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行政院國家科學委員會補助專題研究計畫成果報告

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※ 變異數估計式在Stochastic Domination法則底下之研究 ※

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計畫編號：NSC-90-2118-M-163-001
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一、中文摘要

假設 X_1, X_2, $\cdots$, X_n 是 i.i.d. 且來自 $N_p(\mu,\sigma^2 I_p)$ 之觀察值，其中 $\mu\in\Re^p$ 和 $\sigma^2>0$ 全然不知。本研究利用充分性簡化原始問題，並從決策理論架構底下來探討變異數之估計。其中常被使用之變異數估計式是最大概似估計式 $\hat{\sigma}^2_{MLE}=S/(m+p+1)$。在 quadratic 和 entropy 損失函數底下，σ^2 的最佳 affine 估計式分別是 $\hat{\sigma}^2_q=S/(m+1)$ 和 $\hat{\sigma}^2_e=S/(m-1)$；除此之外，亦有變異數估計式 $\hat{\sigma}^2_\phi=S/(m-1)\{1-\phi(U)\}$ 其中 $U=S/(S+\|X\|^2)$ 且 $0\le\phi(\cdot)\le1$ 是某些特定之函數。在 entropy 損失函數底下，不同 $\phi(\cdot)$ 之選擇，可以得到一些較佳於 $\hat{\sigma}^2_e=S/(m-1)$ 之估計式 $\hat{\sigma}^2_\phi$。近幾年來，以上所提之變異數估計式 $\hat{\sigma}^2_{MLE}$、$\hat{\sigma}^2_q$、$\hat{\sigma}^2_e$、$\hat{\sigma}^2_\phi$ 之間的性質在降低期望風險函數準則底下，已經有許多結果及定理被提出；另外在 "Pitman Nearness Criterion" (PNC) 底下，J.J. Lin et.al.(2002)，以上所提之變異數估計式 $\hat{\sigma}^2_{MLE}$、$\hat{\sigma}^2_q$、$\hat{\sigma}^2_e$、$\hat{\sigma}^2_\phi$ 之間的性質也有一些結論及發現。本研究針對以上所提之變異數估計式，探討了各個變異數估計式之間的性質，在 "Stochastic Domination 法則" (SDC) 底下看看是否有其中的變異數估計式會比另一個變異數估計式來得好。

關鍵詞：Stochastic Domination 法則、quadratic 損失函數，entropy 損失函數，風險函數。

二、英文摘要

Abstract

For estimating a normal variance under squared error loss function it is well known that the best affine (location and scale) equivariant estimator, which is better than the maximum likelihood estimator as well as the unbiased estimator is also inadmissible. The improved estimators, e.g., Stein type, Brown type and Brewster-Zidek type, are all scale equivariant but not location invariant. Lately a good amount of research has been done to compare the improved estimators in terms of risk, some of the estimators are examined in terms of Pitman Nearness Criterion and have made some interesting observations in the process. However, very little attention had been paid to compare these estimators in terms of Stochastic Domination criterion. In this research we take a comprehensive study in terms of Stochastic Domination criterion to compare various variance estimators.

Keywords: Affine equivariance, loss function, risk function, non-central chi-square distribution.

三、緣由與目的

Assume that we have independent random observations X and S such that $X=(X_1,X_2,\cdots,X_p)'$ follows a

$N_p(\mu, \sigma^2 I_p)$ (p-dmensional normal) distribution and (S/σ^2) follows a χ^2_{m-1} (Chi-square with $(m-1)$ d.f.) distribution. Consider the problem of estimation of σ^2 efficiently.

The above-described model is encountered if one has independent and identically distributed (iid) observations X_1, $X_2, \cdots, X_n$ from a $N_p(\mu, \sigma^2 I_p)$ distribution. The data can be reduced by sufficiency principle, and one needs to focus only on $X = \sqrt{n}\overline{X}$, $\overline{X} = (\sum_{i=1}^{n} X_i / n)$ and $S = \sum_{i=1}^{n} \left\| X_i - \overline{X} \right\|^2$. Note that X follows $N_p(\mu, \sigma^2 I_p)$ and $S/\sigma^2 \sim \chi^2_{m-1}$ with $\mu = \sqrt{n}\theta$ and $(m-1) = p(n-1)$.

