行政院國家科學委員會專題研究計畫成果報告

在型 II 設限雙參數指數分配下逐步應力加速試驗的推論
Inference on Step-Stress Accelerated Tests under Two-Parameter Exponential Distribution and Type II Censoring

計畫編號：NSC 89-2118-M-032-016
執行期限：89年8月1日至90年7月31日
主持人：吳碩傑　淡江大學統計學系
計畫參與人員：阮建豐 政治大學統計系碩士班

1　中文摘要

隨著科技的進步，許多壽命試驗在結束時常常只收集到很少產品壽命資料，甚至可能沒有一個測試元件發生故障。在這種情況下，加速壽命試驗是可靠度試驗中常用來使測試元件提早發生故障，並獲得足夠的產品壽命資料的一個方法。本研究主要是探討在型 II 設限下逐步應力加速壽命試驗模式中參數的統計推論，我們假設產品的壽命分配是具有雙參數的指數壽命分配，並且假設產品的故障率函數是應力的對數線性函數 (log-linear function)。本研究將求出模式中參數的最大概似估計量，並進一步試著求導出參數的區間估計。最後，我們將舉一例子來闡述本研究中所提出的推論過程。

關鍵詞：加速壽命試驗；逐步應力；最大概似法；信賴區域。

Abstract With today's high technology, many products last so long that life testing at usual conditions is impractical. Many products can be life tested at high stress conditions to yield failures quickly. This study presents the inferences of parameters on the step-stress model in accelerated life testing with type II censoring. A two-parameter exponential failure time distribution with a hazard function that is a log-linear function of stress and a cumulative exposure model are considered. We obtain the maximum likelihood estimators of the model parameters and construct their confidence region. A numerical example will be investigated to illustrate the proposed inferential procedure.

Keywords: Accelerated life test; confidence region; maximum likelihood method; simple step-stress; Type II censoring.

2　Introduction

Accelerated life test is often used for reliability analysis. Test units are run at higher-than-usual stress conditions in order to yield failures quickly. A model relating life length to stress is fitted to the accelerated failure times and then extrapolated to estimate the lifetime distribution under usual conditions.

The accelerated life test is usually operated using constant stress, step-stress, or varying-stress. The step-stress scheme applies to test units such that the stress on a unit can be changed at a pre-specified time. Generally, a test unit starts at a specified low stress. If the unit does not fail at a specified time, stress on it is raised and held a specified time. Stress is repeatedly increased and held, until the test unit fails. (see Xiong (1998))

In the literature, Nelson (1980) derived the maximum likelihood estimators for the parameters of Weibull distribution under the inverse power law using the breakdown time observations of an electrical insulation. Shaked and Singpurwalla (1983) proposed a model based on shock models and wear processes, and obtained nonparametric estimator for the lifetime distribution at usual condition. Miller and Nelson (1983) investigated the optimum simple step-stress accelerated life test plans for the case where the lifetimes of test units have an exponential distribution and are observed continuously until all test units fail. Bai *et al.* (1989) also studied the optimum simple step-stress accelerated life tests where a prespecified censoring time is involved. Nelson (1990) provided an extensive and comprehensive source for theory and examples for accelerated life tests. Tang *et al.* (1996) obtained a general expression for computing the maximum likelihood estimator of stress-dependent distribution parameters under multiple censoring and linear cumulative exposure model. Xiong (1998) presented the inferences of parameters on the simple step-stress model in accelerated life testing with type II censored exponential data.

In this study, we consider parameter estimations for the simple step-stress accelerated life test with (1) Type II censoring, (2) a two-parameter exponential lifetime distribution at a constant stress, and (3) the cumulative exposure model. In Section 3, we describe the model and some necessary assumptions. We use the maximum likelihood method to obtain the point estimators of the model parameters in Section 4. The confidence regions for the parameters are derived in Section 5. A simulated data set is studied to illustrate the inferential procedure in Section 6.

3 Model and Assumptions

Let us consider the following simple step-stress accelerated life-testing scheme: Suppose n randomly selected units are simultaneously placed on a life test at stress setting x_1; the failure times of those that fail in a time interval $[0, \tau]$ are observed; starting from time τ, the surviving units are put to a different stress setting x_2 $(x_1 < x_2)$; at the time of the r-th failure, the life test is stopped. For any stress, the failure time distribution of the test unit is a two-parameter exponential distribution. At stress level x_i, the hazard function of a test unit is a log-linear function of stress. That is,

$$\log\left(\frac{1}{\theta_i}\right) = \beta_0 + \beta_1 x_i, \quad i = 1, 2. \qquad (1)$$

