such as, in communication engineering, (Dyer and Whisenand (1973a, 1973b)), in life testing of electrovacum devices, (Polovko 1968), etc. The probability density function of the Rayleigh distribution is given by

$$f(x|\theta) = \frac{x}{\theta^2} \exp\left\{-\frac{x^2}{2\theta^2}\right\}, \qquad x > 0, \tag{1}$$

where $\theta > 0$ is the parameter. An important characteristic of the Rayleigh distribution is that its failure rate is an increasing linear function of time. This property makes it a suitable model for components which possibly have no manufacturing defects but age rapidly (see Polovko (1968)) with time. Inferences for the Rayleigh distribution were discussed by several authors such as Kong and Fei (1996), Howlader and Hossain (1995), and Fernández (2000).

Progressive type II censoring is a generalization of type II censoring. In a type II censoring, a total of n units is put on a life test, but instead of continuing until all n units have failed, the life test is stopped at the time of the m-th $(1 \leq m \leq n)$ unit failure. If an experimenter desires to remove live units at points other than the final termination point of a life test, the type II censoring scheme will not be of use to the experimenter. Type II censoring does not allow for units to be removed from the life test before the final termination point. However, this allowance will be desirable, as in the case of accidental breakage of test units, in which the loss of units at points other than the termination point may be unavoidable.

Consider an experiment in which n independent units are placed on a test at time zero, and the failure times of these units are recorded. Suppose that m failures are going to be observed. When the first failure is observed, r_1 of the surviving units are randomly selected and removed. At the second observed failure, r_2 of the surviving units are randomly selected and removed. This experiment stops at the time when the m-th failure is observed and the remaining $r_m = n - r_1 - r_2 - \cdots - r_{m-1} - m$ surviving units are all removed. The m ordered observed failure times are called progressively type II censored order statistics of size m from a sample of size n with censoring scheme $(r_1, \ldots, r_m)$.

Suppose that the failure times of the n independent units originally on a test are identically distributed with probability density function $f(x)$ and cumulative distribution function $F(x)$. Let $X_{1:m:n}, \ldots, X_{m:m:n}$ be a progressively type II censored sample from $f(x)$ with censoring scheme $(r_1, \ldots, r_m)$. The joint probability density function of all m progressively type II censored order statistics is given by Balakrishnan and Aggarwala (2000),

$$f_{X_{1:m:n},\ldots,X_{m:m:n}}(x_{1:m:n}, \ldots, x_{m:m:n}) = c \prod_{i=1}^{m} f(x_{i:m:n})[1 - F(x_{i:m:n})]^{r_i}, \tag{2}$$

where $c = n(n-r_1-1)\cdots(n-r_1-\cdots-r_{m-1}-m+1)$. When data are obtained by progressive censoring, inference problems for various distributions have been studied by several authors including Wong (1993), Balasooriya and Saw (1998), and Wu (2003).

2 Prior and Posterior Distributions

Let $X_{1:m:n}, \ldots, X_{m:m:n}$ be a progressively type II censored sample from a Rayleigh distribution with parameter θ. According to (1) and (2), the likelihood function is given by

$$L(\theta) \propto \frac{1}{\theta^{2m}} \exp\left\{-\frac{1}{2\theta^2}\sum_{i=1}^{m}(r_i+1)x_{i:m:n}^2\right\}. \tag{3}$$

It is easy to obtain the maximum likelihood estimator of θ to be

$$\hat{\theta} = \sqrt{\frac{1}{2m}\sum_{i=1}^{m}(r_i+1)X_{i:m:n}^2}. \tag{4}$$

By the invariance property of the maximum likelihood estimator, we can obtain the maximum likelihood estimator of reliability function $R(t|\theta)$ to be

$$\hat{R}_t = \exp\left\{-\frac{t^2}{2\hat{\theta}^2}\right\}. \tag{5}$$

In the Bayesian approach, θ is considered a random variable having some specified distribution. In this paper, we consider conjugate prior distribution of the form

$$\Pi(\theta) = \frac{a^b}{\Gamma(b)2^{b-1}}\,\theta^{-2b-1}\exp\left\{-\frac{a}{2\theta^2}\right\}, \qquad \theta > 0, \tag{6}$$

