Minimum BER Block-Based Precoder Design for Zero-Forcing Equalization: An Oblique Projection Framework

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***Abstract*—This work devises a minimum bit error rate (BER) block-based precoder used in block transmission systems with the proposed cascaded zero-forcing (ZF) equalizer. The study framework is developed as follows. For a block-based precoder, a received signal model is formulated for the two redundancy schemes, viz., trailing-zeros (TZ) and cyclic-prefix (CP). By exploiting the property of oblique projection, a cascaded equalizer for block transmission systems is proposed and implemented with a scheme, in which the inter-block interference (IBI) is completely eliminated by the oblique projection and followed by a matrix degree-of-freedom for inter-symbol interference (ISI) equalization. With the available channel state information at the transmitter side, the matrix for ISI equalization of the cascaded equalizer is utilized to design an optimum block-based precoder, such that the BER is minimized, subject to the ISI-free and the transmission power constraints. Accordingly, the cascaded equalizer with the ISI-free constraint yields a cascaded ZF equalizer. Theoretical derivations and simulation results confirm that the proposed framework not only retains identical BER performance to previous works for cases with sufficient redundancy, but also allows their results to be extended to the cases of insufficient redundancy.**

***Index Terms*—Block transmission systems, cascaded equalizer, channel state information, inter-block interference (IBI), inter-symbol interference (ISI), oblique projection, precoder, zero-forcing (ZF).**

I. Introduction

INTER-SYMBOL interference (ISI) induced by the channel often significantly impairs simple receiver performance. To alleviate this effect, block transmissions are widely adopted [1], [19]. In such a transmission scheme, the transmitted data stream is divided into consecutive equal size blocks and redundancy is added between blocks. Given proper selection of redundancy length, the inter-block interference (IBI) can be completely removed. Examples of this block transmission include discrete multitone (DMT) [16], [18] and orthogonal frequency division multiplexing (OFDM) modulations [4], [15], [17], which have been adopted in standards [14] and [5]. Previous studies have demonstrated that many existing modulations can be formulated within a unifying multirate filterbank transceiver model [2], [7], [9], [32]. Building on this framework, the FIR filterbanks used in the transmitter/receiver are usually known as FIR filterbanks precoder/equalizer (or are simply termed precoder/equalizer). Particularly, the precoder/equalizer with the property of perfect reconstruction (PR) is a zero-forcing (ZF) precoder/equalizer [2], [23], [26]. A situation in which the length of the FIR fitlerbanks does not exceed block-plus-redundancy/block size is referred to as a zero-order (or a block-based) precoder/equalizer. To facilitate discussion, the nonblock-based transceiver generally represents that both the precoder and the equalizer are not restricted to being zero-order. Relaxing the PR constraint reveals the potential of filtered multitone (FMT) modulation in many applications [24]. However, this study restricts itself to the problem of PR with a (FIR filterbanks) ZF equalizer, especially for a block-based precoder over physical single-input–single-output (SISO) channels.

Consider a system in which the serial data stream is divided into blocks of M data symbols, and every P transmitted symbols contain $P-M$ redundancy. Notably, the PR is impossible for $P=M$ for a nonideal channel [9], [23]. Hence, P satisfies $P-M \geq 1$, and 1 is the minimum redundancy. In [23], Xia demonstrated a general condition for the existence of the non block-based transceiver. Furthermore, when the precoder is restricted to being block-based, the necessary and sufficient conditions of the nonblock-based ZF equalizer have been studied in [2]. Assume that the FIR channel is of order L. With sufficient redundancy (i.e., $P-M \geq L$), [2] verified that a block-based equalizer can completely eliminate IBI. Additionally, in [2] and [23], any precoder with redundancy of the form trailing-zeros (TZ) is shown to be a channel-independent precoder provided $P-M \geq L$ is satisfied. In the case of minimum redundancy, i.e., 1, the PR test condition on channel zeros is given in [2] and [23]. Subsequently, Lin *et al.* proposed a method [27] based on a given P and the zeros of the channel, in which the minimum redundancy for the existence of the nonblock-based transceiver can be determined exactly. In [11] and [27], it was demonstrated that the minimum redundancy for a block-based transceiver is given by $P-M \geq \lceil L/2 \rceil$, where $\lceil\rceil$ denotes the ceiling integer. For such block-based transceivers employing redundancy TZ of $L/2$, the channel matrix for the PR test differed from the matrix used in [2]. Interestingly, in [26] Kung *et al.* also observed the reduced channel matrix for the PR test. Recently, in some aspects, Pohl *et al.*, in [29], extended the results of Lin *et al.* in [27] and Kung *et al.* in [26] to the PR problem of physical multiple-input–multiple-output (MIMO) channels. However, for the physical SISO channels, [29] did not include the case of employing a block-based ZF equalizer for a block-based precoder with redundancy TZ of $L/2$ [29, Ex. 4].

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For the case of ZF equalization, the joint design of linear block-based transceiver has also been studied for the physical SISO channels [2], [10], and [20]. The previous designs assume sufficient redundancy and make the channel state information available in the transmitter for exploitation. Particularly, by exploiting the property of convexity, Ding *et al.* in [10] obtained a minimum bit error rate (min-BER) precoder for block transmission systems with ZF equalization. Their results closely corresponded to the convex optimization framework developed by Palomar *et al.* in [25] under the ZF constraint. Rather than the ZF constraint, designs with other criteria are addressed in [2], [11], [20], [37]–[39]. Again, the aforementioned works considered a block-based precoder with sufficient redundancy. However, designs with smaller redundancy increase spectral efficiency.

The previous discussion demonstrates that the min-BER block-based precoder for ZF equalization has not been solved for insufficient redundancy, i.e., $P - M < L$. Particularly, a block-based transceiver with redundancy TZ of $L/2$, suggested in [11] and [27], was not considered in [10] for min-BER precoder design. Moreover, if the channel scenario is allowed for PR under the case of minimum redundancy, how to design a block-based min-BER precoder for ZF equalization is of priority concern. To solve these problems, this study exploits the property of oblique projections to devise a new equalizing scheme, and then designs an optimum block-based precoder to minimize the BER. A follow up study allows the results in [2] and [10] to be extended to the case of insufficient redundancy. Originally, the oblique projections for signal processing applications were proposed by Behrens *et al.* in [12]. Oblique projections were seldom used in literature and have recently received attention in [13], [28], and [31]. For block transmission systems, the oblique projection has been applied to implement the nonblock-based ZF equalizer of [2] via a decomposition approach [6], [21]. In fact, from the perspective of the oblique projection framework, the conventional OFDM modulation, using sufficient cyclic-prefix (CP) redundancy, is nothing but a special design to avoid IBI. Additionally, its receiver indeed performs a block-based ZF equalizer. Interestingly, this observation is, perhaps beyond the expectations of Behrens and Scharf in [12]. That is, the transceiver design here is more general and can be adopted for completely canceling IBI whether for sufficient or insufficient redundancy.

The rest of this paper is organized as follows. For a block-based precoder, Section II formulates a general received signal model for block transmission systems, which can be determined once the parameters P, M, and the redundancy scheme are clarified and the channel order L is known. Section III then reviews the mathematics of oblique projection operators that are widely used in this paper. Given knowledge of the associated parameters, the received signal model can be constructed and the ZF equalizer subsequently expressed by oblique projection. Next, Section IV introduces a new equalizing scheme and addresses the optimum design of the min-BER block-based precoder. Section V summarizes simulation results that confirm the validity of this study. Conclusions are finally drawn in Section VI.

Notation: Boldface lowercase letters denote column vectors, boldface uppercase letters indicate matrices, and italics denote scalars. $[\mathbf{A}]_{mn}$ and $[\mathbf{a}]_m$ represent the (m,n)th and mth elements of matrix $\mathbf{A}$ and vector $\mathbf{a}$, respectively. Matrix and vector entries are indexed starting from 0. $\mathbf{A}^{-1}$ and $\mathbf{A}^{+}$ denote the inverse matrix and the pseudo-inverse matrix of $\mathbf{A}$. $(\cdot)^H$ and $(\cdot)^T$ indicate Hermitian transpose and transpose, respectively. $\langle\mathbf{A}\rangle$ denotes the subspace spanned by the columns of a matrix $\mathbf{A}$, while $\langle\mathbf{A}\rangle^{\perp}$ denotes the orthogonal complement of $\langle\mathbf{A}\rangle$. The magnitude of a scalar is denoted by $\|\cdot|$. $\mathrm{tr}(\cdot)$ and $\|\|_F$ stand for the trace and Forbenius norm of a matrix, respectively. $\mathbf{I}_a$ denotes the identity matrix of size a. $O(O_{b\times c})$ generally denotes the null matrix ($b \times c$ null matrix). $E[\cdot]$ denotes the statistical expectation. $\mathrm{vec}([\,\mathbf{a}_1 \;\; \mathbf{a}_2 \;\; \cdots \;\; \mathbf{a}_n\,])$ represents $[\,\mathbf{a}_1^T \;\; \mathbf{a}_2^T \;\; \cdots \;\; \mathbf{a}_n^T\,]^T$. $\otimes$ denotes the Kronecker product, while $\delta(i)$ represents the Kronecker delta function. The special notation $\mathbf{a} \sim \mathcal{CN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ indicates that $\mathbf{a}$ is complex Gaussian distributed with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. Similarly, $a \sim \mathcal{CN}(\mu, \sigma^2)$ is used for scalars.

II. Signal Model of Block Transmission Systems

This study addresses a block-based precoder and the zero-forcing (ZF) equalization, particularly for physical SISO channels. Hence, the transmitter model developed in [2] is adopted, along with a received signal model formulated for block transmission systems using ZF equalization. Some assumptions made in [2] are introduced as follows.

<?Pub _kern Amount="-2.5pt"?>A0)During block transmissions, the channel is assumed to be linear and time invariant (LTI).

A1) The channel $h(l)$ is a causal Lth-order FIR channel, i.e., $h(l) = 0, l \notin [0, L]$ and $h(0), h(L) \neq 0$.

A2) The length of each transmitted block of symbols P and that of the data symbols M are chosen to satisfy $P > M$ and $P > L$. Generally, redundancy is required to smaller than M, i.e., $P - M < M$.