Similarly, in a linear model setup $Y_{n\times1} = X_{n\times p}\beta_{p\times1} + \in_{n\times1}$ where $\in_{n\times1}$ follows $N_n(0, \sigma^2 I_n)$ distribution, let $\hat{\beta}$ be the least squares estimate of β and $M_{p\times p}$ be such that $M'M = (X'X)$, then $(M\hat{\beta})$ plays the role of X and S plays the role of error sum of squares (SSE) for suitable choices of θ and m.

In classical statistics, usual estimators of σ^2 are (i) the unique minimum variance unbiased estimator (UMVUE) of σ^2 given by

$$\hat{\sigma}^2_u = S/(m-1); \qquad (1.1)$$

and (ii) the maximum likelihood estimator (MLE) of σ^2 given by

$$\hat{\sigma}^2_{ml} = S/(m+p-1). \qquad (1.2)$$

In a decision-theoretic setup the two most commonly used loss functions are

$$L_S(\hat{\sigma}^2, \omega) = (\hat{\sigma}^2/\sigma^2 - 1)^2 \qquad (1.3)$$

$$L_E(\hat{\sigma}^2, \omega) = (\hat{\sigma}^2/\sigma^2) - \ln(\hat{\sigma}^2/\sigma^2) - 1. \qquad (1.4)$$

where $\hat{\sigma}^2$ is an estimator of σ^2 and $\omega = (\theta, \sigma^2)$. The loss functions L_S and L_E are called respectively the squared error loss (SEL) and the entropy loss (EL).

If we consider the group $\mathcal{G}_{\mathcal{A}}$ of affine transformation. (i.e., $(X,S) \to (aX+b, a^2S)$, $a>0$, $b \in \Re^p$ = p - dimensional real space), then the affine equivariant estimators have the form $\hat{\sigma}^2_c = cS$, where $c>0$ is a constant. Since the group $\mathcal{G}_{\mathcal{A}}$ (and the corresponding induced group $\mathcal{G}_{\mathcal{A}}$ acting on $\Omega = \{\omega = (\theta, \sigma^2) \mid \theta \in R^p$, $\sigma^2 > 0\}$ such that $(\theta, \sigma^2) \to (a\theta + b, a^2\sigma^2)$, $a>0$, $b \in R^p$) is transitive, the affine equivariant estimator $\hat{\sigma}^2_c$ has constant risk on Ω. Therefore, one can find the best affine equivariant estimator (BAEE) of σ^2 by minimizing the risk of $\hat{\sigma}^2_c$ with respect to (wrt) c. The BAEEs of σ^2 under L_S and L_E are respectively

$$\hat{\sigma}^2_S = S/(m+1)$$

and $\hat{\sigma}^2_E = \hat{\sigma}^2_u = S/(m-1)$ (1.5).

Interestingly, $\hat{\sigma}^2_S$ $(\hat{\sigma}^2_E)$ is inadmissible under L_S (L_E), and improved and therefore is inadmissible under L_S (L_E), and improved estimators are only scale equivariant but not location invariant. Stein (1964) showed that under L_S, an improved estimator of σ^2 can be found which is uniformly better than $\hat{\sigma}^2_S$. Brown (1968) proposed a similar but somewhat different estimator of σ^2 under L_S. However, both $\hat{\sigma}^2_{S(S)}$ and $\hat{\sigma}^2_{S(B)}$ are nonanalytic and hence inadmissible. Brown's technique was further extended by Brewster and Zidek (1974) who obtained an admissible improved estimator of σ^2.

For a comprehensive review on normal variance estimation and related topics see Pal, Ling and Lin (1998). Estimator analogous to (1.6), (1.7) and (1.8) under the loss L_E can be derived.

While emphasis had been given to compare various variance estimators in terms of risk, the attention had also been paid to do the same in terms of another important criterion namely, the Pitman nearness criterion (PNC).

Comparison of these three affine equivariant estimators, e.g., $\hat{\sigma}^2_{ml}$, $\hat{\sigma}^2_u = \hat{\sigma}^2_E$ and $\hat{\sigma}^2_S$, in terms of PNC has been studied by Lin, Pal and Chang (2002). It appears that $\hat{\sigma}^2_u$ the UMVUE (as well as BAEE under L_E) is the best among these three popular estimators.