where $a > 0$ and $b > 0$. This density is known as the square-root inverted-gamma distribution. It follows, from (3) and (6), that the posterior distribution of θ is given by

$$\Pi(\theta|\boldsymbol{x}) = \frac{[a+\sum_{i=1}^{m}(r_i+1)x_{i:m:n}^2]^{b+m}}{2^{b+m-1}\Gamma(b+m)}\,\theta^{-2(b+m)-1}\exp\left\{-\frac{1}{2\theta^2}\left[a+\sum_{i=1}^{m}(r_i+1)x_{i:m:n}^2\right]\right\}, \tag{7}$$

for $\theta > 0$, zero elsewhere. Substituting $\theta^2 = -t^2/(2\log s)$ into (7), we obtain the posterior probability density function of $R(t|\theta)$ as

$$\Pi(s|\boldsymbol{x}) = \frac{1}{\Gamma(b+m)}\left[\frac{a+\sum_{i=1}^{m}(r_i+1)x_{i:m:n}^2}{t^2}\right]^{b+m}(-\log s)^{b+m-1}\, s^{\frac{a+\sum_{i=1}^{m}(r_i+1)x_{i:m:n}^2}{t^2}-1}, \tag{8}$$

for $0 < s < 1$, zero elsewhere.

3 Bayesian Estimation

3.1 Bayes Estimators

In order to derive Bayes estimators we must first specify a loss function which represents the cost involved in using the estimate $\tilde{\theta}$ when the true value is θ. The squared error loss is appropriate when decisions become gradually more damaging for larger errors. Under squared error loss, the Bayes estimator of θ is the posterior mean

$$\tilde{\theta} = E(\theta|\boldsymbol{X}) = \sqrt{\frac{1}{2}\left[a + \sum_{i=1}^{m}(r_i+1)X_{i:m:n}^2\right]}\ \frac{\Gamma(b+m-\frac{1}{2})}{\Gamma(b+m)}. \tag{9}$$

Another problem of interest is that of estimating reliability function $R(t|\theta)$ with fixed $t > 0$. For squared error loss, the Bayes estimator of $R(t|\theta)$ is given by

$$\tilde{R}_t = E[R(t|\theta)|\boldsymbol{X}] = \left[\frac{a + \sum_{i=1}^{m}(r_i+1)X_{i:m:n}^2}{a + \sum_{i=1}^{m}(r_i+1)X_{i:m:n}^2 + t^2}\right]^{b+m}. \tag{10}$$

The highest posterior density (HPD) estimation is another method in popular use from the Bayesian perspective. This parameter estimation is based on the maximum likelihood principle and, hence the mode of posterior density will be the HPD estimator. Since the posterior density (7) is unimodal, we can obtain the HPD estimator of θ as

$$\theta^* = \sqrt{\frac{a + \sum_{i=1}^{m}(r_i+1)X_{i:m:n}^2}{2(b+m)+1}}.$$

From (8), the HPD estimator of $R(t|\theta)$ is

$$R_t^* = \exp\left\{-\frac{(b+m-1)t^2}{a + \sum_{i=1}^{m}(r_i+1)X_{i:m:n}^2 - t^2}\right\}.$$

3.2 HPD Credible Intervals

A $100(1-\alpha)\%$ Bayesian credible interval for the parameter θ is any interval (ℓ, u) satisfying

$$P(\ell < \theta < u|\boldsymbol{x}) = 1 - \alpha. \tag{11}$$

This two-sided interval (ℓ, u) can be chosen in different ways. The most frequent use is the HPD credible interval. A $100(1-\alpha)\%$ HPD credible interval chooses (ℓ, u) to consist of all values of θ with $\Pi(\theta|\boldsymbol{x}) > C_\alpha$, where C_α is chosen such that (11) holds.