A3) At the transmitter end, each block of M data symbols has a block-based precoder F_0. Particularly, the precoder is assumed to be a $P \times M$ matrix and is of rank M. At the receiver end, a $(Q-1)$th order ZF equalizer is assumed; That is, the ZF equalizer is an $M \times QP$ matrix.

In fact, A0) suggests an application scenario in which the channel state is almost static or varies slowly. Intuitively, for a specific channel, the assumption A0) is reasonable if the smaller transmitted block size with the same relative redundancy (i.e., the same $P - M/P$, see [2] and [27]) is employed. Additionally, $h(0)$, $h(L)$ of A1) guarantees that the channel order is time invariant. Here, continuous time Nyquist transmitting and receiving filters are assumed. These effects are included in the channel model $h(l)$ [8].

Fig. 1 illustrates the transmitter and channel model. To describe the model, the nth block of M data symbols are denoted by the $M \times 1$ vector $\mathbf{s}(n) = [s(nM), s(nM+1), \ldots, s(nM+M-1)]^T$. From A3), the $P \times 1$ vector of the transmitted symbols $\mathbf{u}(n) = [u(nP), u(nP+1), \ldots, u(nP+P-1)]^T$ is given by

$$\mathbf{u}(n) = \mathbf{F}_0\mathbf{s}(n). \tag{1}$$

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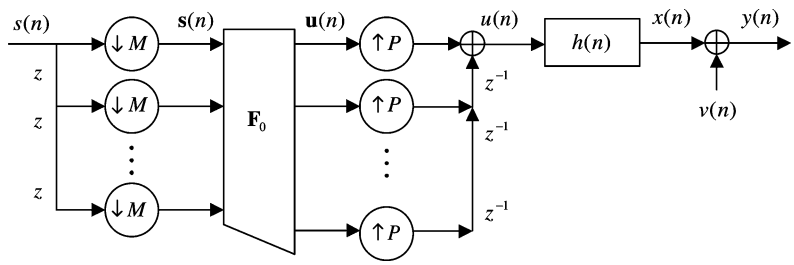

Fig. 1. Discrete-time baseband equivalent transmitter and channel model. The advance elements and downsamplers parse the input data stream $s(n)$ into blocks of M symbols. The $P \times M$ precoder is denoted by $\mathbf{F}_0$. The upsamplers and delay elements unblock the size-P transmitted blocks of $\mathbf{u}(n)$. The $h(n)$ and $v(n)$ denote the channel and additive noise, respectively.

At the receiver end, the $P\times 1$ vector $\mathbf{y}(n) = [y(nP), y(nP+1), \ldots, y(nP+P-1)]^T$ denotes the nth block of the received symbols. If the $P \times 1$ vectors of channel output, and additive noise are denoted by $\mathbf{x}(n) = [x(nP), x(nP+1), \ldots, x(nP+P-1)]^T$, and $\mathbf{v}(n) = [v(nP), v(nP+1), \ldots, v(nP+P-1)]^T$, respectively, then from A0)–A2) and (1), $\mathbf{y}(n)$ can be written as

$$\begin{aligned}\mathbf{y}(n) &= \mathbf{x}(n) + \mathbf{v}(n)\\ &= \mathbf{H}_0\mathbf{u}(n) + \mathbf{H}_1\mathbf{u}(n-1) + \mathbf{v}(n)\\ &= \mathbf{H}_0\mathbf{F}_0\mathbf{s}(n) + \mathbf{H}_1\mathbf{F}_0\mathbf{s}(n-1) + \mathbf{v}(n)\end{aligned} \tag{2}$$

where $\mathbf{H}_0$ and $\mathbf{H}_1$ are the $P \times P$ Toeplitz channel matrices and are defined as $[\mathbf{H}_0]_{k,l} = h(k-l)$ and $[\mathbf{H}_1]_{k,l} = h(P+k-l)$, respectively, for $k, l \in [0, P-1]$. In (2), for the block $\mathbf{s}(n)$, $\mathbf{H}_1$ models IBI from the previous block $\mathbf{s}(n-1)$, whereas $\mathbf{H}_0$ models ISI within the symbols of block $\mathbf{s}(n)$.

For a $(Q-1)$th-order ZF equalizer given in A3), consecutive Q blocks of $\mathbf{y}(n)$ in (2) can be stacked into a $QP \times 1$ vector $\bar{\mathbf{y}}_Q(n)$ for equalization while denoting $\bar{\mathbf{y}}_Q(n) = \mathrm{vec}([\mathbf{y}(n-Q+1), \ldots, \mathbf{y}(n)])$. Similarly, denoting by $\bar{\mathbf{x}}_Q(n)$ and $\bar{\mathbf{v}}_Q(n)$ the $QP \times 1$ vectors $\bar{\mathbf{x}}_Q(n) = vec([\mathbf{x}(n-Q+1), \ldots, \mathbf{x}(n)])$ and $\bar{\mathbf{v}}_Q(n) = vec([\mathbf{v}(n-Q+1), \ldots, \mathbf{v}(n)])$, respectively. Then $\bar{\mathbf{y}}_Q(n) = \bar{\mathbf{x}}_Q(n) + \bar{\mathbf{v}}_Q(n)$. Furthermore, to capture the effect of the $(Q+1)$ blocks of transmitted symbols in a matrix form, the $(Q+1)M \times 1$ vector $\bar{\mathbf{s}}_{Q+1}(n) = vec([\mathbf{s}(n-Q), \ldots, \mathbf{s}(n)])$, and the $(Q+1)P \times 1$ vector $\bar{\mathbf{u}}_{Q+1}(n) = vec([\mathbf{u}(n-Q), \ldots, \mathbf{u}(n)])$ are defined. Using the previous notations, $\bar{\mathbf{y}}_Q(n)$ can be expressed as

$$\begin{aligned}\bar{\mathbf{y}}_Q(n) &= \mathcal{H}\bar{\mathbf{u}}_{Q+1}(n) + \bar{\mathbf{v}}_Q(n)\\ &= \mathcal{H}\mathcal{F}_{Q+1}\bar{\mathbf{s}}_{Q+1}(n) + \bar{\mathbf{v}}_Q(n)\end{aligned} \tag{3}$$

where $\mathcal{H}$ is a $QP \times (Q+1)P$ block Toeplitz channel matrix whose first block of rows is $[\mathbf{H}_1, \mathbf{H}_0, \mathbf{O}_{P\times(Q-1)P}]$, and $\mathcal{F}_{Q+1} = \mathbf{I}_{Q+1} \otimes \mathbf{F}_0$.

Since the precoder F_0 is of rank M as stated by A3) and is utilized to introduce the redundancy, it can be decomposed into a $P \times M$ full column rank redundancy generating matrix $\mathbf{F}_C$ and a square full rank matrix of dimension M, $\mathbf{F}$. That is, $\mathbf{F}_0 = \mathbf{F}_C\mathbf{F}$ and $\mathbf{F}^{-1}$ exists. Particularly, two redundancy schemes are considered, i.e., TZ, and CP. For CP systems, $\mathbf{F}_C = \mathbf{F}_{\mathrm{CP}} = [\mathbf{I}_M \mid [\mathbf{I}_{P-M} \mid \mathbf{O}]^T]^T$ is used, whereas for TZ systems, $\mathbf{F}_C = \mathbf{F}_{\mathrm{TZ}} = [\mathbf{I}_M \mid \mathbf{O}_{M\times(P-M)}]^T$ is used. The matrix $\mathbf{F}$ can also be referred to as the precoder since $\mathbf{F}_C$ is fixed.

For ease of demonstrating the new ZF equalizer proposed in Section III, (3) is reformulated as follows. Let H of (3) be decomposed into the $QP \times P$, $QP \times (Q-1)P$, and $QP \times P$ submatrices from left to right

$$\begin{aligned}\mathcal{H} &= \left[\begin{array}{c|ccc|c} \mathbf{H}_1 & \mathbf{H}_0 & \mathbf{O} & \ldots & \mathbf{O}\\ \mathbf{O} & \mathbf{H}_1 & \mathbf{H}_0 & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & \mathbf{O}\\ \mathbf{O} & \ldots & \mathbf{O} & \mathbf{H}_1 & \mathbf{H}_0 \end{array}\right]\\ &= [\mathbf{D}_2 \mid \mathbf{D}_1 \mid \mathbf{D}_0].\end{aligned} \tag{4}$$

Substituting (4) and $\mathbf{F}_0 = \mathbf{F}_C\mathbf{F}$ into (3) and using the properties of the Kronecker product [36] yields

$$\begin{aligned}\bar{\mathbf{y}}_Q(n) = &\left[\mathbf{U}_2 \mid \mathbf{U}_1\right](\mathbf{I}_Q \otimes \mathbf{F})\bar{\mathbf{s}}_Q(n-1)\\ &+ \mathbf{U}_0\mathbf{F}\mathbf{s}(n) + \bar{\mathbf{v}}_Q(n)\end{aligned} \tag{5}$$

where

$$\mathbf{U}_2 = \mathbf{D}_2\mathbf{F}_C, \quad \mathbf{U}_1 = \mathbf{D}_1(\mathbf{I}_{Q-1} \otimes \mathbf{F}_C), \quad \mathbf{U}_0 = \mathbf{D}_0\mathbf{F}_C \tag{6}$$

and $\bar{\mathbf{s}}_Q(n-1)$ is defined similarly to $\bar{\mathbf{s}}_{Q+1}(n)$, described before. Notably, $\mathbf{H}_1$, like $\mathbf{D}_2$ in (4), has nonzero elements only in its $L \times L$ top right submatrix. Therefore, the size of the nonzero columns of $\mathbf{U}_2$ defined in (6) depends strongly on $\mathbf{F}_C$. Restated, the selection of the redundancy scheme and the induced redundancy (related to the parameters L, P, and M) affect the structure of $\mathbf{U}_2$ in (5). Herein, a $M \times A$ dynamic selection matrix $\mathbf{\Psi}$ (where the column dimension A is dynamic) can be defined to extract the nonzero columns of $\mathbf{U}_2$. Consequently, applying $\mathbf{\Psi}$ into (5) yields a general received signal model

$$\bar{\mathbf{y}}_Q(n) = \mathbf{Z}\mathbf{b}(n) + \mathbf{U}_0\mathbf{F}\mathbf{s}(n) + \bar{\mathbf{v}}_Q(n) \tag{7}$$

where $\mathbf{Z} = [\mathbf{U}_2\mathbf{\Psi} \mid \mathbf{U}_1]$, and $\mathbf{b}(n) = \left[\mathbf{s}^T(n-Q)\mathbf{F}^T\mathbf{\Psi} \mid \bar{\mathbf{s}}_{Q-1}^T(n-1)(\mathbf{I}_{Q-1} \otimes \mathbf{F})^T\right]^T$. In fact, $\mathbf{Z}$ and $\mathbf{U}_0$ can be regarded as the effective channel matrices, which combine the LTI channel matrix H and the redundancy generating matrix $\mathbf{F}_C$. Moreover, for block $\mathbf{s}(n)$, $\mathbf{Z}$ includes the IBI effect that is caused by $\mathbf{H}_1$. A general received signal model given by (7) has been formulated to devise a

