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Abstract

If the process observations are autocorrelated, the performance of control chart is influenced significantly. This autocorrelation leads to a large false alarm rate. This paper considers the problem of monitoring the mean of AR(1) process with random error. We provide a simple algorithm to improve the estimation results of process parameters. Simulation results show that the proposed method can produce stable and adequate estimates for the AR(1) process with random error even though the sample size is small.

Keywords: Autoregressive moving average model; Exponentially weighted moving average control chart; First-order autoregressive model; Maximum likelihood estimation; Shewhart control chart.

1 Introduction

1.1 Problem

In the applications of control chart, the assumption usually made is that the process observations $X_1, X_2, \ldots$ are independent and identically distributed (iid) random variables from a normal distribution with mean μ and variance σ_ϵ^2. These process observations can be expressed as

$$X_k = \mu + \epsilon_k, \quad k = 1, 2, \ldots,$$

where the random error terms $\epsilon_1, \epsilon_2, \ldots$ are iid normal random variables with mean 0 and variance σ_ϵ^2. When the process is in-control, the mean μ is often assumed to be the target value. This value can be changed to some other values when a special cause occurs. However, observations from a process are often autocorrelated, and this autocorrelation significantly affects the performance of control charts under independent assumption.

1.2 Literature

If the process observations are autocorrelated, in general, there are two methods to construct control charts. The first method uses the Shewhart control chart but adjusts the control limits to account for the autocorrelation (*e.g.*, see Vasilopoulos and Stamboulis (1978) and VanBrackle and Reynolds (1997)). The second method considers a time series model to fit the process data and then uses the residuals to develop control charts. Control charts based on residuals have been investigated by several authors such as Abraham and Kartha (1979), Alwan (1991), Alwan and Roberts (1988), Harris and Ross (1991), Montgomery (2000) and Lu and Reynolds (1999a, 1999b, 2001).

Reynolds *et al.* (1996) studied the properties of fixed sampling interval and variable sampling interval $\overline{X}$ control charts for a first-order autoregressive (AR(1)) process with random error. The observations of this process can be written as

$$X_k = \mu_k + \epsilon_k, \quad k = 1, 2, \ldots, \tag{1}$$

where μ_k is the mean at time t_k and $\epsilon_1, \epsilon_2, \ldots$ are iid normal random variables with mean 0 and variance σ_ϵ^2. If μ_k follows the AR(1) process and the $(k-1)$-th and k-th samples are one time unit apart, then μ_k can be written as

$$\mu_k = (1-\phi)\xi + \phi\mu_{k-1} + \alpha_k, \quad k = 1, 2, \ldots, \tag{2}$$

where ξ is a parameter, ϕ is the autoregressive parameter and α_k's are iid normal random variables with mean 0 and variance σ_α^2, and α_k's and ϵ_k's are independent. The process is stationary if $|\phi| < 1$. However, in practice, for most process of interest in control chart applications, ϕ is nonnegative. This paper assumes that $0 \le \phi < 1$. The model determined by equations (1) and (2) is a special case of a first-order autoregressive moving average (ARMA(1,1)) process. If $\phi = 0$, then the AR(1) process with random error still holds and the means at different times are independent. The distribution of means μ_k's depends on the starting point μ_0. If the starting point μ_0 follows a normal distribution with mean ξ and variance $\sigma_\mu^2 = \frac{\sigma_\alpha^2}{1-\phi^2}$, then μ_k is normally distributed with mean ξ and variance σ_μ^2. Hence, it is easy to show that X_k, $k = 1, 2, \ldots$ are normally distributed with mean ξ and variance $\sigma_X^2 = \sigma_\mu^2 + \sigma_\epsilon^2$. Notice that the variance σ_X^2 consists of the variance due to the AR(1) process and the variance due to the random error. Let ψ be the proportion of variance due to the AR(1) process, that is,

$$\psi = \frac{\sigma_\mu^2}{\sigma_\mu^2 + \sigma_\epsilon^2}. \tag{3}$$

We can show that the correlation between X_k and X_{k-1} is $\rho = \phi\psi$. When $\psi = 1$, the sequence $X_1, X_2, \ldots$ is an AR(1) process. This model has been discussed, for example, by Harris and Ross (1991), Wardell *et al.* (1994) and Lu and Reynolds (1999a, 1999b, 2001).