Due to the unimodality of (7), the $100(1-\alpha)\%$ HPD credible interval (ℓ, u) for θ must satisfy the following two equations.

$$\int_{\ell}^{u} \Pi(\theta|\boldsymbol{x})d\theta = 1 - \alpha. \tag{12}$$

and

$$\Pi(\ell|\boldsymbol{x}) = \Pi(u|\boldsymbol{x}). \tag{13}$$

From (12) and (13) and after some algebraic computation, the $100(1-\alpha)\%$ HPD credible interval (ℓ, u) for θ is given by the simultaneous solution of the equations $\Gamma_{\mathrm{I}}(\nu_1, b+m) - \Gamma_{\mathrm{I}}(\nu_2, b+$

$m) = 1 - \alpha$ and $(u/\ell)^{2(b+m)+1} = \exp\{\nu_1 - \nu_2\}$, where $\nu_1 = \frac{1}{2\ell^2}\left[a + \sum_{i=1}^{m}(r_i+1)x_{i:m:n}^2\right]$, $\nu_2 = \frac{1}{2u^2}\left[a + \sum_{i=1}^{m}(r_i+1)x_{i:m:n}^2\right]$, and $\Gamma_{\rm I}(\nu_i, b+m) = \frac{1}{\Gamma(b+m)}\int_0^{\nu_i} z^{b+m-1}e^{-z}dz$, the incomplete gamma function. Similarly, the $100(1-\alpha)\%$ HPD credible interval (ℓ_R, u_R) for $R(t|\theta)$ must satisfy $\Gamma_{\rm I}(-\omega\log\ell_R, b+m) - \Gamma_{\rm I}(-\omega\log u_R, b+m) = 1-\alpha$ and $(\log u_R/\log\ell_R)^{b+m-1} = (\ell_R/u_R)^{\omega-1}$, where $\omega = \left[a + \sum_{i=1}^{m}(r_i+1)x_{i:m:n}^2\right]/t^2$.

4 Predicting Future Observations

It is often of interest to predict the k-th failure time in a future sample of size N from the same distribution. Let $Y_{(1)} < \cdots < Y_{(N)}$ be the order statistics in a sample of size N with lifetimes distributed as (1). The probability density function of the k-th $(1 \le k \le N)$ order statistic is

$$f(y_{(k)}|\theta) = \frac{N!}{(k-1)!(N-k)!}\,\frac{y_{(k)}}{\theta^2}\left(1 - \exp\left\{-\frac{y_{(k)}^2}{2\theta^2}\right\}\right)^{k-1}\exp\left\{-(N-k+1)\frac{y_{(k)}^2}{2\theta^2}\right\}, \tag{14}$$

for $y_{(k)} > 0$, zero elsewhere. By forming the product of (7) and (14), and integrating out θ over the set $\{\theta; 0 < \theta < \infty\}$, the predictive distribution of $Y_{(k)}$, given $\boldsymbol{X}$, is

$$\begin{aligned} f(y_{(k)}|\boldsymbol{x}) = {} & \frac{2(N!)(b+m)}{(k-1)!(N-k)!}\,y_{(k)}\left[a + \sum_{i=1}^{m}(r_i+1)x_{i:m:n}^2\right]^{b+m} \\ & \sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}\left[a + \sum_{i=1}^{m}(r_i+1)x_{i:m:n}^2 + (N-k+j+1)y_{(k)}^2\right]^{-(b+m+1)}, \end{aligned}$$

for $y_{(k)} > 0$, zero elsewhere. Under squared error loss, the Bayes predictive estimator of $Y_{(k)}$ is the expectation of the predictive distribution, that is,

$$\tilde{Y}_{(k)} = E(Y_{(k)}|\boldsymbol{X}) = \frac{N!\sqrt{\frac{\pi}{2}}}{(k-1)!(N-k)!}\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}\left(N-k+j+1\right)^{-\frac{3}{2}}\tilde{\theta},$$

where $\tilde{\theta}$ is the Bayes estimator of θ given in (9).