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$(Q-1)$th-order ZF equalizer. Notably, the proposed signal model differs from that of [2] in that the concatenation effect of the matrix $\mathbf{D}_2$ and the redundancy scheme $\mathbf{F}_C$ is considered in (7); in contrast, [2] focuses only on ZF equalization using model $C\mathcal{F}_Q$, where C is the lower $(QP-L)\times QP$ submatrix of $[\,\mathbf{D}_1 \;\;|\;\; \mathbf{D}_0\,]$ in (4) and $\mathcal{F}_Q = \mathbf{I}_Q \otimes \mathbf{F}_0$ ([2, Eq. (24)–(29)]).

For subsequent ease of discussion, two symbols are defined (in boldface), i.e., **TZ-Systems** and **CP-Systems**, for constructing the TZ and CP received signal models, respectively. The symbols are defined as follows.

TZ-Systems: Substitute $\mathbf{F}_C = \mathbf{F}_{\mathrm{TZ}}$ and $\mathbf{\Psi} = \mathbf{\Psi}_{\mathrm{TZ}}$ into (6) and (7), we have $\bar{\mathbf{y}}_Q(n) = \bar{\mathbf{y}}_{Q(\mathrm{TZ})}(n)$.

CP-Systems: Substitute $\mathbf{F}_C = \mathbf{F}_{\mathrm{CP}}$ and $\mathbf{\Psi} = \mathbf{\Psi}_{\mathrm{CP}}$ into (6) and (7), we have $\bar{\mathbf{y}}_Q(n) = \bar{\mathbf{y}}_{Q(\mathrm{CP})}(n)$.

Section III, based on (7), demonstrates that $\mathbf{\Psi}_{\mathrm{TZ}}$ (or $\mathbf{\Psi}_{\mathrm{CP}}$) and the minimum Q can be determined to construct a TZ (or CP) received signal model for ZF equalization once P, M, and L are specified. Moreover, the corresponding ZF equalizer, based on an oblique projection, is derived.

III. ZF Equalizer With Oblique Projection

This section focuses on devising a ZF equalizer based on (7) in the absence of noise. Designs in the presence of additive noise will be investigated in Section IV. First, an overview of the oblique projection operators that will be used extensively in the rest of this paper is presented.

A. Oblique Projections

The basic notations for *projections* (including orthogonal and nonorthogonal) are introduced in the following.

Definition 1—Projections [12]: A matrix $\mathbf{P}$ is a projection if and only if it is idempotent, i.e., $\mathbf{P}^2 = \mathbf{P}$.

Definition 2—Orthogonal Projections [12]: An orthogonal projection has a null space that is orthogonal to its range. A matrix is an orthogonal projection if and only if it is Hermitian symmetry, i.e., $\mathbf{P}^H = \mathbf{P}$. An orthogonal projection is denoted by $\mathbf{P_H}$ with a subscript referring to the range space $\langle\mathbf{H}\rangle$, where $\mathbf{P_H}$ is termed the projector onto $\langle\mathbf{H}\rangle$. Similarly, $\mathbf{P}_{\mathbf{H}}^{\perp}$ denotes the orthogonal complement projection with range $\langle\mathbf{H}\rangle^{\perp}$ and null $\langle\mathbf{H}\rangle$. If $\mathbf{P_H}$ has a null space $\langle\mathbf{A}\rangle = \langle\mathbf{H}\rangle^{\perp}$, then $\mathbf{P_H H} = \mathbf{H}$ and $\mathbf{P_H A} = \mathbf{O}$.

Definition 3—Oblique Projections [12]: The projection matrices that are not orthogonal are referred to as oblique (nonorthogonal) projections. Oblique projections are not symmetric. An oblique projection is denoted by $\mathbf{E_{HA}}$, with a double subscript referring first to the range $\langle\mathbf{H}\rangle$ and second to the null $\langle\mathbf{A}\rangle$. The $\mathbf{E_{HA}}$ is a projection which projects onto $\langle\mathbf{H}\rangle$ along a direction parallel to $\langle\mathbf{A}\rangle$, and has the properties $\mathbf{E_{HA}H} = \mathbf{H}$ and $\mathbf{E_{HA}A} = \mathbf{O}$.

B. ZF Equalizer for Block Transmission Systems

This subsection describes how to construct a received signal model for TZ systems (or CP systems) with the specified parameters P, M, and L and how to employ the oblique projection to devise a ZF equalizer. The following assumption is made, in addition to A0)–A3).

A4) The channel state information (CSI) is fully known at the receiver.

Moreover, the ZF equalizer is devised by first offering the following definition.

Definition 4—ZF Equalizer: Consider the additive noise $\bar{\mathbf{v}}_Q(n)$ of $\bar{\mathbf{y}}_Q(n)$ in (7) to be absent (as in the noise-free case or in a case of sufficiently high signal-to-noise ratio); the $M\times QP$ matrix $\mathcal{G}$ is a ZF (or perfect reconstruction, PR) equalizer if the condition $\hat{\mathbf{s}}(n) = \mathcal{G}\bar{\mathbf{y}}_Q(n) = \mathbf{s}(n-d)$ holds, where $\hat{\mathbf{s}}(n)$ denotes the nth block of equalized symbols and d represents the system delay. Since d does not affect the existence and uniqueness of $\mathcal{G}$, then without a loss of generality, we take $d=0$ in this paper, $\hat{\mathbf{s}}(n) = \mathbf{s}(n)$, i.e., (see [2, fn. 2]). The PR condition suggests that a ZF equalizer possesses the following two properties:

P0) to remove IBI from the previous block to be IBI-free;

P1) to equalize ISI within the symbols of $\mathbf{s}(n)$ to be ISI-free.

As demonstrated in Theorem 1, given specified P, M, and L, $\mathbf{\Psi}_{\mathrm{TZ}}$ of the **TZ-Systems** can be determined to construct the received signal model of TZ systems, i.e., $\bar{\mathbf{y}}_{Q(\mathrm{TZ})}(n)$. Moreover, based on the constructed $\bar{\mathbf{y}}_{Q(\mathrm{TZ})}(n)$, the conditions (e.g., minimum block size Q) under which an oblique projection can be defined are provided to devise an equalizer with properties P0) and P1) of Definition 4, i.e., a ZF equalizer, for retrieving $\mathbf{s}(n)$. The same applies to Theorem 2 for CP systems.

Theorem 1—TZ Systems: Assume A0)–A4) hold and the triplet (P, M, L) is known for TZ systems. For $P-M<L$, $\mathbf{\Psi}_{\mathrm{TZ}} = [\,\mathbf{O}_{(L+M-P)\times(P-L)} \;\;\; \mathbf{I}_{(L+M-P)}\,]^T$, while for $P-M\geq L$, $\mathbf{\Psi}_{\mathrm{TZ}}$ is any $M\times A$ matrix with positive integer A. Based on the definition of **TZ-Systems**, $\bar{\mathbf{y}}_{Q(\mathrm{TZ})}(n)$ is constructed. In the absence of noise and for a given $\mathbf{F}$, if the block size Q is chosen to satisfy the condition

$$\begin{cases} QP \geq QM + L + M - P, & \text{if } P-M<L \\ Q \geq 1, & \text{if } P-M\geq L \end{cases} \tag{8}$$

and if the composite matrix $[\,\mathbf{Z} \;\;|\;\; \mathbf{U}_0\,]$ of $\bar{\mathbf{y}}_{Q(\mathrm{TZ})}(n)$ has full column rank, then there exists a unique $(Q-1)$th-order ZF equalizer $\mathcal{G}_{\mathrm{ob(TZ)}}$ such that $\mathcal{G}_{\mathrm{ob(TZ)}}\bar{\mathbf{y}}_{Q(\mathrm{TZ})}(n) = \mathbf{s}(n)$. The ZF equalizer is given by

$$\mathcal{G}_{\mathrm{ob(TZ)}} = \mathbf{F}^{-1}\mathbf{U}_0^{+}\mathbf{E}_{\mathbf{U}_0\mathbf{Z}} = \mathbf{F}^{-1}\mathbf{M}_{\mathbf{U}_0\mathbf{Z}}^{\#} \tag{9}$$

where $\mathbf{E}_{\mathbf{U}_0\mathbf{Z}}$ is the oblique projection from Definition 3 and $\mathbf{M}_{\mathbf{U}_0\mathbf{Z}}^{\#} = \mathbf{U}_0^{+}\mathbf{E}_{\mathbf{U}_0\mathbf{Z}}$ is the oblique pseudo-inverse corresponding to $\mathbf{E}_{\mathbf{U}_0\mathbf{Z}}$ [12], [13].