Box *et al.* (1994) showed that the AR(1) process with random error is equivalent to an ARMA(1,1) process. The ARMA(1,1) process can be written as

$$X_k = (1-\phi)\xi + \phi X_{k-1} + \gamma_k - \theta\gamma_{k-1}, \quad k = 1, 2, \ldots,$$

where ϕ and θ are the parameters and γ_k's are iid normal random variables with mean 0 and variance σ_γ^2. If $0 < \phi < 1$ and $\sigma_\epsilon^2 > 0$, Reynolds *et al.* (1996) gave equations for expressing the ARMA(1,1) parameters in terms of the parameters in the AR(1) process with random error. That is,

$$\theta = \frac{\sigma_\alpha^2 + (1+\phi^2)\sigma_\epsilon^2}{2\phi\sigma_\epsilon^2} - \frac{1}{2}\sqrt{\left(\frac{\sigma_\alpha^2 + (1+\phi^2)\sigma_\epsilon^2}{\phi\sigma_\epsilon^2}\right)^2 - 4} \tag{4}$$

and

$$\sigma_\gamma^2 = \frac{\phi}{\theta}\sigma_\epsilon^2. \tag{5}$$

Conversely, if the parameters in the ARMA(1,1) process satisfy $0 \le \theta \le \phi < 1$ and $\sigma_\gamma^2 > 0$, then the parameters in the AR(1) process with random error can be expressed in terms of the ARMA(1,1) parameters as follows:

$$\sigma_\alpha^2 = \frac{(\phi-\theta)(1-\phi\theta)}{\phi}\sigma_\gamma^2 \tag{6}$$

and

$$\sigma_\epsilon^2 = \frac{\theta}{\phi}\sigma_\gamma^2. \tag{7}$$

When observations are from the AR(1) process with random error, it is more convenient to treat these observations as from an ARMA(1,1) process. Standard time series estimation techniques (*e.g.*, see Box *et al.* (1994)) can be used to estimate the parameters. Then equations (6) and (7) can be used to estimate the parameters in the AR(1) process with random error as long as the estimates of the parameters in the ARMA(1,1) model satisfy $0 \leq \hat{\theta} \leq \hat{\phi} < 1$ and $\hat{\sigma}_\gamma^2 > 0$. The estimation result depends on a precise parameter estimation in the ARMA(1,1) process, and the performance of the parameter estimation in the ARMA(1,1) process depends on the sample size. Lu and Reynolds (1999a) indicated that the estimation results of ϕ and θ in the ARMA(1,1) process are unstable if the sample size is not large enough. We also find that there is a large possibility to produce negative estimates of ϕ and θ when the sample size is small. Actually, when the parameter constraint $0 \leq \theta \leq \phi < 1$ is violated, the control chart can not be constructed adequately. Some researchers may suggest to set $\hat{\theta}$ to be zero when it is negative. However, if $\hat{\theta} = 0$, the estimate of ψ is always equal to 1. This will result in an inadequate control chart with high false alarm rate because the chart parameters depend on the estimates of ϕ, ψ and σ_X^2 of the AR(1) process with random error.

1.3 Overview

In this paper, a simple algorithm based on conditional maximum likelihood estimation method is provided in Section 2. In Section 3, some simulation results are provided to evaluate the performance of the proposed method. Conclusions are made in Section 4.