The $100(1-\alpha)\%$ HPD prediction interval (ℓ_k, u_k) for $Y_{(k)}$ should simultaneously satisfy $\int_{\ell_k}^{u_k} f(y_{(k)}|\boldsymbol{x})dy_{(k)} = 1-\alpha$ and $f(\ell_k|\boldsymbol{x}) = f(u_k|\boldsymbol{x})$. After some algebraic simplification, the

Table 1: Progressively type II censored sample from Rayleigh distribution

i	1	2	3	4	5	6	7	8	9	10
x_i	0.1970	0.3029	0.5786	0.9758	1.0066	1.3734	1.4159	1.5209	2.0482	2.2496
r_i	2	0	0	2	0	0	0	2	0	4

$100(1-\alpha)\%$ HPD prediction interval (ℓ_k, u_k) satisfies

$$
\begin{aligned}
1-\alpha = {} & \frac{N!}{(k-1)!(N-k)!}\left[a+\sum_{i=1}^{m}(r_i+1)x_{i:m:n}^2\right]^{b+m}\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}\frac{1}{N-k+j+1} \\
& \left\{\left[a+\sum_{i=1}^{m}(r_i+1)x_{i:m:n}^2+(N-k+j+1)\ell_k^2\right]^{-(b+m)}\right. \\
& \left.-\left[a+\sum_{i=1}^{m}(r_i+1)x_{i:m:n}^2+(N-k+j+1)u_k^2\right]^{-(b+m)}\right\},
\end{aligned}
$$

and

$$
\begin{aligned}
& \sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}u_k\left[a+\sum_{i=1}^{m}(r_i+1)x_{i:m:n}^2+(N-k+j+1)u_k^2\right]^{-(b+m+1)} \\
& = \sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}\ell_k\left[a+\sum_{i=1}^{m}(r_i+1)x_{i:m:n}^2+(N-k+j+1)\ell_k^2\right]^{-(b+m+1)}.
\end{aligned}
$$

5 Numerical Example and Simulation Study

5.1 Illustrative Example

Consider a progressively type II censored sample of size $m = 10$ from a sample of size $n = 20$ with censoring scheme $\boldsymbol{r} = (2, 0, 0, 2, 0, 0, 0, 2, 0, 4)$ from Rayleigh distribution with parameter θ. It is assumed that the prior distribution of θ is a square-root inverted-gamma distribution given in (6) with $a = 7.0$ and $b = 2.0$. Table 1 is a progressively type II censored sample. This sample was simulated by using the following algorithm.

Step 1. For the given values of prior parameters (a, b), generate θ from the square-root inverted-gamma distribution.

Step 2. Using θ obtained in Step 1, generate a progressively type II censored sample of size m from a sample of size n with censoring scheme $\boldsymbol{r} = (r_1, \ldots, r_m)$ from Rayleigh distribution according to the algorithm presented in Balakrishnan and Aggarwala (2000, pp. 32-33).

Table 2: Bayes predictive estimates and HPD prediction intervals

k	$\tilde{Y}_{(k)}$	(l_k, u_k)
1	0.6010	$(0.0719, 1.0068)$
2	0.9260	$(0.3081, 1.3886)$
3	1.1924	$(0.5086, 1.6865)$
4	1.4379	$(0.6897, 1.9625)$
5	1.6794	$(0.8628, 2.2379)$

From (4) and (5), we obtained the maximum likelihood estimates of θ and $R(t = 2|\theta)$ to be $\hat{\theta} = 1.4957$ and $\hat{R}_{t=2} = 0.4090$, respectively. From (9) and (10), we determined the Bayes estimates of θ and $R(t = 2|\theta)$ to be $\tilde{\theta} = 1.5163$ and $\tilde{R}_{t=2} = 0.4092$. Similarly, we can calculate the HPD estimates of θ and $R(t = 2|\theta)$ to be $\theta^* = 1.4386$ and $R^*_{t=2} = 0.3979$. To obtain the 90% HPD credible intervals for θ and $R(t|\theta)$ we need to use the Newton-Raphson method to solve the equations in Section 3.2. The 90% HPD credible intervals for θ and $R(t = 2|\theta)$ are $(1.0699, 1.7393)$ and $(0.1860, 0.5261)$, respectively.