Proof: From Section II, it is known that A0)–A3) hold for building $\bar{\mathbf{y}}_Q(n)$ of (7). The signal model of TZ systems is rewritten in the absence of noise ($\bar{\mathbf{v}}_Q(n)=0$) as follows to facilitate the subsequent derivation of a ZF equalizer, based on the definition of **TZ-Systems**

$$\bar{\mathbf{y}}_{Q(\mathrm{TZ})}(n) = \mathbf{Z}\mathbf{b}(n) + \mathbf{U}_0\mathbf{F}\mathbf{s}(n) \tag{10}$$

where

$$\mathbf{Z} = [\,\mathbf{U}_2\mathbf{\Psi}_{\mathrm{TZ}} \;\;|\;\; \mathbf{U}_1\,]$$

and

$$\mathbf{b}(n) = \left[\,\mathbf{s}^T(n-Q)\mathbf{F}^T\mathbf{\Psi}_{\mathrm{TZ}} \;\;|\;\; \bar{\mathbf{s}}_{Q-1}^T(n-1)(\mathbf{I}_{Q-1}\otimes\mathbf{F})^T\right]^T.$$

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If P, M, and L are known, then $\mathbf{\Psi}_{\mathrm{TZ}}$ of $\mathbf{Z}$ is determined as follows. The fact that $\mathbf{H}_1$ has nonzero elements in its $L \times L$ top right submatrix [see two lines below (2)] implies that $\mathbf{U}_2$ contains $(L+M-P)$ nonzero columns for $P-M < L$ and $\mathbf{U}_2$ is a null matrix for $P-M \geq L$. This is because $\mathbf{U}_2 = \mathbf{D}_2\mathbf{F}_{\mathrm{TZ}}$ and $\mathbf{D}_2 = \left[\mathbf{H}_1^T, \mathbf{O}_{P\times(Q-1)P}\right]^T$ [see (4)]. Therefore, extracting the nonzero columns of $\mathbf{U}_2$ requires that $\mathbf{\Psi}_{\mathrm{TZ}} = \left[\mathbf{O}_{(L+M-P)\times(P-L)} \quad \mathbf{I}_{(L+M-P)}\right]^T$ is obtained for $P-M < L$; in addition, $\mathbf{\Psi}_{\mathrm{TZ}}$ is any $M \times A$ matrix with positive integer A for $P-M \geq L$.

Consequently, given $\mathbf{\Psi}_{\mathrm{TZ}}$, $\bar{\mathbf{y}}_{Q(\mathrm{TZ})}(n)$ of (10) is constructed. It follows that for $P-M < L$, $[\mathbf{Z} \quad \mathbf{U}_0]$ of $\bar{\mathbf{y}}_{Q(\mathrm{TZ})}(n)$ is a $QP\times(QM+L+M-P)$ composite matrix and is guaranteed to be a tall matrix if Q is chosen to satisfy $QP \geq (QM+L+M-P)$. Similarly, for $P-M \geq L$, $[\mathbf{Z} \mid \mathbf{U}_0] = [\mathbf{U}_1 \mid \mathbf{U}_0]$ is a tall matrix if $QP \geq QM$ holds. In this case, $Q \geq 1$ is obtained by using the fact that $P > M$ in A2). Assume that A4) holds and the $[\mathbf{Z} \mid \mathbf{U}_0]$ of $\bar{\mathbf{y}}_{Q(\mathrm{TZ})}(n)$ is tall and has full rank. From Definition 3, an oblique projection $\mathbf{E}_{\mathbf{U}_0\mathbf{Z}}$ can be defined with range $\langle\mathbf{U}_0\rangle$ and a null space that includes $\langle\mathbf{Z}\rangle$ [12]. Hence, $\mathbf{E}_{\mathbf{U}_0\mathbf{Z}}\mathbf{U}_0 = \mathbf{U}_0$ and $\mathbf{E}_{\mathbf{U}_0\mathbf{Z}}\mathbf{Z} = \mathbf{O}$ are immediately obtained. As indicated in Section II, for the block $\mathbf{s}(n)$, $\mathbf{Z}$ includes the IBI effect that is caused by $\mathbf{H}_1$. Consequently, the first term of (10) (regarded as the structured noise in [12] and [13]) can be completely removed by $\mathbf{E}_{\mathbf{U}_0\mathbf{Z}}$ and the second term is undisturbed [12] if $\bar{\mathbf{y}}_{Q(\mathrm{TZ})}(n)$ is preprocessed with $\mathbf{E}_{U_oZ}$. That is

$$\mathbf{E}_{\mathbf{U}_0\mathbf{Z}}\bar{\mathbf{y}}_{Q(\mathrm{TZ})}(n) = \mathbf{O} + \mathbf{U}_0\mathbf{F}\mathbf{s}(n). \tag{11}$$

Since $\mathbf{U}_0$ is assumed to be full column rank to construct $\mathbf{E}_{\mathbf{U}_0\mathbf{Z}}$, and since $\mathbf{F}^{-1}$ exists as described in Section II, $\mathbf{U}_0\mathbf{F}$ of (11) is also full column rank. Hence

$$\begin{aligned}\mathbf{s}(n) &= (\mathbf{U}_0\mathbf{F})^{+}\mathbf{E}_{\mathbf{U}_0\mathbf{Z}}\bar{\mathbf{y}}_{Q(\mathrm{TZ})}(n)\\ &= \mathbf{F}^{-1}\mathbf{U}_0^{+}\mathbf{E}_{\mathbf{U}_0\mathbf{Z}}\bar{\mathbf{y}}_{Q(\mathrm{TZ})}(n)\end{aligned} \tag{12}$$

where $\mathbf{F}^{-1}\mathbf{U}_0^{+}$ gives a unique solution to the linear system (11). Accordingly, based on the results of (11) and (12), $\mathcal{G}_{\mathrm{ob(TZ)}} = \mathbf{F}^{-1}\mathbf{U}_0^{+}\mathbf{E}_{\mathbf{U}_0\mathbf{Z}}$, with the properties P0) and P1) of Definition 4 (via $\mathbf{E}_{\mathbf{U}_0\mathbf{Z}}$ and $\mathbf{F}^{-1}\mathbf{U}_0^{+}$, respectively), provides a unique ZF equalizer such that $\mathcal{G}_{\mathrm{ob(TZ)}}\bar{\mathbf{y}}_{Q(\mathrm{TZ})}(n) = \mathbf{s}(n)$. Based on [13], we have $\mathcal{G}_{\mathrm{ob(TZ)}} = \mathbf{F}^{-1}\mathbf{M}^{\#}_{\mathbf{U}_0\mathbf{Z}}$, where $\mathbf{M}^{\#}_{\mathbf{U}_0\mathbf{Z}} = \mathbf{U}_0^{+}\mathbf{E}_{\mathbf{U}_0\mathbf{Z}}$ represents the oblique pseudo-inverse corresponding to $\mathbf{E}_{\mathbf{U}_0\mathbf{Z}}$. From [12] and [13], the formulas for constructing $\mathbf{E}_{\mathbf{U}_0\mathbf{Z}}$ and $\mathbf{M}^{\#}_{\mathbf{U}_0\mathbf{Z}}$ are as follows:

$$\mathbf{E}_{\mathbf{U}_0\mathbf{Z}} = \mathbf{U}_0\left(\mathbf{U}_0^H\mathbf{P}_{\mathbf{Z}}^{\perp}\mathbf{U}_0\right)^{-1}\mathbf{U}_0^H\mathbf{P}_{\mathbf{Z}}^{\perp} \tag{13a}$$

and

$$\mathbf{M}^{\#}_{\mathbf{U}_0\mathbf{Z}} = \left(\mathbf{U}_0^H\mathbf{P}_{\mathbf{Z}}^{\perp}\mathbf{U}_0\right)^{-1}\mathbf{U}_0^H\mathbf{P}_{\mathbf{Z}}^{\perp} \tag{13b}$$

where $\mathbf{U}_0$ and $\mathbf{Z}$ are given in $\bar{\mathbf{y}}_{Q(\mathrm{TZ})}(n)$ and $\mathbf{P}_{\mathbf{Z}}^{\perp} = \mathbf{I}_{\mathrm{QP}} - \mathbf{Z}\mathbf{Z}^{+}$. This completes the proof.

For CP systems, Theorem 2 is obtained.

Theorem 2—CP Systems: Assume A0)–A4) hold and the triplet (P, M, L) is known for CP systems. For $M \leq L$, $\mathbf{\Psi}_{\mathrm{CP}}$ is an identity matrix $\mathbf{I}_M$, while for $M > L$, $\mathbf{\Psi}_{\mathrm{CP}}$ is a $M \times L$ matrix and

$$\mathbf{\Psi}_{\mathrm{CP}} = \left[\left[\mathbf{I}_{(P-M)} \quad \mathbf{O}_{(P-M)\times(2M-P)}\right]^T \;\middle|\; \left[\mathbf{O}_{(L+M-P)\times(P-L)} \quad \mathbf{I}_{(L+M-P)}\right]^T\right].$$

Based on the definition of **CP-Systems**, $\bar{\mathbf{y}}_{Q(\mathrm{CP})}(n)$ is constructed. In the absence of noise and for a given $\mathbf{F}$, if the block size Q is chosen to satisfy the condition

$$QP \geq QM + \varphi, \text{ where } \varphi = \begin{cases} L, & \text{if } M > L \\ M, & \text{if } M \leq L \end{cases} \tag{14}$$

and if the composite matrix $[\mathbf{Z} \mid \mathbf{U}_0]$ of $\bar{\mathbf{y}}_{Q(\mathrm{CP})}(n)$ has full column rank, then there exists a unique $(Q-1)$th-order ZF equalizer $\mathcal{G}_{\mathrm{ob(CP)}}$ such that $\mathcal{G}_{\mathrm{ob(CP)}}\bar{\mathbf{y}}_{Q(\mathrm{CP})}(n) = \mathbf{s}(n)$. The ZF equalizer is given by $G_{\mathrm{ob(CP)}} = \mathbf{F}^{-1}\mathbf{U}_0^{+}\mathbf{E}_{\mathbf{U}_0\mathbf{Z}} = \mathbf{F}^{-1}\mathbf{M}^{\#}_{\mathbf{U}_0\mathbf{Z}}$, where $\mathbf{E}_{\mathbf{U}_0\mathbf{Z}}$ and $\mathbf{M}^{\#}_{\mathbf{U}_0\mathbf{Z}}$ are constructed using the formulas in (13).

Proof: The derivation resembles that of Theorem 1, with [21] providing the reference for $M \leq L$ and [6] providing the reference for $M > L$.

Remarks

The following remarks are made regarding the ZF equalizer of Theorems 1 and 2.