2 Main Results

Assume that $X_1, X_2, \ldots$ are observations from equations (1) and (2). For any integer $k \geq 1$, let $Y_k = X_k - \xi$ and $\mu_k^{'} = \mu_k - \xi$. Then equations (1) and (2) can be rewritten as

$$Y_k = \mu_k^{'} + \epsilon_k, \tag{8}$$

and

$$\mu_k^{'} = \phi\mu_{k-1}^{'} + \alpha_k, \tag{9}$$

for $k = 1, 2, \ldots$. Assume that μ_0 is normally distributed with mean ξ and variance σ_μ^2. Using equations (8) and (9), we can show that the mean of an individual observation Y_k is 0 and the variance is $\sigma_Y^2 = \sigma_X^2 = \sigma_\mu^2 + \sigma_\epsilon^2 = \frac{\sigma_\alpha^2}{1-\phi^2} + \sigma_\epsilon^2$. By equation (3), it is easy to show that the covariance of Y_k and Y_{k-1} is $\phi\sigma_\mu^2$ and the correlation between Y_k and Y_{k-1} is $\rho = \phi\psi$. From equation (9), we have

$$\mu_k^{'} = \phi^k\mu_0^{'} + \sum_{j=1}^{k}\phi^{k-j}\alpha_j, \quad k = 1, 2, \ldots,$$

where the starting value $\mu_0^{'}$ is normally distributed with mean 0 and variance $\frac{\sigma_\alpha^2}{1-\phi^2}$. This implies that $E(\mu_k^{'}) = 0$ and $Var(\mu_k^{'}) = \frac{\sigma_\alpha^2}{1-\phi^2}$, $k = 1, 2, \ldots$. When $i \leq j$, we can show that

$$Cov(\mu_i^{'}, \mu_j^{'}) = \frac{\phi^{j-i}}{1-\phi^2}\sigma_\alpha^2.$$

When $i > j$, $Cov(\mu_i^{'}, \mu_j^{'}) = Cov(\mu_j^{'}, \mu_i^{'})$. Hence, the vector $\boldsymbol{\mu} = (\mu_1^{'}, \mu_2^{'}, \ldots, \mu_n^{'})^T$ is normally distributed with mean vector $\mathbf{0}$ and variance-covariance matrix $\sigma_\alpha^2 \boldsymbol{V}$, where $\boldsymbol{V}$ is an $n \times n$ symmetric matrix with entry $v_{ij} = \frac{\phi^{j-i}}{1-\phi^2}$. Let $\boldsymbol{\epsilon}$ denote the vector of random error terms $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)^T$. Then, $\boldsymbol{\epsilon}$ is normally distributed with mean vector $\mathbf{0}$ and variance-covariance matrix $\sigma_\epsilon^2 \boldsymbol{I}$, where $\boldsymbol{I}$ is an $n \times n$ identity matrix. Now, let $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_n)^T$. It follows that the data vector $\boldsymbol{Y} = \boldsymbol{\mu} + \boldsymbol{\epsilon}$ has a normal distribution with mean vector $\mathbf{0}$ and variance-covariance matrix $\sigma_\alpha^2 \boldsymbol{V} + \sigma_\epsilon^2 \boldsymbol{I}$.

Let $\delta = \frac{\sigma_\alpha^2}{\sigma_\epsilon^2}$. Since $\sigma_\alpha^2 = \sigma_\mu^2(1-\phi^2)$, $\psi = \frac{\sigma_\mu^2}{\sigma_\mu^2 + \sigma_\epsilon^2}$ and $\rho = \phi\psi$, we can get $\delta = \frac{(1-\phi^2)\rho}{\phi - \rho}$. Let $\boldsymbol{W}_{\rho,\phi} = \delta \boldsymbol{V} + \boldsymbol{I}$. Then the likelihood function is given by

$$\begin{aligned} L(\sigma_\epsilon^2, \phi, \rho; \boldsymbol{y}) &= f(\boldsymbol{y}; \sigma_\epsilon^2, \phi, \rho) \\ &= \frac{1}{(2\pi)^{n/2} |\sigma_\epsilon^2 \boldsymbol{W}_{\rho,\phi}|^{1/2}} \exp\left\{ -\frac{1}{2\sigma_\epsilon^2} \boldsymbol{y}^T \boldsymbol{W}_{\rho,\phi}^{-1} \boldsymbol{y} \right\}. \end{aligned}$$