The β_0 and β_1 (> 0) are unknown parameters. Therefore the hazard rate of a test unit at low stress is smaller than that at high stress. Furthermore, failures occur according to a cumulative exposure model. That is, the remaining life of a unit depends only on the exposure it has seen, and the unit does not remember how the exposure was accumulated. (see Miller and Nelson (1983))

From previous assumptions, the cumulative distribution function of a test unit under simple step-stress test is:

$$G(t) = \begin{cases} F_1(t) & \text{for } \mu \le t < \tau \\ F_2(s + t - \tau) & \text{for } \tau \le t < \infty, \end{cases}$$

where $F_i(t) = 1 - \exp(-\frac{t-\mu}{\theta_i})$, $i = 1, 2$, and $s = \frac{\theta_2}{\theta_1}(\tau - \mu) + \mu$ is the solution of $F_2(s) = F_1(\tau)$. Hence, the probability density function of a test unit is

$$f(t) = \begin{cases} \frac{1}{\theta_1}\, e^{-\frac{t-\mu}{\theta_1}} & \text{for } \mu \le t < \tau \\ \frac{1}{\theta_2}\, e^{-\frac{1}{\theta_1}(\tau-\mu) - \frac{1}{\theta_2}(t-\tau)} & \text{for } \tau \le t < \infty. \end{cases} \tag{2}$$

4 Maximum Likelihood Estimators

Suppose $T_{11} < T_{12} < \cdots < T_{1n_1} < T_{21} < T_{22} < \cdots < T_{2n_2}$ are the lifetimes of the completely observed units to fail. That is, n_i failure times T_{ij}, $j = 1, 2, \ldots, n_i$, of the test units are observed while testing at stress x_i, $i = 1, 2$, and $r = n_1 + n_2$ is the total number of failures. Thus, the likelihood function for T_{ij}, $j = 1, 2, \ldots, n_i$, $i = 1, 2$, is

$$\begin{aligned} L(\theta_1, \theta_2, \mu) &= \frac{n!}{(n-r)!}\, \frac{1}{\theta_1^{n_1}}\, \frac{1}{\theta_2^{n_2}} \\ &\exp\left\{ -\frac{1}{\theta_1}\left[\sum_{j=1}^{n_1} t_{1j} + (n - n_1)\tau - n\mu\right] \right. \\ &\left. -\frac{1}{\theta_2}\left[\sum_{j=1}^{n_2} t_{2j} - n_2\tau + (n - r)(t_{2n_2} - \tau)\right] \right\}, \end{aligned}$$

where $t_{ij} \ge \mu$, $n_1 > 0$ and $n_2 > 0$.

Since $\mu \le t_{11} < \cdots < t_{1n_1} < t_{21} < \cdots < t_{2n_2}$, it is easy to see that the maximum likelihood estimator (MLE) of μ is $\hat{\mu} = T_{11}$. Substitute $\hat{\mu}$ for μ and (1) for θ_1 and θ_2; the log likelihood function is a function of unknown parameters β_0 and β_1:

$$\begin{aligned} \log L(\beta_0, \beta_1) &\propto n_1(\beta_0 + \beta_1 x_1) + n_2(\beta_0 + \beta_1 x_2) \\ &- U_1\, e^{\beta_0 + \beta_1 x_1} - U_2\, e^{\beta_0 + \beta_1 x_2}, \end{aligned}$$

where

$$U_1 = \sum_{j=1}^{n_1} t_{1j} + (n - n_1)\tau - n\hat{\mu}, \tag{3}$$

and

$$U_2 = \sum_{j=1}^{n_2} t_{2j} - n_2\tau + (n - r)(t_{2n_2} - \tau). \tag{4}$$

Let $\frac{\partial}{\partial\beta_0} \log L(\beta_0, \beta_1) = 0$ and $\frac{\partial}{\partial\beta_1} \log L(\beta_0, \beta_1) = 0$. We then find that the MLEs for β_0 and β_1 are

$$\hat{\beta}_0 = \frac{1}{x_2 - x_1}\left[x_1 \log\left(\frac{U_2}{n_2}\right) - x_2 \log\left(\frac{U_1}{n_1}\right)\right],$$

and

$$\hat{\beta}_1 = \frac{1}{x_2 - x_1} \log\left(\frac{n_2 U_1}{n_1 U_2}\right),$$

respectively.

5 Confidence Regions

In this section, the joint confidence regions for μ, β_0, and β_1 are given. Let random variable Y be defined as

$$Y = \begin{cases} \frac{T-\mu}{\theta_1} & \text{if } \mu \le T < \tau \\ \frac{\tau-\mu}{\theta_1} + \frac{T-\tau}{\theta_2} & \text{if } \tau < T < \infty, \end{cases} \tag{5}$$

where T has the probability density function in (2). It is easy to prove that Y has an exponential distribution with mean 1.