1) For TZ systems, based on Theorem 1, (8) can be rewritten as $P \geq M + \lceil L/(Q+1)\rceil$ for $P-M < L$ (i.e., insufficient redundancy). Similarly, for CP systems, from Theorem 2, (14) can be rewritten as $P \geq M + \lceil M/Q\rceil$ for $M \leq L$. In these two cases, our results demonstrate that a smaller Q is obtained compared to the condition, $P \geq M + \lceil L/Q\rceil$, as shown in [2, a1.1]. Moreover, if Q is set to equal 1 for TZ systems with $P-M < L$, the so called minimum redundancy can be obtained for a block-based transceiver given by Lin *et al.* in [11] and [27], namely, $P-M \geq \lceil L/2\rceil$.
2) Consider $P = M+L$ for TZ systems, then from Theorem 1, Q can be set to 1 (i.e., block-based), and $\mathbf{\Psi}_{\mathrm{TZ}}$ can be any $M\times A$ matrix with positive integer A. In this case, $\mathbf{D}_2$, and $\mathbf{D}_0$ of (4) becomes $\mathbf{H}_1$, and $\mathbf{H}_0$, respectively, while $\mathbf{D}_1$ of (4) is a null matrix. Based on the definition of **TZ-Systems**, $\mathbf{Z}$ is a null matrix and $\mathbf{U}_0 = \mathbf{H}_0\mathbf{F}_{\mathrm{TZ}}$ contains the first M columns of $\mathbf{H}_0$. Consequently, from Definition 2, $\mathbf{P}_{\mathbf{Z}}^{\perp} = \mathbf{I}_P$. Substituting $\mathbf{P}_{\mathbf{Z}}^{\perp} = \mathbf{I}_P$ into $\mathbf{E}_{\mathbf{U}_0\mathbf{Z}}$ of (13a) and thus from (9), $\mathcal{G}_{\mathrm{ob(TZ)}} = \mathbf{F}^{-1}\left(\mathbf{U}_0^H\mathbf{U}_0\right)^{-1}\mathbf{U}_0^H$, which is the same ZF solution given in [2] (see [2, Th. 2]). From the perspective of the oblique projection framework, [2, Th. 2] is included in Theorem 1.
3) Considering $P = M+L$ for CP systems, then $P-M < M$ of A2) results in a situation where $M > L$. Thus, from Theorem 2, Q can be set to 1 (i.e., block-based) and $\mathbf{\Psi}_{\mathrm{CP}} = [\mathbf{I}_L\mathbf{O}_{L\times(M-L)}]^T$. In this case, as is now demonstrated, $\mathcal{G}_{\mathrm{ob(CP)}}$ of Theorem 2 can be simplified. From Remark 2, for $Q = 1$, then it is immediately found that $\mathbf{D}_2 = \mathbf{H}_1$ and $\mathbf{D}_0 = \mathbf{H}_0$, while $\mathbf{D}_1$ is null. Based on these analytical results, according to the definition of

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CP-Systems, $\mathbf{U}_2 = \mathbf{H}_1\mathbf{F}_{\rm CP}$ and $\mathbf{U}_0 = \mathbf{H}_0\mathbf{F}_{\rm CP}$ can be obtained and $\mathbf{U}_1$ is null. Additionally, the $P \times L$ matrix $\mathbf{Z} = \mathbf{U}_2\mathbf{\Psi}_{\rm CP} = \mathbf{H}_1\left[\mathbf{I}_M \;\vdots\; [\mathbf{I}_{P-M} \;\vdots\; \mathbf{O}]^T\right]^T [\mathbf{I}_L\mathbf{O}_{L\times(M-L)}]^T =$ (column permutation of the nonzero columns of $\mathbf{H}_1$) can be obtained. Correspondingly, $\langle\mathbf{Z}\rangle = \langle$last L columns of $\mathbf{H}_1\rangle$. Since the $L \times L$ top submatrix of the last L columns of $\mathbf{H}_1$ is an upper triangular Toeplitz matrix, then based on the Definition 2, $\mathbf{P}_{\mathbf{Z}}^{\perp}$ is readily obtained, and is given by

$$\mathbf{P}_{\mathbf{z}}^{\perp} = \left[\begin{array}{c|c} \mathbf{O}_{L\times L} & \mathbf{O}_{L\times L}C \\ \hline \mathbf{O}_{M\times L} & \mathbf{I}_M \end{array}\right]. \tag{15}$$

From Theorem 2,

$$\mathcal{G}_{\rm ob(CP)} = \mathbf{F}^{-1}\mathbf{M}^{\#}_{\mathbf{U}_0\mathbf{Z}} = \mathbf{F}^{-1}\left(\mathbf{U}_0^H\mathbf{P}_{\mathbf{Z}}^{\perp}\mathbf{U}_0\right)^{-1}\mathbf{U}_0^H\mathbf{P}_{\mathbf{Z}}^{\perp}.$$

To simplify $\mathcal{G}_{\rm ob(CP)}$, by using $\mathbf{U}_0 = \mathbf{H}_0\mathbf{F}_{\rm CP}$, (15), and the structure of $\mathbf{H}_0$ [see one line below (2)], $\mathbf{U}_0^H\mathbf{P}_{\mathbf{Z}}^{\perp}$ can be reduced to $\mathbf{U}_0^H\mathbf{P}_{\mathbf{Z}}^{\perp} = \mathbf{K}_h^H\left[\mathbf{O}_{M\times L} \quad \mathbf{I}_M\right]$, where $\mathbf{K}_h$ is a $M \times M$ circulant Toeplitz matrix with the first column $[h(L), 0, \ldots, h(0), \ldots, h(L-1)]^T$. Therefore, based on the idempotent and Hermitian properties of $\mathbf{P}_{\mathbf{Z}}^{\perp}$ (see Definition 1 and 2)

$$\begin{aligned}\mathcal{G}_{\rm ob(CP)} &= \mathbf{F}^{-1}\left[(\mathbf{U}_0^H\mathbf{P}_{\mathbf{Z}}^{\perp})(\mathbf{U}_0^H\mathbf{P}_{\mathbf{Z}}^{\perp})^H\right]^{-1}(\mathbf{U}_0^H\mathbf{P}_{\mathbf{Z}}^{\perp}) \\ &= \mathbf{F}^{-1}\mathbf{K}_h^{-1}\left[\mathbf{O}_{M\times L} \quad \mathbf{I}_M\right].\end{aligned} \tag{16}$$

If DFT-based is considered, i.e., $\mathbf{F} = \mathbf{F}_{\mathbf{D}}$, and $\mathbf{F}_{\mathbf{D}}$ is given by

$$[\mathbf{F}_{\mathbf{D}}]_{ij} = \frac{1}{\sqrt{M}}e^{j(2\pi/M)kl} \text{ for } 0 \le k, \quad l \le M-1 \tag{17}$$

the eigenvalue-decomposition of $\mathbf{K}_h$ can be expressed as $\mathbf{K}_h = \mathbf{F}_{\mathbf{D}}\mathbf{\Omega}_h\mathbf{F}_{\mathbf{D}}^{-1}$, where $\mathbf{\Omega}_h$ denotes a diagonal matrix, whose diagonal elements are the M-point DFT of the entries of the first column of $\mathbf{K}_h$ [33]. Based on the previous, the block-based ZF equalizer of (16) for DFT-based CP systems can be obtained as follows:

$$\mathcal{G}_{\rm ob(CP)(DFT)} = \mathbf{\Omega}_h^{-1}\mathbf{F}_{\mathbf{D}}^{-1}[\mathbf{O}_{M\times L} \quad \mathbf{I}_M]. \tag{18}$$

Interestingly, $\mathcal{G}_{\rm ob(CP)(DFT)}$ has been widely used in conventional DFT-based CP systems. That is, $[\mathbf{O}_{M\times L} \quad \mathbf{I}_M]$ removes symbols with IBI from the previous block, $\mathbf{F}_{\mathbf{D}}^{-1}$ denotes inverse DFT, and $\mathbf{\Omega}_h^{-1}$ implements one-tap equalizer on each subchannel. Obviously, from the perspective of the oblique projection framework, our results demonstrate that the conventional CP-OFDM equalizing steps are an implementation of a block-based ZF equalizer, as indicated in (18).

Based on the results in this section, one may ask why the blocks should not be made larger from the beginning, to ensure that the redundancy is always sufficient. Yes, if blocks of size $Q'P$ instead of P are used, then the increased redundancy (for a fixed relative redundancy) can yield a block-based ZF equalizer provided $Q'P - Q'M \ge L$ is satisfied. Nevertheless, in this case, *both* the transmitter and the receiver would have to suffer a longer delay for precoding and equalizing *each* size $-P$ block. In contrast, without any modification of the transmitter (such as a spectral mask or a precoding delay), the receiver can simply stack Q size$-P$ blocks for ZF equalization [where Q satisfies (8) or (14)] once it knows that the estimated channel order exceeds the transmitter-induced redundancy. Notably, minimum Q is no larger than Q'. Moreover, applying the pipelining hardware implementation to the Q blocks ZF filterbanks equalizer yields only an initial buffer delay for retrieving size $-P$ blocks.

IV. New Equalizing Scheme and Minimum BER Precoder Design

This section describes the optimum design in the presence of additive noise. In particular, this study designs an optimum block-based precoder (we mean $\mathbf{F}$ here, as explained in Section II) for minimizing BER, subject to the transmission power constraint, for block transmission systems with ZF equalization. Hence, the following assumptions are made.

A5) The additive noise $v(n)$ is assumed to be stationary and white, and $v(n) \sim CN(0, \sigma_v^2)$.

A6) The CSI is available in the transmitter end.

Unlike other optimum designs that deal only with cases of sufficient redundancy, the design presented herein is expected to be applicable to systems with sufficient or insufficient redundancy. Hence, a new equalizing scheme is proposed.

A. New Equalizing Scheme

The idea of a cascaded equalizer is defined as follows to eliminate IBI in case the transmitter-induced redundancy is insufficient, and to simultaneously provide a similar ISI equalization mechanism, as adopted in previous works on cases of sufficient redundancy.

