行政院國家科學委員會專題研究計畫成果報告

韋伯分配中有限制的失敗-設限壽命檢驗

Limited Failure-Censored Life Test for the Weibull Distribution

計畫編號: NSC88-2118-M032-008

執行期限:87 年 8 月 1 日至 88 年 7 月 31 日

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1. 中文摘要

在壽命檢驗中，我們常以達到一個預設的失敗產品數為檢驗過程停止的標準。然而，當檢驗的時間成本過高時，要求所有檢驗產品必須達到一定的數目失敗才能將檢驗停止的壽命檢驗計畫便不適用。當所檢驗的產品壽命服從指數分配時，Balasooriya 提出了一套失敗−設限可靠度抽樣計畫，此計畫可有效的節省時間及成本。本文主要是延伸 Balasooriya 提出的失敗−設限可靠度抽樣計畫，除了將產品壽命從指數分配推廣到韋伯分配外，並在 r 條檢驗設備可以同時進行檢驗的情況下提出一個比 Balasooriya 所提出的失敗−設限可靠度抽樣計畫更為有效率的方法。此外，本文也將討論此一新的抽樣計畫中參數的估計過程。

關鍵詞:壽命檢驗抽樣計畫; 蒙地卡羅模擬; 順序統計量;韋伯分配;最佳平方和不偏估計量。

Abstract

A lifetime test is usually stopped when a predetermined number of specimens failed and testing method is to test n specimens simultaneously. However, this test plan is inappropriate when the duration of experiment is important. When the failure-time of test items is exponential distributed, Balasooriya provides a failure-censored reliability sampling plan that can save time and money efficiently. In this paper, we extend the failure-censored reliability sampling plan suggested by Balasooriya to Weibull distribution. We also propose a more efficient life test procedure to replace Balasooriya's procedure when r (the total censoring number) test lines (each contains n specimens) can be run simultaneously. Moreover, when we only have a test line and the replicate times of specimen batches in Balasooriya are limited to $k(<r)$. We propose a limited failure-censored life test that has an advantage on saving time and money. We also discuss some feasible estimation procedures of unknown parameters for the Weibull distribution.

Key Words: Life test sampling plans; Monte Carlo simulation; Order statistic; Weibull distribution, BLUE.

2. Introduction

An important quality characteristic is the lifetime of a product in reliability analysis. Lifetime data often come with a feature that creates special problems in the analysis of the data. This feature is known as censoring. Generally speaking, censoring usually applies when exact lifetimes are known for only a portion of the products and the remainder of the lifetimes is known only to exceed certain value under a life test. In fact, there are several types of censoring processes. In life testing, experiments involving Type II censoring (failure-censored) are often used. It is designed so that a total of n units is placed on test, but instead of continuing until all n units have failed. The test is stopped at the time of the rth unit failure. Such test can save time and money, because it may take a long time for all units to fail in some life testing (Lawless [2], Schneider [5]).

Sometimes the lifetime of a product is quite high. Thus, a Type II censoring life test plan for such a product may be still time consuming. Johnson [3] proposed a sampling plan. This plan is the experimenter might decide to group the test units into several sets, each as an assembly of test units and then run all the test units simultaneously until the first failure in each group. In such a sampling plan, one usually needs that test facilities are scare but test material is relatively cheap. Balasooriya [1] examined the failure-censored sampling plan for the two-parameter exponential distribution based on testing r random samples, each of size n, one after the other. The suggested procedure is based on exact results and only the first failure time of each sample is needed. Hence, the sample size of proposal sampling plan is rn.

In some situations, the test facilities can only be used k ($k<r$) times. That is, the number of failures that we observed is less than the required number of observations. Hence, we cannot only take the first failure time in each sample, especially when k is small.

In this paper, we first extend the results of Balasooriay [1] to the Weibull distribution and we suggest a new and more efficient procedure than Balasooriya[1] when r test lines can be run simultaneously. Moreover, when the test facilities can be repeated by using only k times, we propose a procedure called limited failure-censored life test to get r failures and overcome this problem.

In Section 3, we will extend the limited failure-censored life test to Weibull distribution. We also prove that this method is the best for saving time when the value of the shape parameter in Weibull distribution is equal to 1. If the value of shape parameter in Weibull distribution is not equal to 1, the formula of expected duration of experiment is very complicated. When the value of shape parameter in Weibull distribution is no more than 1, our method still works. We will use the Monte Carlo simulation to evaluate these cases in the last section. As the value of shape parameter in Weibull distribution is larger than 1, our method will depend on the censoring rate. We can not get the unique optimal solution and we will not discuss it in this paper. Moreover, we also discuss how to get three feasible parameters' estimations and confidence intervals' estimation procedures when a limited failure-censored reliability sampling plan be performed. If the distribution of lifetimes is exponential distribution, then our estimator for scale parameter is still the BLUE. If the distribution of lifetimes is Weibull distribution, then we can use the pooled transformed linear estimators to estimate the unknown parameters. Further, two alternative minimal mean square error estimators are also proposed. Finally, the corresponding confidence intervals' estimations will be derived for associate parameters. The related results are displayed in Wu *et al*. [6].