To derive the confidence regions for μ, β_0, and β_1, the following lemma is necessary.

Lemma 1. *Let $Y_{(1)}, \ldots, Y_{(r)}$ be the first r ordered observations of a random sample of size n from the exponential distribution with mean 1. Let $D = \sum_{i=1}^{r} Y_{(i)} + (n-r)Y_{(r)} - nY_{(1)}$. Then, $Y_{(1)}$ and D are independent, and $2nY_{(1)}$ and $2D$ are distributed as $\chi^2_{(2)}$ and $\chi^2_{(2r-2)}$, respectively.*

This lemma is a simpler version of the Theorem 3.5.1 in Lawless (1982). Hence, the proof can be easily obtained.

The next two theorems provide a joint confidence region for the parameters μ and β_0, and a joint confidence region for the parameters μ and β_1. The proofs can be found in Wu (2001). In the following discussion, let $F_{\alpha(\nu_1,\nu_2)}$ be the upper α percentage point of the F distribution with ν_1 and ν_2 degrees of freedom and let $\chi^2_{\alpha(\nu)}$ be the upper α percentage point of the chi-square distribution with ν degrees of freedom.

Theorem 1. *Suppose that T_{ij}, $j = 1, 2, \ldots, n_i$, $i = 1, 2$, are the ordered failure times of a sample with size n from a distribution which has density function in (2). Then for any $0 < \alpha < 1$, $n_1 > 0$, and $n_2 > 0$, the following inequalities determine a $1-\alpha$ joint confidence region for μ and β_1.*

$$\begin{cases} \hat{\mu} - \dfrac{n_1\hat{\theta}_1}{n(n_1-1)} F_{\frac{\alpha}{4}(2,2n_1-2)} < \mu \\ \qquad < \hat{\mu} - \dfrac{n_1\hat{\theta}_1}{n(n_1-1)} F_{1-\frac{\alpha}{4}(2,2n_1-2)} \\ \\ \dfrac{1}{x_2-x_1} \log\left(\dfrac{1}{U_2}\left[\dfrac{n(r-1)(\hat{\mu}-\mu)}{F_{\frac{\alpha}{4}(2,2r-2)}} - U_1\right]\right) \\ \qquad < \beta_1 < \\ \dfrac{1}{x_2-x_1} \log\left(\dfrac{1}{U_2}\left[\dfrac{n(r-1)(\hat{\mu}-\mu)}{F_{1-\frac{\alpha}{4}(2,2r-2)}} - U_1\right]\right) \end{cases}$$

where $r = n_1 + n_2$, $\hat{\mu} = T_{11}$, $\hat{\theta}_1 = \frac{1}{n_1}\left[\sum_{j=1}^{n_1} T_{1j} + (n-n_1)T_{1n_1} - nT_{11}\right]$, and U_1 and U_2 are defined in (3) and (4), respectively.

Theorem 2. *Suppose that T_{ij}, $j = 1, 2, \ldots, n_i$, $i = 1, 2$, are the ordered failure times of a sample with size n from a distribution which has density function in (2). Then for any $0 < \alpha < 1$, $n_1 > 0$, and $n_2 > 0$, the following inequalities determine a $1-\alpha$ joint confidence region for μ and β_0.*

$$\begin{cases} \hat{\mu} - \dfrac{n_1\hat{\theta}_1}{n(n_1-1)} F_{\frac{\alpha}{6}(2,2n_1-2)} < \mu \\ \qquad < \hat{\mu} - \dfrac{n_1\hat{\theta}_1}{n(n_1-1)} F_{1-\frac{\alpha}{6}(2,2n_1-2)} \\ \\ \dfrac{1}{x_2-x_1}\left[x_2 \log\left(\dfrac{\chi^2_{1-\frac{\alpha}{6}(2)}}{2n(\hat{\mu}-\mu)}\right) - x_1 \log\left(\dfrac{1}{U_2}\right.\right. \\ \qquad \left.\left.\left(\dfrac{1}{2}\chi^2_{\frac{\alpha}{6}(2r-2)} - \dfrac{U_1}{2n(\hat{\mu}-\mu)}\chi^2_{1-\frac{\alpha}{6}(2)}\right)\right)\right] \\ \quad < \beta_0 < \\ \dfrac{1}{x_2-x_1}\left[x_2 \log\left(\dfrac{\chi^2_{\frac{\alpha}{6}(2)}}{2n(\hat{\mu}-\mu)}\right) - x_1 \log\left(\dfrac{1}{U_2}\right.\right. \\ \qquad \left.\left.\left(\dfrac{1}{2}\chi^2_{1-\frac{\alpha}{6}(2r-2)} - \dfrac{U_1}{2n(\hat{\mu}-\mu)}\chi^2_{\frac{\alpha}{6}(2)}\right)\right)\right] \end{cases}$$

where $r = n_1 + n_2$, $\hat{\mu} = T_{11}$, $\hat{\theta}_1 = \frac{1}{n_1}\left[\sum_{j=1}^{n_1} T_{1j} + (n-n_1)T_{1n_1} - nT_{11}\right]$, and U_1 and U_2 are defined in (3) and (4), respectively.