The log-likelihood function may then be expressed as

$$\ell(\sigma_\epsilon^2, \phi, \rho; \boldsymbol{y}) = -\frac{n}{2}\log(2\pi) - \frac{1}{2}\log|\boldsymbol{W}_{\rho,\phi}| - \frac{n}{2}\log(\sigma_\epsilon^2) - \frac{1}{2\sigma_\epsilon^2}\boldsymbol{y}^T \boldsymbol{W}_{\rho,\phi}^{-1}\boldsymbol{y}, \tag{10}$$

and thus the maximum likelihood estimator (MLE) of σ_ϵ^2, found by maximizing (10), is,

$$\hat{\sigma}_\epsilon^2(\rho, \phi) = \frac{1}{n}\boldsymbol{y}^T \boldsymbol{W}_{\rho,\phi}^{-1}\boldsymbol{y}. \tag{11}$$

Substitute (11) into (10), we have

$$\ell(\phi, \rho; \boldsymbol{y}) \propto -n\log(\boldsymbol{y}^T \boldsymbol{W}_{\rho,\phi}^{-1}\boldsymbol{y}) - \log|W_{\rho,\phi}|. \tag{12}$$

Since it is hard to obtain the MLEs of ϕ and ρ simultaneously from (12), we suggest to find a suitable estimator of ρ first. The best candidate for estimating ρ is the sample autocorrelation coefficient

$$\hat{\rho} = \frac{\sum_{k=2}^n Y_k Y_{k-1}}{\sum_{k=2}^n Y_k^2}.$$

Substitute $\hat{\rho}$ into (12) and obtain

$$\ell(\phi, \hat{\rho}; \boldsymbol{y}) \propto -n\log(\boldsymbol{y}^T \boldsymbol{W}_{\hat{\rho},\phi}^{-1}\boldsymbol{y}) - \log|W_{\hat{\rho},\phi}|. \tag{13}$$

Finally, the estimate of ϕ is obtained by maximizing (13). Because $\phi > \rho$, searching the estimate of ϕ will be restricted in the interval $(\hat{\rho}, 1)$.

Suppose that the initial in-control observations $X_1, X_2, \ldots, X_n$ are collected. We can summarize the above estimation procedure as follows:

Step 1. Compute $\hat{\xi} = \bar{X}$ and obtain $Y_i = X_i - \hat{\xi}$, $i = 1, 2, \ldots, n$.

Step 2. Calculate the sample autocorrelation coefficient $\hat{\rho}$.

Step 3. Substitute $\hat{\rho}$ into (12) to obtain (13).

Step 4. Find the estimate of ϕ, say $\hat{\phi}$, in the interval $(\hat{\rho}, 1)$ so that (13) is maximized.

Step 5. Substitute $\hat{\rho}$ and $\hat{\phi}$ into (11) and get the estimate of σ_ϵ^2, say $\hat{\sigma}_\epsilon^2$.

The estimates of the other parameters in the AR(1) process with random error can be obtained as follows:

$$\begin{aligned}
\hat{\sigma}_\alpha^2 &= \hat{\delta}\hat{\sigma}_\epsilon^2 = \frac{(1-\hat{\phi}^2)\hat{\rho}}{\hat{\phi}-\hat{\rho}}\hat{\sigma}_\epsilon^2, \\
\hat{\sigma}_\mu^2 &= \frac{\hat{\sigma}_\alpha^2}{1-\hat{\phi}^2}, \\
\hat{\sigma}_X^2 &= \hat{\sigma}_\mu^2 + \hat{\sigma}_\epsilon^2,
\end{aligned}$$

and

$$\hat{\psi} = \frac{\hat{\sigma}_\mu^2}{\hat{\sigma}_X^2}.$$

Using the estimates $\hat{\phi}$, $\hat{\psi}$ and $\hat{\sigma}_X^2$, and the tables provided by Lu and Reynolds (1999a) and Reynolds *et al.* (1996), an $\bar{X}$ control chart or an exponentially weighted moving average (EWMA) control chart for autocorrelated data can be constructed at early stage.