Notations:

$X_{1:n}^{(i)}$, is the first order statistics in ith sample, each of size n, i=1,2,...,r.

[x], the largest integer less than x.

BLUE, is the best linear unbiased estimator.

3. Design of Limited Failure-censored Reliability Sampling Plan

3.1. The Failure-censored Sampling Plan

Let X be the lifetime of a product, and it is Weibull distributed with probability density function

$$f(x;\beta,\alpha)=\frac{\beta}{\alpha}(\frac{x}{\alpha})^{\beta-1}\exp\{-(\frac{x}{\alpha})^{\beta}\},\quad x>0, \qquad (1)$$

where β is the shape parameter and α is the scale parameter. This distribution was first suggested by Weibull [7] and its applicability to various failure situations are discussed again by Weibull [8]).

Assume that $X_{1:n}^{(i)}$, i=1,2,...,r are the first order statistics of r samples, each of size n from (1) in ith test line, then $X_{1:n}^{(i)}$, i=1,2,...,r, can be considered as a random sample of size r with probability density function.

$$f(x_{1:n};\beta,\alpha n^{-1/\beta})=\frac{n\beta}{\alpha}(\frac{x_{1:n}}{\alpha})^{\beta-1}\}\cdot \exp\{-n(\frac{x_{1:n}}{\alpha})^{\beta}\},\quad x_{1:n}>0. \qquad (2)$$

Hence, the distribution of $X_{1:n}^{(i)}$ is Weibull with scale parameter $\alpha n^{-1/\beta}$, i=1,2,...,r. In some situations, if the lifetime of a product is quite length and test facilities are scarce but testing materials are relatively cheap, then one can test rn experimental units by testing r (the predetermined total censoring number) sets, each containing n experimental units, one after the other. Assume the test stops when the first failure in each replicate is achieved (i.e., the total failure items are r), and the probability distribution of lifetime of specimens is assumed to be (1). Let T_1 be the total testing time we need, i.e., $T_1=\sum_{i=1}^{r}X_{1:n}^{(i)}$, then the expected duration of the test is denoted by

$$E(T_1)=r\alpha n^{-1/\beta}\Gamma(1+\frac{1}{\beta}). \qquad (3)$$

Alternatively, when the test facilities are available to run r test lines (each contains n specimens) simultaneously. According to the test procedure of Balasooriya [1], the duration of the experiment, say T_2 is equal to the rth order statistic of $X_{1:n}^{(i)}$, i=1,2,...r(i.e., the largest order statistic of $X_{1:n}^{(1)}$, $X_{1:n}^{(2)}$,..., $X_{1:n}^{(r)}$, where $X_{1:n}^{(j)}$ is the first order statistic in the jth test line). We can use the expected value of the rth order statistic in (1) (see *e.g.,* Johnson *et al* ([4], p.638)) to get

$$E(X_{r:n})=\alpha n\binom{n-1}{r-1}\Gamma(1+\frac{1}{\beta})\cdot \sum_{i=0}^{r-1}(-1)^{i}\binom{r-1}{i}/(n-r+i+1)^{1+1/\beta}. \qquad (4)$$

Then, we can show that the expectation of T_2 as follows:

$$E(T_2)=r\alpha n^{-1/\beta}\Gamma(1+\frac{1}{\beta})\cdot$$

$$\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}/(i+1)^{1+1/\beta}. \quad (5)$$

In this situation, These rn specimens are tested in r test lines independently and simultaneously. The duration of the experiment using the rth order statistic in all rn specimens will be shorter than to take the first order statistics in each test line separately, then again to take the largest order statistic among them. Hence, we suggest to take the rth order statistic on all rn test specimens, and revise $E(T_2)$ as following equation (6). It will be the best in this experimental situation.

$$E(T_2)=\alpha rn\binom{rn-1}{r-1}\Gamma(1+\frac{1}{\beta}) \cdot$$

$$\sum_{i=0}^{r-1}(-1)^i\binom{r-1}{i}/(rn\text{-}r\text{+}i+1)^{1+1/\beta} \quad (6)$$