Definition 5—A Cascaded Equalizer: If the equalizing scheme for block transmission systems can be *definitely* formulated into a cascade configuration, in which IBI cancellation is followed by a block-based ISI equalization, it is termed a cascaded equalizer. Particularly, if both properties P0) and P1) of a ZF equalizer (see Definition 4) hold, a cascaded ZF equalizer is defined.

Recall the proof of Theorem 1 for TZ systems. The oblique projection was employed to preprocess the received signal $\bar{\mathbf{y}}_{Q(\rm TZ)}(n)$ of (10). Consequently, as indicated in (11), the term $\mathbf{Zb}(n)$, which contains IBI, is completely eliminated; meanwhile, the term $\mathbf{U}_0\mathbf{Fs}(n)$, which contains the desired block of symbols $\mathbf{s}(n)$, remains undisturbed. Interestingly, the matrix $\mathbf{U}_0$ defined in (6) yields the fact that the upper $(Q-1)P \times M$ submatrix of $\mathbf{U}_0$ is empty [see (4) and (6)]. Therefore, only the lower $P \times 1$ submatrix of the preprocessed signal $\mathbf{E}_{\mathbf{U}_0\mathbf{Z}}\bar{\mathbf{y}}_{Q(\rm TZ)}(n)$ in (10) contains information about $\mathbf{s}(n)$ and can be extracted and equalized independently.

Based on the previous, a new equalizing scheme is proposed for TZ systems. To describe this scheme, the following preliminary conditions are required to hold.

C0) The (P, M, L) is given, and then $\mathbf{\Psi}_{\rm TZ}$ is determined based on the result of Theorem 1.

C1) The block size Q is decided to satisfy the condition in (11) of Theorem 1.

C2) The received signal model $\bar{\mathbf{y}}_{Q(\rm TZ)}(n)$ is constructed using the definition of **TZ-Systems**. Additionally, the matrix $[\mathbf{Z} \;\vdots\; \mathbf{U}_0]$ of $\bar{\mathbf{y}}_{Q(\rm TZ)}(n)$ is assumed to have full column rank.

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With C0)–C2), the oblique projection $\mathbf{E}_{\mathbf{U}_0\mathbf{Z}}$ can be constructed by (13a). The proposed equalizing scheme is then implemented as follows. In the first stage, $\bar{\mathbf{y}}_{Q(\mathrm{TZ})}(n)$ is preprocessed by $\mathbf{E}_{\mathbf{U}_0\mathbf{Z}}$. Then, an IBI-free received signal $\mathbf{E}_{\mathbf{U}_0\mathbf{Z}}\bar{\mathbf{y}}_{Q(\mathrm{TZ})}(n) = \mathbf{U}_0\mathbf{F}\mathbf{s}(n) + \mathbf{E}_{\mathbf{U}_0\mathbf{Z}}\bar{\mathbf{v}}_Q(n)$ is obtained [cf. (7) and (11)]. Subsequently, the extraction matrix $\mathbf{S}_e = [\mathbf{O}_{P\times(Q-1)P}\mathbf{I}_P]$ can be defined to extract the lower $P \times 1$ submatrix of the previous signal. Denoted by $\mathbf{y}_{\mathrm{TZ}}(n)$ the reduced received signal, we have

$$\begin{aligned}\mathbf{y}_{\mathrm{TZ}}(n) &= \mathbf{S}_e\mathbf{E}_{\mathbf{U}_0\mathbf{Z}}\bar{\mathbf{y}}_{Q(\mathrm{TZ})}(n)\\ &= \bar{\mathbf{H}}_0\mathbf{F}\mathbf{s}(n) + \mathbf{v}'(n)\end{aligned} \tag{19}$$

where $\bar{\mathbf{H}}_0 = \mathbf{S}_e\mathbf{U}_0 = \mathbf{H}_0\mathbf{F}_{\mathrm{TZ}}$ is a $P \times M$ matrix; $\mathbf{v}'(n) = \mathbf{S}_e\mathbf{E}_{\mathbf{U}_0\mathbf{Z}}\bar{\mathbf{v}}_Q(n)$ is the oblique projected noise and $\mathbf{v}'(n) \sim CN(\mathbf{0}, \sigma_v^2\mathbf{S}_e\mathbf{E}_{\mathbf{U}_0\mathbf{Z}}\mathbf{E}_{\mathbf{U}_0\mathbf{Z}}^H\mathbf{S}_e^H)$ as A5) holds. Notably, $\bar{\mathbf{H}}_0$ of (19) is of full column rank due to the full rank assumption of $\mathbf{U}_0$ in C2). Next, in the second stage, a block-based (i.e., a $M \times P$ matrix) equalizer $\mathbf{G}$ can be designed such that the nth block of equalized symbols, denoted by $\hat{\mathbf{s}}(n)$, can be obtained from (19), i.e.,

$$\hat{\mathbf{s}}(n) = \mathbf{G}\mathbf{y}_{\mathrm{TZ}}(n) = \mathbf{G}\bar{\mathbf{H}}_0\mathbf{F}\mathbf{s}(n) + \mathbf{G}\mathbf{v}'(n). \tag{20}$$

Based on the results of (19) and (20) and the Definition 5, we demonstrate that the new equalizing scheme is a cascaded equalizer. Notably, the proposed equalizing scheme is not necessarily a cascaded ZF equalizer because property P0) of a ZF equalizer only holds in the first stage and $\mathbf{G}$ offers a matrix degree-of-freedom in the second stage. Since the proposed equalizing scheme was not initially developed only for the cases of insufficient redundancy, as expected, it can be applied to all systems with sufficient or insufficient redundancy. Although the proposed scheme was developed for TZ systems, a cascaded equalizer can easily be obtained for CP systems by simply replacing $\mathbf{\Psi}_{\mathrm{TZ}}$, (8), Theorem 1, **TZ-Systems**, and $\bar{\mathbf{y}}_{Q(\mathrm{TZ})}(n)$ in C0)–C2) with $\mathbf{\Psi}_{\mathrm{CP}}$, (14), Theorem 2, **CP-Systems**, and $\bar{\mathbf{y}}_{Q(\mathrm{CP})}(n)$, respectively, and following the previous two stages.

B. Minimum BER Precoder

As indicated in (19) and (20), applying the proposed equalizing scheme eventually yields a block-based equalizer, matrix $\mathbf{G}$, for ISI equalization, for both sufficient and insufficient redundancy. Interestingly, this result is quite similar to those of the equations used in other works to optimize precoders with sufficient redundancy [2], [10], [20], [25], and others. Notably, unlike in previous works that assume that the noise is full-rank, this study addresses the fact that the oblique-projected noise $\mathbf{v}'(n)$ in (19) is a reduced rank noise, as we will discuss later. Therefore, the results associated with previous designs in [2] and [10] cannot be applied directly to the design problem herein without modifications. Additionally, [10] employs different dimensionalities of $\mathbf{G}$ for TZ and CP systems [10, Sec. II-B], while the proposed new equalizing scheme yields the same dimensionality of $\mathbf{G}$ for both TZ and CP systems.

Since the proposed cascaded equalizer can eliminate IBI in the first stage, particularly for a precoder with insufficient redundancy, the minimum BER block-based precoder $\mathbf{F}$ can be designed for ZF equalization with constrained transmission power by beginning from the second stage, as in (20). The following assumption is made in proceeding the design.

A7) The data symbols $s(n)$ are zero-mean, uncorrelated and with average power 1, i.e., $E[\mathbf{s}(k)\mathbf{s}^H(l)] = \mathbf{I}_M\delta(k-l)$. Also, $s(n)$ and $v(n)$ are mutually uncorrelated. For simplicity, equiprobable BPSK modulation is adopted. Meanwhile, the zero threshold detection is used in the receiver.

The transmission power equals the power used to transmit the block of data symbols, and is expressed as $\mathrm{tr}\left(E\left[\mathbf{F}_0\mathbf{s}(n)(\mathbf{F}_0\mathbf{s}(n))^H\right]\right) = \mathrm{tr}\left(\mathbf{F}_0\mathbf{F}_0^H\right)$. For simplicity, the power consumed by CP is neglected and thus $\mathrm{tr}\left(\mathbf{F}_0\mathbf{F}_0^H\right) = \mathrm{tr}(\mathbf{F}\mathbf{F}^H)$ is used for both TZ and CP systems (see [10] for details). Actually, neglecting the power of CP is reasonable if less CP is used in the case of insufficient redundancy.

As discussed in Section IV-A, the proposed cascaded equalizer must also exhibit the property **P1)** of a ZF equalizer to achieve ZF equalization. Consequently, by imposing an ISI-free constraint, $\mathbf{G}\bar{\mathbf{H}}_0\mathbf{F} = \mathbf{I}_M$ on $\mathbf{G}$ for a given $\mathbf{F}$, the proposed cascaded equalizer can yield a cascaded ZF equalizer. The design problem is then formulated as follows: For TZ systems, given (20), an optimum precoder $\mathbf{F}$ is designed such that the average BER is minimized, subject to ISI-free and to transmission power constraints, i.e., $\mathbf{G}\bar{\mathbf{H}}_0\mathbf{F} = \mathbf{I}_M$ and $\mathrm{tr}(\mathbf{F}\mathbf{F}^H) \le P_T$, where P_T is a constant power.