Furthermore, consider a future sample of size $N = 5$ from the same distribution. Using the formula in Section 4, Bayes predictive estimates and the corresponding 90% HPD prediction intervals for the k-th, $1 \leq k \leq 5$, failure times are shown in Table 2. It is easy to see that the length of the HPD prediction interval increases as k increases. This implies that the prediction is less precise as a larger order statistic is considered.

5.2 Simulation Results

In the following, the maximum likelihood estimates and Bayes estimates of the parameter θ and the $R(t|\theta)$ are compared via Monte Carlo simulation. Using the method given in Section 5.1, the progressively type II censored samples from Rayleigh distribution with parameter θ having square-root inverted-gamma prior density were generated for $(a, b) = (2, 5)$, $t = 0.5$, and different combinations of n, m, and censoring schemes $\boldsymbol{r}$. Table 3 provides the estimated risks of the maximum likelihood estimators and Bayes estimators. The estimated risks were calculated as the average of squared deviations. All the results were computed over 10000 simulations. From Table 3, we can see that the Bayes estimates are better than their corresponding maximum likelihood estimates for the considered cases. However, more investigations are needed to see the robustness of the choice of the prior.

References

Balakrishnan, N. and Aggarwala, R. (2000). *Progressive Censoring – Theory, Methods, and Applications*. Birkhäuser, Boston.

Table 3: Estimated risks of the MLE and Bayes estimates with prior parameters $(a, b) = (2, 5)$

n	m	censoring scheme	parameter θ MLE	parameter θ Bayes	reliability function $R(t = 0.5\|\theta)$ MLE	reliability function $R(t = 0.5\|\theta)$ Bayes
20	5	(4*0,15)	0.0611	0.0268	0.0490	0.0301
		(15,4*0)	0.0627	0.0275	0.0500	0.0306
	15	(14*0,5)	0.0503	0.0352	0.0505	0.0404
		(5,14*0)	0.0492	0.0345	0.0498	0.0399
25	10	(9*0,15)	0.0524	0.0319	0.0493	0.0363
		(15,9*0)	0.0527	0.0321	0.0500	0.0369
	20	(19*0,5)	0.0485	0.0367	0.0505	0.0423
		(5,19*0)	0.0488	0.0370	0.0513	0.0430
50	20	(19*0,30)	0.0489	0.0370	0.0514	0.0431
		(30,19*0)	0.0484	0.0365	0.0498	0.0417
	30	(29*0,20)	0.0474	0.0391	0.0511	0.0451
		(20,29*0)	0.0472	0.0388	0.0507	0.0447

Balasooriya, U. and Saw, S. L. C. (1998). Reliability sampling plans for the two-parameter exponential distribution under progressive censoring. *Journal of Applied Statistics*, **25**, 707-714.

Dyer, D. D. and Whisenand, C. W. (1973a). Best linear unbiased estimator of the parameter of the Rayleigh distribution – Part I: Small sample theory for censored order statistics. *IEEE Transactions on Reliability*, **22**, 27-34.

Dyer, D. D. and Whisenand, C. W. (1973b). Best linear unbiased estimator of the parameter of the Rayleigh distribution – Part II: Optimum theory for selected order statistics. *IEEE Transactions on Reliability*, **22**, 229-231.

Fernández, A. J. (2000). Bayesian inference from type II doubly censored Rayleigh data. *Statistics & Probability Letters*, **48**, 393-399.

Howlader, H. A. and Hossain, A. (1995). On Bayesian estimation and prediction from Rayleigh based on type II censored data. *Communications in Statistics – Theory & Methods*, **24**, 2249-2259.

Kong, F. and Fei, H. (1996). Limits theorems for the maximum likelihood estimate under general multiply type II censoring. *Annals of the Institute of Statistical Mathematics*, **48**, 731-755.

Polovko, A. M. (1968). *Fundamentals of Reliability Theory.* Academic Press, New York.

Wong, J. Y. (1993). Simultaneously estimating the three Weibull parameters from progressively censored samples. *Microelectronics and Reliability*, **33**, 2217-2224.

Wu, S.-J. (2003). Estimation for the two-parameter Pareto distribution under progressive censoring with uniform removals. *Journal of Statistical Computation and Simulation*, **73**, 125-134.