3.2. The Limited Failure-censored Reliability Sampling Plan

When the test facilities are scarce and the replicate times of specimen batches are limited to k ($<r$), it is inappropriate to use previous sampling plan in a life test. In this situation, it is impossible to only observe the first failure of specimens in each testing line. If we just observe the first failure of specimens in each testing line, then there are k failures. The total number of failures will be less than the predetermined censoring number r. Thus, the test can not be stopped. Therefore, we must accumulate more failure specimens in some testing lines. We call this design is the limited failure-censored reliability sampling plan. According to this design, we have two situations that $\frac{r}{k}$ is an integer and $\frac{r}{k}$ is not an integer, respectively. We will use a theorem and a corollary to discuss these two situations, respectively. When the value of shape parameter β is equal to 1, the Weibull distribution will degenerate to exponential distribution. The expectation of the rth order statistics in n observations can be represented in a simple form as follows

$$E(X_{r:n})=\alpha\sum_{i=1}^{r}\frac{1}{n-i+1}\cdot \quad (7)$$

Let T_3 be the stopping time of the experiment and the total censors in ith test line(each contain n specimens to test) is r_i, i=1,2,…k. That is, $T_3=\sum_{i=1}^{k}X_{r_i:n}^{(i)}$. We can use (7) to get the expected value of T_3,

$$E(T_3)=\alpha\sum_{i=1}^{k}[\frac{1}{n}+\frac{1}{n-1}+\ldots+\frac{1}{n-r_i+1}]. \quad (8)$$

We want to find the value r_i, $i=1,2,\ldots,k$, such that $E(T_3)$ is minimal or, equivalently, to minimize $\sum_{i=1}^{k}[\frac{1}{n}+\frac{1}{n-1}+\ldots+\frac{1}{n-r_i+1}]$. Let $g(r_1,r_2,\ldots,r_k)=\sum_{i=1}^{k}[\frac{1}{n}+\frac{1}{n-1}+\ldots+\frac{1}{n-r_i+1}]$. When $\frac{r}{k}$ is an integer, we can prove the minimal value of $g(r_1,r_2,\ldots,r_k)$ is equal to $g(\frac{r}{k},\frac{r}{k},\ldots,\frac{r}{k})$. Therefore, $g(r_1,r_2,\ldots,r_k)\ \geq\ g(\frac{r}{k},\frac{r}{k},\ldots,\frac{r}{k})$ and "=" hold if $r_i=\frac{r}{k}$, i=1,2,…,k; we thus get the following theorem.

[Theorem 1]

If a batch of n items is tested consecutively, and the replicate times are limited to be k ($<r$).

Suppose $r_i(\geq 1)$ is the stopping number in ith replication, i=1,2,...,k and $\frac{r}{k}$ is an integer. Then the minimal expected complete time of the experiment can be achieved when $r_i = \frac{r}{k}$, i=1,2,...,k.

Next, if $\frac{r}{k}$ is not an integer, then in the first, we assign the $[\frac{r}{k}]$ (the largest integer less than $\frac{r}{k}$) censoring number to each testing line. According Theorem 1, We can show is the optimal design for the integer part in division. Secondary, we try to assign the c_r(as the remainder of $\frac{r}{k}$ and c_r<k) censoring number to all k testing lines such that the censoring number differences among each testing line are as small as possible. Therefore, we assign one specimen to each c_r testing line again, and the other k-c_r testing lines are still to test until $[\frac{r}{k}]$ specimens are censored. We can use Theorem 1 to get the following Corollary

[Corollary]

In the Theorem 1, if the remainder of $\frac{r}{k}$ is equal to c_r, then the optimal selection for total censoring number in each line is to stop in c_r testing lines when $[\frac{r}{k}]+1$ specimens failed , and to stop in the other k-c_r testing lines when $[\frac{r}{k}]$ specimens failed, where $[\frac{r}{k}]$ is defined as above.

In addition, if the value of shape parameter β in Weibull distribution is not equal to 1, then the expected testing time is very complicated. We will use Monte Carlo simulation to discuss this situation. According to our simulation results, we find the optimal stopping rules in limited failure-censored reliability sampling plan depend on the value of the shape parameter in Weibull distribution and the censoring rate. Generally speaking, the rules of our theorem and corollary in this Section are still valid when the value of shape parameter β in Weibull distribution is no more than 1. When β is large than 1, the optimal stopping rules in limited failure-censored reliability sampling plan do not have a unique solution. The rules will change depend on the value of shape parameter β and the censoring rate. Therefore, we will only present the result of Monte Carlo simulations when the value of shape parameter in Weibull distribution is not more than 1. (see Wu *et al*. [6]) The simulation results support our viewpoint.

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