Appendix

Proof of theorem 1

Let $G_j^{(r)} = \frac{\sum_{h=1}^{(n_1+\dots+n_r)(m_1+\dots+m_r)} R_{hj}}{(n_1+\dots+n_r)(m_1+\dots+m_r)}$ and $T_j^{(r)} = \sqrt{n_1+\dots+n_r}[G_j^{(r)} - E(G_j^{(r)})]$ for $r = 1,\dots,s$; $j = 1,\dots,p$. By the conditional expectation, it can be shown that

$$Var(T_j^{(r)}) = \sigma_{j1}^2 + \frac{n_1+\dots+n_r}{m_1+\dots+m_r}\sigma_{j2}^2 + \frac{1}{m_1+\dots+m_r}\left(\frac{p^2-1}{12} - \sigma_{j1}^2 - \sigma_{j2}^2\right)$$

$$Var(T_p^{(r)}) = \sum_{j=1}^{p-1}\left[\sigma_{j1}^2 + \frac{n_1+\dots+n_r}{m_1+\dots+m_r}\sigma_{j2}^2\right] + \sum_{j\neq j'}^{p-1}\left[\sigma_{jj',3} + \frac{n_1+\dots+n_r}{m_1+\dots+m_r}\sigma_{jj',4}\right] + \frac{1}{m_1+\dots+m_r}\left\{\frac{p^2-1}{12} - \sum_{j=1}^{p-1}(\sigma_{j1}^2+\sigma_{j2}^2) - \sum_{j\neq j'}^{p-1}(\sigma_{jj',3}+\sigma_{jj',4})\right\}$$

and

$$Var\left(\sum_{j=1}^{p} jT_j^{(r)}\right) = \sum_{j=1}^{p-1}(p-j)^2\left[\sigma_{j1}^2 + \frac{n_1+\dots+n_r}{(m_1+\dots+m_r)}\sigma_{j2}^2\right] + \sum_{j\neq j'}^{p-1}(p-j)(p-j')\left[\sigma_{jj',3} + \frac{n_1+\dots+n_r}{m_1+\dots+m_r}\sigma_{jj',4}\right] + \frac{1}{m_1+\dots+m_r}\left\{\left(\sum_{j=1}^{p} j^2 - \sum_{j\neq j'}^{p} jj'\right)\left(\frac{p^2-1}{12}\right) - \sum_{j=1}^{p-1}(p-j)^2(\sigma_{j1}^2+\sigma_{j2}^2)\right\} - \frac{1}{m_1+\dots+m_r}\sum_{j\neq j'}^{p-1}(p-j)(p-j')(\sigma_{jj',3}+\sigma_{jj',4})$$

where

$\sigma_{j1}^2 = Var(E[(R_{hj} - \frac{p+1}{2}) \mid \vec{X}_i])$, $\sigma_{j2}^2 = Var(E[(R_{hj} - \frac{p+1}{2}) \mid \vec{Y}_k])$,
$\sigma_{jj',3} = Cov(E[(R_{hj} - \frac{p+1}{2}) \mid \vec{X}_i], E[(R_{hj'} - \frac{p+1}{2}) \mid \vec{X}_i])$,
$\sigma_{jj',4} = Cov(E[(R_{hj} - \frac{p+1}{2}) \mid \vec{Y}_k], E[(R_{hj'} - \frac{p+1}{2}) \mid \vec{Y}_k])$
for $j \neq j'$ $j, j' = 1,2,\dots,p-1$

Let $T_j^{(r)*}$ be the projection of $T_j^{(r)}$, then $T_j^{(r)*} = \sqrt{n_1+\dots+n_r}[G_j^{(r)*} - \frac{p+1}{2}]$. By the projection theorem,

$$\sum_{j=1}^{k} jT_j^{(r)*} = \sqrt{n_1+\dots+n_r}\,\frac{1}{n_1+\dots+n_r}\sum_{i=1}^{(n_1+\dots+n_r)} V(\vec{X}_i) + \sqrt{n_1+\dots+n_r}\,\frac{1}{m_1+\dots+m_r}\sum_{k=1}^{(m_1+\dots+m_r)} U(\vec{Y}_k)$$