Now, consider block-by-block detection, relying on (20), in which case the average BER of the detected symbols P_{ae} is the average of the error probability of each symbol of the block, that is

$$P_{\mathrm{ae}} = \frac{1}{M}\sum_{i=0}^{M-1} P_{\mathrm{ie}} \tag{21}$$

where P_{ie} denotes the error probability of the ith symbol. Since the average power of each equalized symbol is unity (from A7), and the ISI-free constraint $\mathbf{G}\bar{\mathbf{H}}_0\mathbf{F} = \mathbf{I}_M$) and the covariance matrix of the equalized oblique projected noise of (20) is $E[\mathbf{G}\mathbf{v}'(n)\mathbf{v}'^H(n)\mathbf{G}^H]$, the error probability of the ith symbol for the BPSK modulation (we adopt [2, (79)] with unity average power) can be expressed as

$$P_{\mathrm{ie}} = \frac{1}{2}\mathrm{erfc}\left(\frac{1}{\sqrt{(E[\mathbf{G}\mathbf{v}'(n)\mathbf{v}'^H(n)\mathbf{G}^H])_{ii}}}\right) \tag{22}$$

where $\mathrm{erfc}(\eta) \triangleq (2/\sqrt{\pi})\int_\eta^\infty e^{-x^2}dx$ and $(E[\mathbf{G}\mathbf{v}'(n)\mathbf{v}'^H(n)\mathbf{G}^H])_{ii}$ represents the noise variance of the ith equalized symbol of (20). Substituting (22) into (21) and using the covariance matrix of $\mathbf{v}'(n)$, defined below (19), yields

$$P_{\mathrm{ae}} = \frac{1}{2M}\sum_{i=0}^{M-1}\mathrm{erfc}\left(\frac{1}{\sqrt{\sigma_v^2[\mathbf{G}\mathbf{R}_{\mathbf{v}'\mathbf{v}'}\mathbf{G}^H]_{ii}}}\right) \tag{23}$$

where $\mathbf{R}_{\mathbf{v}'\mathbf{v}'} = \mathbf{S}_e\mathbf{E}_{\mathbf{U}_0\mathbf{Z}}\mathbf{E}_{\mathbf{U}_0\mathbf{Z}}^H\mathbf{S}_e^H$ denotes the $P \times P$ normalized covariance matrix of the oblique projected noise. From [10] and [25], it is clear that if the noise variance σ_v^2 is less than $2/(3[\mathbf{G}\mathbf{R}_{\mathbf{v}'\mathbf{v}'}\mathbf{G}^H]_{ii})$ for all i the average BER P_{ae} of (23) is

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a convex function. That is, the convexity of $P_{\rm ae}$ holds given a sufficiently high signal-to-noise ratio (SNR) at each ith equalized symbol. Assume that the convexity of $P_{\rm ae}$ holds. Then, by applying Jensen's inequality [34], the lower bound on $P_{\rm ae}$ can be obtained, i.e.,

$$P_{\rm ae} \geq \frac{1}{2}{\rm erfc}\left(\frac{1}{\sqrt{\frac{\sigma_v^2}{M}\sum_{i=0}^{M-1}[\mathbf{G}\mathbf{R}_{\mathbf{v}'\mathbf{v}'}\mathbf{G}^H]_{ii}}}\right) \tag{24}$$

$$= \frac{1}{2}{\rm erfc}\left(\sqrt{\frac{M}{\sigma_v^2 tr\left(\mathbf{G}\mathbf{R}_{\mathbf{v}'\mathbf{v}'}\mathbf{G}^H\right)}}\right) \triangleq P_{\rm ae\_lb}. \tag{25}$$

1) Minimize Only the Quantity $P_{\rm ae\_lb}$: Interestingly, as demonstrated here, minimizing the quantity $P_{\rm ae\_lb}$ of (25) involves the application of a maximum SNR (max-SNR) criterion, as considered in [2]. The ${\rm erfc}(\eta)$ is a strictly monotone decreasing function with respect to η. Consequently, except for the trivial solution, maximizing the element of the bracket of (25) involves minimizing the quantity $P_{\rm ae\_lb}$. Substituting the oblique projection of (13a) into $\mathbf{R}_{\mathbf{v}'\mathbf{v}'}$ [see the line below (23)], we have

$$\mathbf{G}\mathbf{R}_{\mathbf{v}'\mathbf{v}'}\mathbf{G}^H = \mathbf{G}\bar{\mathbf{H}}_0\left(\mathbf{U}_0^H\mathbf{P}_{\mathbf{Z}}^{\perp}\mathbf{U}_0\right)^{-1}\bar{\mathbf{H}}_0^H\mathbf{G}^H. \tag{26}$$

Since the ISI-free constraint results in (22)–(25), by using (26), and given that $M = |{\rm tr}(\mathbf{I}_M)|^2/M$, the problem of minimizing $P_{\rm ae\_lb}$ of (25) can be formulated as

$$\begin{aligned}\max_{\mathbf{G},\mathbf{F}} \quad & \frac{|{\rm tr}(\mathbf{I}_M)|^2}{M\sigma_v^2 {\rm tr}\left[\mathbf{G}\bar{\mathbf{H}}_0(\mathbf{U}_0^H\mathbf{P}_{\mathbf{Z}}^{\perp}\mathbf{U}_0)^{-1}\bar{\mathbf{H}}_0^H\mathbf{G}^H\right]}\\ \text{subject to} \quad & \mathbf{G}\bar{\mathbf{H}}_0\mathbf{F} = \mathbf{I}_M.\end{aligned} \tag{27}$$

Given (27), the numerator, from A7), represents M times the average total signal power of a block of equalized symbols. Clearly, since scale M also appears in the denominator, problem (27) maximizes the signal (total)-to-noise (total) power ratio, subject to the ISI-free constraint (see Section V-B of [2]). Directly solving problem (27) is difficult since the $P \times P$ covariance matrix $\mathbf{R}_{\mathbf{v}'\mathbf{v}'}$ is not a positive-definite matrix [see the right-hand side of (26)]. The nonpositive definite matrix results from the rank M $\mathbf{E}_{\mathbf{U}_0\mathbf{Z}}$ by Definition 3 [12] and the full column rank $P \times M$ matrix $\bar{\mathbf{H}}_0$. To simplify the objective function of (27), consider $\mathbf{G}$ to be composed of an $M \times M$ full-rank matrix $\mathbf{G}'$ and the $M \times P$ matrix $\bar{\mathbf{H}}_0^H$; that is

$$\mathbf{G} = \mathbf{G}'\bar{\mathbf{H}}_0^H. \tag{28}$$

Since $\mathbf{G}\bar{\mathbf{H}}_0\mathbf{F} = \mathbf{I}_M$ yields rank $(\mathbf{G}\bar{\mathbf{H}}_0\mathbf{F}) = M$, $\mathbf{G} \in \langle\bar{\mathbf{H}}_0^H\rangle$ is obtained based on the fact that $\mathbf{F}$ is a full matrix from A4) and the $P \times M$ matrix $\bar{\mathbf{H}}_0$ is of rank M. Hence, (28) has no loss of generality. By substituting (28) into (27), the formulated problem of (27) can be rewritten as

$$\begin{aligned}\max_{\mathbf{G}',\mathbf{F}} \quad & \frac{|tr(\mathbf{I}_M)|^2}{M\sigma_v^2 tr\left[\mathbf{G}'\bar{\mathbf{H}}_0^H\bar{\mathbf{H}}_0\left(\mathbf{U}_0^H\mathbf{P}_{\mathbf{Z}}^{\perp}\mathbf{U}_0\right)^{-1}\bar{\mathbf{H}}_0^H\bar{\mathbf{H}}_0\mathbf{G}'^H\right]}\\ \text{subject to} \quad & \mathbf{G}'\bar{\mathbf{H}}_0^H\bar{\mathbf{H}}_0\mathbf{F} = \mathbf{I}_M.\end{aligned} \tag{29}$$

To solve (29) with respect to $\mathbf{G}'$ and $\mathbf{F}$, we invoke the weight inner-product for matrices which is defined as $< \mathbf{X}|\mathbf{Y} > \triangleq {\rm tr}(\mathbf{X}^H\mathbf{D}^H\mathbf{D}\mathbf{Y})$, where $\mathbf{D}$ is a nonsingular matrix. Particularly, if $\mathbf{Y} = \mathbf{X}$, $< \mathbf{X}|\mathbf{X} >= |\mathbf{D}\mathbf{X}|_F^2$. Applying the general Cauchy–Schwarz inequality yields (30) shown at the bottom of the page, where the first equality relied on the constraint in (29) and the choice of $\mathbf{D} = \left[\bar{\mathbf{H}}_0^H\bar{\mathbf{H}}_0\left(\mathbf{U}_0^H\mathbf{P}_{\mathbf{Z}}^{\perp}\mathbf{U}_0\right)^{-1}\bar{\mathbf{H}}_0^H\bar{\mathbf{H}}_0\right]^{1/2}$. Accordingly, the maximum of (30) is achieved if $\mathbf{G}' = \mathbf{F}^H\mathbf{U}_0^H\mathbf{P}_{\mathbf{Z}}^{\perp}\mathbf{U}_0(\bar{\mathbf{H}}_0^H\bar{\mathbf{H}}_0)^{-1}$. Clearly, $\mathbf{G}'$ is justified as a $M \times M$ full-rank matrix, defined in (28), since $\mathbf{F}$, $\mathbf{U}_0^H\mathbf{P}_{\mathbf{Z}}^{\perp}\mathbf{U}_0$ and $(\bar{\mathbf{H}}_0^H\bar{\mathbf{H}}_0)^{-1}$ of $\mathbf{G}'$ are all $M \times M$ full-rank matrices. Substituting $\mathbf{G}'$ into (28) and the constraint of (29), we have

$$\mathbf{G} = \mathbf{F}^H\mathbf{U}_0^H\mathbf{P}_{\mathbf{Z}}^{\perp}\mathbf{U}_0\left(\bar{\mathbf{H}}_0^H\bar{\mathbf{H}}_0\right)^{-1}\bar{\mathbf{H}}_0^H \tag{31}$$

and

$$\mathbf{F}^H\mathbf{U}_0^H\mathbf{P}_{\mathbf{Z}}^{\perp}\mathbf{U}_0\mathbf{F} = \mathbf{I}_M. \tag{32}$$

If A6) holds, then $[\,\mathbf{Z} \;\;|\;\; \mathbf{U}_0\,]$ is known to solve (32). Consider the eigenvalue-decomposition [3]

$$\mathbf{U}_0^H\mathbf{P}_{\mathbf{Z}}^{\perp}\mathbf{U}_0 = \mathbf{\Gamma}\Pi\Gamma^H \tag{33}$$

where $\mathbf{\Gamma}$ and $\mathbf{\Pi}$ denote the $M \times M$ unitary and positive-diagonal matrices, respectively. Given (33), by solving (31) and (32), the optimum $\mathbf{F}$ and $\mathbf{G}$ for minimizing $P_{\rm ae\_lb}$ (i.e., $\max-{\rm SNR}$) are

$$\begin{aligned}\mathbf{F}_{\max-{\rm SNR}} &= \mathbf{\Gamma}\Pi^{-(1/2)}\\ \mathbf{G}_{\max-{\rm SNR}} &= \mathbf{\Pi}^{1/2}\mathbf{\Gamma}^H\left(\bar{\mathbf{H}}_0^H\bar{\mathbf{H}}_0\right)^{-1}\bar{\mathbf{H}}_0^H.\end{aligned} \tag{34}$$

With the power constraint, i.e., ${\rm tr}(\mathbf{F}\mathbf{F}^H) \leq P_T$, (34) is scaled by a constant $\sqrt{P_T/{\rm tr}(\mathbf{\Pi}^{-1})}$.