It appears, quite interestingly, that the unbiased estimator emerges as the most preferable among the three affine equivariant estimators. Besides, the comparison of $\hat{\sigma}^2_u = \hat{\sigma}^2_E$ against $\hat{\sigma}^2_{E(S)}$ and $\hat{\sigma}^2_{E(BZ)}$, Stein type and Brewster-Zidek type improved estimators under L_E respectively, have been also undertaken by Lin, J. J., Pal, N. and Chang, C. H. (2002). On the other hand, several properties of the Pitman's measurement of closeness has been criticized by Robert, Hwang and Strawderman (1993) and defended by Ghosh, Keating, Sen (1993), including the lack of transitivity, its incompatibility with the Stochastic Domination Criteria and the difficulties of the use of its joint probability distribution of the estimators.

In this research, SDC is defined as given in Definition 2.1, for our variance estimation problem.

Definition 2.1: Given two estimators, say $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$, of σ^2, $\hat{\sigma}_1^2$ is said to be better than $\hat{\sigma}_2^2$ in terms of SDC (Stochastic Domination Criterion) if

$$P_r\left[\left|\hat{\sigma}_1^2-\sigma^2\right|\le d\right] \ge P_r\left[\left|\hat{\sigma}_2^2-\sigma^2\right|\le d\right]$$

$\forall\, d>0$

There are nothing known about the above mentioned variance estimators $\hat{\sigma}^2_{ml}$ 、$\hat{\sigma}^2_u$ 、$\hat{\sigma}^2_s$ under "Stochastic Domination Criteria" (SDC) . Therefore we study and do the comparisons among theses normal variance estimators to see if any estimators is better than others most of the time in terms of "Stochastic Domination Criteria" (SDC).

四、結果與討論

COMPARISON OF AFFINE EQUIVARLANT ESTIMATORS

2.1.Case I— For $d. \ge 1$ $(d \ge \sigma^2)$

$$\Delta(\hat{\sigma}_1^2 \mid \hat{\sigma}_2^2) = Q(\hat{\sigma}_1^2 \mid d.,m,p,1) - Q(\hat{\sigma}_2^2 \mid d.,m,p,1)$$

$$= \int_0^{\frac{1+d.}{c_1}} (x^2_{m-1} pdf)dx - \int_0^{\frac{1+d.}{c_2}} (x^2_{m-1} pdf)dx .$$

Result 2.1: For any two variance estimators $\hat{\sigma}_i^2 = c_i S$ and $\hat{\sigma}_j^2 = c_j S$, if $c_i \le c_j$ then $P_r\left[\left|\hat{\sigma}_1^2-\sigma^2\right|\le d\right] \ge P_r\left[\left|\hat{\sigma}_2^2-\sigma^2\right|\le d\right]$ for $d \ge \sigma^2$. In other words, $\hat{\sigma}_i^2$ dominates $\hat{\sigma}_j^2$ in SDC if $c_i \le c_j$

Remark 2.1:

In case . for $d \ge \sigma^2$

(i) $\hat{\sigma}^2_{ml}$ is better than $\hat{\sigma}^2_s$ in SDC

(ii) $\hat{\sigma}^2_s$ is better than $\hat{\sigma}^2_u$ in SDC

(iii) $\hat{\sigma}^2_{ml}$ is better than $\hat{\sigma}^2_u$ in SDC

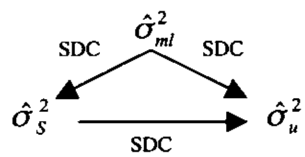

Figure 2.1 Comparison of $\hat{\sigma}^2_u$, $\hat{\sigma}^2_s$ and $\hat{\sigma}^2_{ml}$

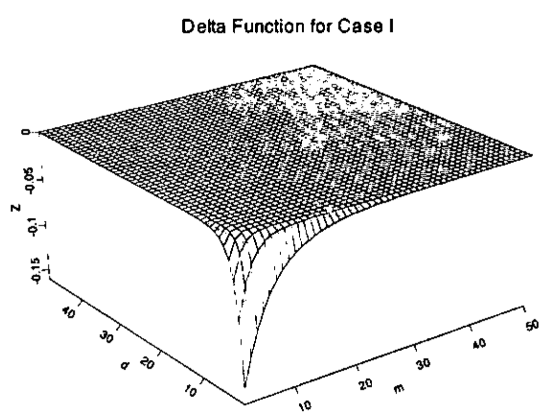

Figure 2.2 3D graph of $\Delta(\hat{\sigma}^2_u \mid \hat{\sigma}^2_s)$ or case I.