3 Simulation Study

In order to evaluate the performance of the proposed method, some simulation studies are conducted. The data sets with different correlations are generated from the AR(1) process with random error for sample sizes 50, 100, 200 and 300. The values of parameters are as follows: (i) $\xi = 0$, $\phi = 0.25, 0.75$, $\sigma_\mu^2 = 0.4$ and $\sigma_\epsilon^2 = 0.6$, and (ii) $\xi = 0$, $\phi = 0.60, 0.80$, $\sigma_\mu^2 = 0.75$ and $\sigma_\epsilon^2 = 0.25$. Accordingly, we have $\sigma_X^2 = \sigma_\mu^2 + \sigma_\epsilon^2 = 1$, $\psi = \frac{\sigma_\mu^2}{\sigma_X^2} = 0.40, 0.75$, and $\rho = \phi\psi =$0.1, 0.3, 0.45, 0.6, respectively. The statistical software R is used to generate data sets and do parameter estimation. Each combination is repeated 3000 times, and then the average and mean squared error of estimates are computed. Tables 1 to 4 of Tsai *et al.* (2004) show the simulation results by using the standard time series maximum likelihood estimation method (STSMLE) and the proposed method.

The STSMLE can be used to estimate the parameters in the ARMA(1,1) model, and then equations (6) and (7) are used to determine estimates of the parameters of the AR(1) process with random error. Notice that this technique is fulfilled only if the condition $0 \le \theta \le \phi < 1$ is met. However, if the true value of θ is small, there is a high possibility to get a negative estimate of θ. Even though the true value of θ is not very small, it is still possible to obtain an estimate of ϕ which is smaller than the estimate of θ. These two situations result in a failure in parameter estimation.

From Tables 1 to 4 of Tsai *et al.* (2004), we can see the following results:

1. The mean values of estimates of σ_X^2 and ψ by using the STSMLE are very far from the true values, especially for small ρ.

2. The STSMLE always underestimates σ_X^2 for all combinations of parameters, and becomes worse when n is small.

3. When $\psi < 0.5$, the STSMLE seriously overestimates ψ. Actually, the STSMLE obtains estimates which are larger than 0.5 whenever the true value of ψ is.

4. When ρ is large, the mean values of estimates of ϕ obtained by using the STSMLE are closer to the true values of ϕ than the proposed method. However, they get to close when n is large.

5. The STSMLE has a higher failure rate than the proposed method.

6. The STSMLE will have a bad performance of control chart due to it always seriously underestimate σ_X^2 and seriously overestimate ψ. For example, if the Shewhart control chart is considered, the control limits are $\overline{X} \pm h\hat{\sigma}_X$. Table 7 of Reynolds *et al.* (1996) indicates that a large value of ψ results in a small value of h. Since both h and $\hat{\sigma}_X$ are smaller than the true values of parameters, we will get a small in-control average run length (ARL) which leads to a high false alarm rate.

From Tables 1 to 4 of Tsai *et al.* (2004) we also see that the proposed method improves the performance of parameter estimation. For the case of $\psi = 0.4$, The Table 2 of Tsai *et al.* (2004) shows the proposed method produce a good estimation and lower failure rate if the sample size $n \geq 200$. However, in the Table 1 of Tsai *et al.* (2004), when ρ is small, the performance of parameter estimation is not as good as that in the Table 2 of Tsai *et al.* (2004). A larger sample size is needed to obtain good estimation results. For the case of $\psi = 0.75$, Tables 3 and 4 of Tsai *et al.* (2004) show that only 100 observations is enough to produce a good estimation.

4 Conclusions

In this paper, an alternative estimation method is provided to estimate the parameters of the AR(1) process with random error directly without reference to the ARMA(1,1) model. The proposed method performs better than the standard time series estimation method especially when the sample size is small. According to the simulation results, only 200 observations are enough. In practice, most process of interest in control chart applications, ϕ is nonnegative. Therefore, we only consider the case of $0 \leq \phi < 1$. If the sample autocorrelation $\hat{\rho}$ is negative, then the proposed method fails. However, from our simulation, such failure seldom occurs. Hence, the proposed method is adequate to be used in the initial state of correlated process.

Until a large sample, say 1000 or more observations, are accumulated, the standard time series estimation method may be used for the future monitoring on process.

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