6 An Illustrative Example

To illustrate the use of the method given in this paper, the following are the simulated data from model (2).

These data are simulated by generating a sample from the exponential distribution

Table 1: Simulated failure time data

stress	failure times
x_1	51.65, 58.91, 59.28, 77.16, 82.53
	85.39, 85.85
x_2	90.99, 92.08, 92.09, 96.47, 96.52
	97.95, 100.38, 101.27, 113.78
	122.73, 124.42, 130.43, 143.15

with mean 1, and then the transformation (5) is used to get the sample from model (2). We choose $n = 25$, $r = 20$, $\beta_0 = -5.8$, $beta_1 = 1.4$, $x_1 = 0.5$, $x_2 = 1.5$, $\mu = 50.0$, and $\tau = 90.0$.

The MLEs of μ, β_0, and β_1 are $\hat{\mu} = 51.6475$, $\hat{\beta}_0 = -5.3397$, and $\hat{\beta}_1 = 1.1293$, respectively. To construct a 90% joint confidence region for μ and β_1, note that $F_{0.975(2,12)} = 0.0254$, $F_{0.025(2,12)} = 5.0959$, $F_{0.975(2,38)} = 0.0253$, and $F_{0.025(2,38)} = 4.0713$. Then, by Theorem 1, a 90% joint confidence region for the parameters μ and β_1 is determined by the following inequalities:

$$\begin{cases} 26.0012 < \mu < 51.5198 \\ \log(10.4353 - 0.2343\mu) < \beta_1 \\ \quad < \log(1942.6199 - 37.6453\mu). \end{cases}$$

Furthermore, $F_{0.0167(2,12)} = 5.8716$, $F_{0.9833(2,12)} = 0.0168$, $\chi^2_{0.0167(2)} = 8.1887$, $\chi^2_{0.9833(2)} = 0.0336$, $\chi^2_{0.0167(38)} = 58.8282$, and $\chi^2_{0.9833(38)} = 21.8561$. Thus, a 90% confidence region for μ and β_0 can be determined by the following inequalities:

$$\begin{cases} 22.0970 < \mu < 51.5628 \\ 1.5\log\left(\dfrac{0.0007}{51.6475 - \mu}\right) - 0.5\log\left(0.0591 - \right. \\ \left. \dfrac{0.0011}{51.6475 - \mu}\right) < \beta_0 < 1.5\log\left(\dfrac{0.1638}{51.6475 - \mu}\right) \\ -\, 0.5\log\left(0.0219 - \dfrac{0.2728}{51.6475 - \mu}\right) \end{cases}$$

by Theorem 2.

References

Bai, D. S., Kim, M. S. and Lee, S. H. (1989). Optimum simple step-stress accelerated life tests with censoring, *IEEE Transactions on Reliability*, **38**, 528-532.

Lawless, J. F. (1982). *Statistical Models and Methods for Lifetime Data*, Wiley, New York.

Miller, R. and Nelson, W. B. (1983). Optimum simple step stress plans for accelerated life testing, *IEEE Transactions on Reliability*, **32**, 59-65.

Nelson, W. B. (1980). Accelerated life testing – step-stress models and data analysis, *IEEE Transactions on Reliability*, **29**, 103-108.

Nelson, W. (1990). *Accelerated Testing: Statistical Models, Test Plans, and Data Analysis*, Wiley, New York.

Ross, S. (1998). *A First Course in Probability*, 5th ed., Prentice-Hall.

Shaked, M. and Singpurwalla, N. D. (1983). Inference for step-stress accelerated life tests, *Journal of Statistical Planning and Inference*, **7**, 295-306.

Tang, L. C., Sun, Y. S., Goh, T. N., and Ong, H. L. (1996). Analysis of step-stress accelerated-life-test data: a new approach, *IEEE Transactions on Reliability*, **45**, 69-74.

Wu, S.-J. (2001). Parameter estimation of the two-parameter exponential distribution under step-stress accelerated test with censoring. Submitted.

Xiong, C. (1998). Inferences on a simple step-stress model with type-II censored exponential data, *IEEE Transactions on Reliability*, **47**, 142-146.