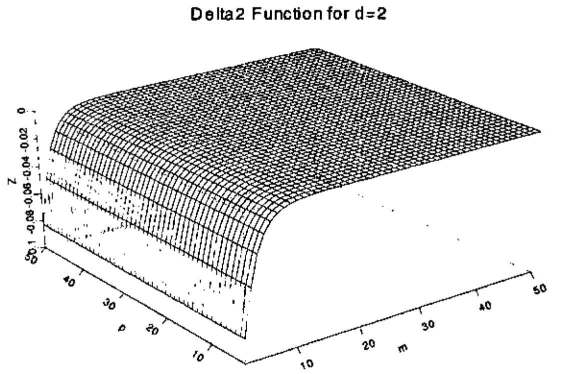

Figure 2.3 3D graph of $\Delta\left(\hat{\sigma}_u^2 \middle| \hat{\sigma}_{ml}^2\right)$ **for** $p \geq 3$ **and** $d_* = 2$

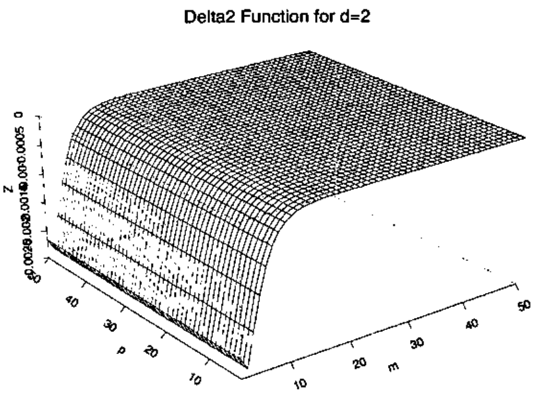

Figure 2.4 3D graph of $\Delta\left(\hat{\sigma}_s^2 \middle| \hat{\sigma}_{ml}^2\right)$ **for** $p \geq 3$ **and** $d_* = 2$

2.2.Case II— for $d_* < 1 \quad (d < \sigma^2)$

$$Q(\hat{\sigma}_i^2 | d, m, p, \sigma^2) \quad = Q(\hat{\sigma}_i^2 | d_*, m, p, 1) \quad =$$

$$\begin{cases} \int_{(1-d_*)/c_i}^{(1+d_*)/c_i} (\chi_{m-1}^2 pdf)dx, \; for\; 0 < d_* < 1 ......(2.5) \\ \int_0^{(1+d_*)/c_i} (\chi_{m-1}^2 pdf)dx, \; for\; d_* \geq 1 ....(2.6) \end{cases}$$

To simplify the notation, let G(t) denote the *cdf* of χ_{m-1}^2 distribution, i.e., $G(t) = \int_0^t (\chi_{m-1}^2 pdf)dx$, then for $d_* < 1$,

$$\Delta(\hat{\sigma}_1^2 | \hat{\sigma}_2^2) = G\left(\frac{1+d_*}{c_1}\right) - G\left(\frac{1-d_*}{c_1}\right) - G\left(\frac{1+d_*}{c_2}\right) + G\left(\frac{1-d_*}{c_2}\right).$$

.the problem is studied in cases(A)-(C).The values of the difference $\Delta(\hat{\sigma}_1^2 | \hat{\sigma}_2^2)$ of the comparision of the above three mentioned estimators are provided in tables and graphs .

(A) Comparison of $\hat{\sigma}_u^2$ and $\hat{\sigma}_S^2$.

(B) Comparison of $\hat{\sigma}_u^2$ and $\hat{\sigma}_{ml}^2$.

(C) Comparison of $\hat{\sigma}_S^2$ and $\hat{\sigma}_{ml}^2$.

For saving the space, the graph tables are omitted.

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