where $V(\vec{X}_i) = \sum_{j=1}^{p-1}(p-j)E[(R_{hj} - E(R_{hj})) \mid \vec{X}_i]$. $U(\vec{Y}_k) = \sum_{j=1}^{p-1}(p-j)E[(R_{hj} - E(R_{hj})) \mid \vec{Y}_k]$, $\sum_{i=1}^{(n_1+\dots+n_r)} V(\vec{X}_i)$ and $\sum_{k=1}^{(m_1+\dots+m_r)} U(\vec{Y}_k)$ are independent , and $\sum_{j=1}^{p} jT_j^{(r)*} \xrightarrow{D} N(0, \sum_{j=1}^{p-1}(p-j)^2(\sigma_{j1}^2 + \lambda\sigma_{j2}^2) + \sum_{j\neq j'}^{p-1}(p-j)(p-j')(\sigma_{jj',3} + \lambda\sigma_{jj',4}))$.

Thus,

$Var(\sum_{j=1}^{p} jT_j^{(r)}) \to \sum_{j=1}^{p-1}(p-j)^2(\sigma_{j1}^2 + \lambda\sigma_{j2}^2) + \sum_{j\neq j'}^{p-1}(p-j)(p-j')(\sigma_{jj',3} + \lambda\sigma_{jj',4})$ for $n_1 \asymp m_1, n_2 \asymp m_2, \dots, n_r \asymp m_r$, $\frac{n_1+\dots+n_r}{m_1+\dots+m_r} \to \lambda$ as $n_1,\dots,n_r, m_1,\dots,m_r \to \infty$. Thus, the null limiting distribution of

$$\sum_{r=1}^{s} a_r W_{1r} = \sum_{r=1}^{s} a_r\left[\frac{L_{1r} - E(L_{1r})}{\sqrt{Var(L_{1r})}}\right] = \sum_{r=1}^{s} a_r\left\{\frac{\sum_{j=1}^{p} jT_j^{(r)*}}{\sqrt{Var(\sum_{j=1}^{p} jT_j^{(r)})}}\right\} + a_1\frac{o_p(\frac{1}{\sqrt{n_1}})}{\frac{1}{\sqrt{n_1}}\sqrt{Var(\sum_{j=1}^{p} jT_j^{(1)})}} + \dots + a_s\frac{o_p(\frac{1}{\sqrt{n_1+\dots+n_r}})}{\frac{1}{\sqrt{n_1+\dots+n_s}}\sqrt{Var(\sum_{j=1}^{p} jT_j^{(s)})}}$$

is normal. By the result of the null limiting distribution for any arbitrary linear combination $\sum_{r=1}^{s} a_r W_{1r}$ is normal, we can obtain that the joint null limiting distribution of $\vec{W}_1 = (W_{11},\dots,W_{1s})'$ is multivariate normal.

Proof of the theorem 2

By the previous result, it can be shown that $W_{2r} = \frac{L_{2r} - E(L_{2r})}{\sqrt{Var(L_{2r})}} = \frac{\sum_{j=1}^{p_r} jT_j^*}{\sqrt{Var(\sum_{j=1}^{p_r} jT_j)}} + \frac{o_p(\frac{1}{\sqrt{n}})}{\frac{1}{\sqrt{n}}\sqrt{Var(\sum_{j=1}^{p_r} jT_j)}}$, where $T_j = \sqrt{n}\left\{\frac{\sum_{h=1}^{nm} R_{hj}}{nm} - E\left\{\frac{\sum_{h=1}^{nm} R_{hj}}{nm}\right\}\right\}$. Since T_j^* is the project of T_j and can be written as $T_j^* = \sqrt{n}\left\{\frac{1}{n}\sum_{i=1}^{n} E(R_{hj} - \frac{p+1}{2} \mid \vec{X}_i) + \frac{1}{m}\sum_{k=1}^{m} E(R_{hj} - \frac{p+1}{2} \mid \vec{Y}_k)\right\}$. For any constants b_1, b_2, b_3, it shows that the null limiting distribution of $\sum_{r=1}^{3} b_r W_{2r}$ is normal as $n \asymp m$ and $\frac{n}{m} \to \infty$. Hence, the proof is completed.