2) Minimize $P_{\rm ae\_lb}$ and Equalize the Jensen's Inequality of $P_{\rm ae}$: As mentioned previously, the Jensen's inequality of

$$\begin{aligned}\frac{|tr(\mathbf{I}_M)|^2}{tr[\mathbf{G}'\bar{\mathbf{H}}_0^H\bar{\mathbf{H}}_0(\mathbf{U}_0^H\mathbf{P}_{\mathbf{Z}}^{\perp}\mathbf{U}_0)^{-1}\bar{\mathbf{H}}_0^H\bar{\mathbf{H}}_0\mathbf{G}'^H]} &= \frac{\left|< \mathbf{G}'^H|[\bar{\mathbf{H}}_0^H\bar{\mathbf{H}}_0(\mathbf{U}_0^H\mathbf{P}_{\mathbf{Z}}^{\perp}\mathbf{U}_0)^{-1}\bar{\mathbf{H}}_0^H\bar{\mathbf{H}}_0]^{-1}\bar{\mathbf{H}}_0^H\bar{\mathbf{H}}_0\mathbf{F} >\right|^2}{\left\|[\bar{\mathbf{H}}_0^H\bar{\mathbf{H}}_0(\mathbf{U}_0^H\mathbf{P}_{\mathbf{Z}}^{\perp}\mathbf{U}_0)^{-1}\bar{\mathbf{H}}_0^H\bar{\mathbf{H}}_0]^{1/2}\mathbf{G}'^H\right\|_F^2}\\ &\leq \left\|[\bar{\mathbf{H}}_0^H\bar{\mathbf{H}}_0(\mathbf{U}_0^H\mathbf{P}_{\mathbf{Z}}^{\perp}\mathbf{U}_0)^{-1}\bar{\mathbf{H}}_0^H\bar{\mathbf{H}}_0]^{-1/2}\bar{\mathbf{H}}_0^H\bar{\mathbf{H}}_0\mathbf{F}\right\|_F^2\end{aligned} \tag{30}$$

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(24) is valid only when $\left[\mathbf{G}\mathbf{R}'_{\mathbf{v}'\mathbf{v}}\mathbf{G}^H\right]_{ii}$ is less than $2/3\sigma_v^2$ for all i. Furthermore, the equality in (24) can hold if and only if $[\mathbf{G}\mathbf{R}'_{\mathbf{v}'\mathbf{v}}\mathbf{G}^H]_{ii}$ are equal for all i. Therefore, by considering the convexity constraint, the design goal should be formulated as follows:

$$\min_{\mathbf{F},\mathbf{G}} \quad P_{\text{ae\_lb}} \tag{35a}$$

$$\text{subject to} \quad \mathbf{G}\bar{\mathbf{H}}_0\mathbf{F} = \mathbf{I}_M, \tag{35b}$$

$$[\mathbf{G}\mathbf{R}_{\mathbf{v}'\mathbf{v}'}\mathbf{G}^H]_{ii} = [\mathbf{G}\mathbf{R}_{\mathbf{v}'\mathbf{v}'}\mathbf{G}^H]_{jj} \leq \frac{2}{3\sigma_v^2}, \tag{35c}$$

$$\text{for all } 0 \leq i, \quad j \leq M-1,$$

$$\text{and } tr(\mathbf{F}\mathbf{F}^H) \leq P_T \tag{35d}$$

where $P_{\text{ae\_lb}}$ is given by (25). We will demonstrate that a particular choice of $\mathbf{G}$ and $\mathbf{F}$ can simplify the problem of (35) and achieve the design goal. To satisfy (35b), we consider

$$\mathbf{G} = \mathbf{F}^{-1}\left(\bar{\mathbf{H}}_0^H\bar{\mathbf{H}}_0\right)^{-1}\bar{\mathbf{H}}_0.^H \tag{36}$$

In (36), $\mathbf{F}^{-1}$ exists as described in Section II. Furthermore, applying the singular value decomposition [3], [22] to parameterize $\mathbf{F}$ [2], [10], and [25] yields

$$\mathbf{F} = \mathbf{W}^H\mathbf{\Xi}\mathbf{K} \tag{37}$$

where $\mathbf{W}$ and $\mathbf{K}$ are both square unitary matrices of dimension M, and $\mathbf{\Xi}$ is a $M \times M$ positive diagonal matrix. By substituting (33), (36), and (37) into (26), (26) can be rewritten as

$$\mathbf{G}\mathbf{R}'_{\mathbf{v}'\mathbf{v}}\mathbf{G}^H = \mathbf{K}^H\mathbf{\Xi}^{-1}\mathbf{W}\mathbf{\Gamma}\mathbf{\Pi}^{-1}\mathbf{\Gamma}^H\mathbf{W}^H\mathbf{\Xi}^{-1}\mathbf{K}. \tag{38}$$

Since the relation between $\mathbf{F}$ and $\mathbf{G}$ is given by (36), minimizing $P_{\text{ae\_lb}}$ in (35a) is equivalent to minimizing the denominator, i.e., $\text{tr}(\mathbf{G}\mathbf{R}'_{\mathbf{v}'\mathbf{v}}\mathbf{G}^H)$ [see (25)]. Given this fact and based on substituting (37) into (35d), (38) into (35c) and $\text{tr}(\mathbf{G}\mathbf{R}_{\mathbf{v}'\mathbf{v}'}\mathbf{G}^H)$, the problem (35) can be reformulated as follows:

$$\min_{\mathbf{W},\mathbf{\Xi},\mathbf{K}} \quad \text{tr}(\mathbf{\Xi}^{-2}\mathbf{W}\mathbf{\Gamma}\mathbf{\Pi}^{-1}\mathbf{\Gamma}^H\mathbf{W}^H) \tag{39a}$$

$$\text{subject to} \quad [\mathbf{K}^H\mathbf{\Xi}^{-1}\mathbf{W}\mathbf{\Gamma}\mathbf{\Pi}^{-1}\mathbf{\Gamma}^H\mathbf{W}^H\mathbf{\Xi}^{-1}\mathbf{K}]_{ii} = [\mathbf{K}^H\mathbf{\Xi}^{-1}\mathbf{W}\mathbf{\Gamma}\mathbf{\Pi}^{-1}\mathbf{\Gamma}^H\mathbf{W}^H\mathbf{\Xi}^{-1}\mathbf{K}]_{jj} \leq \frac{2}{3\sigma_v^2}, \text{ for all } 0 \leq i, \quad j \leq M-1, \tag{39b}$$

$$\text{and } \text{tr}(\mathbf{\Xi}^2) \leq P_T \tag{39c}$$

where (39a) and (39c) were derived using the relation $\text{tr}(\mathbf{X}\mathbf{Y}) = \text{tr}(\mathbf{Y}\mathbf{X})$ for compatible matrices $\mathbf{X}$ and $\mathbf{Y}$. The design goal of (35) is then reduced to (39), and (39) is similar to the design problem, formulated in [10] and [25] (see [10, (28)] [25, (51)], respectively). By applying [10, Lemma 1], the optimum $\mathbf{K}_{\text{opt}}$ of (37) is given by $\mathbf{K}_{\text{opt}} = \mathbf{F}_{\mathbf{D}}$, where $\mathbf{F}_{\mathbf{D}}$ denotes the normalized DFT in (20), while by applying the Appendix of [10], the optimum $\mathbf{W}$ and $\mathbf{\Xi}$ of (37) are obtained by $\mathbf{W}_{\text{opt}} = \mathbf{\Gamma}^H$ and $\Xi_{\text{opt}} = \sqrt{P_T/\text{tr}(\mathbf{\Pi}^{-1/2})}\mathbf{\Pi}^{-1/4}$, respectively. Notably, the optimum $\mathbf{K}_{\text{opt}}$ adopted herein is not the only one choice. For example, the $M \times M$ normalized Hadamard matrix is also an optimum solution when M is to the power of two (see [10] and [25] for details). With the previous results, as well as (37), and (36), the optimum min-BER precoder $\mathbf{F}$ and the equalizer $\mathbf{G}$, subject to a bound on transmission power, are given by

$$\mathbf{F}_{\min-\text{BER}} = \sqrt{\frac{P_T}{\text{tr}(\mathbf{\Pi}^{-1/2})}}\mathbf{\Gamma}\mathbf{\Pi}^{-1/4}\mathbf{F}_{\mathbf{D}}$$

$$\mathbf{G}_{\min-\text{BER}} = \left(\sqrt{\frac{P_T}{\text{tr}(\mathbf{\Pi}^{-1/2})}}\right)^{-1} \times \mathbf{F}_{\mathbf{D}}^{-1}\mathbf{\Pi}^{1/4}\mathbf{\Gamma}^H\left(\bar{\mathbf{H}}_0^H\bar{\mathbf{H}}_0\right)^{-1}\bar{\mathbf{H}}_0^H. \tag{40}$$

Fig. 2 illustrates the new transceiver model of TZ systems employing a cascaded ZF equalizer and a min-BER precoder. As stated earlier, the optimum designs derived in this subsection are all applicable to CP systems since a cascaded equalizer and the results of (19) and (20) can also be obtained for CP systems. Although the previous designs are derived for BPSK, these results can be extended to other high-order constellations. A brief explanation is as follows. For an N-ary QAM constellation (assuming that a Gray mapping from bits to symbols is used) under additive Gaussian noise, the BER of the ith equalized symbol of (20) can be approximated by $P_{\text{ie(QAM)}} \approx \alpha\text{erfc}\left(\sqrt{\varepsilon_s/\beta(E[\mathbf{G}\mathbf{v}'(n)\mathbf{v}'^H(n)\mathbf{G}^H])_{ii}}\right)$, where $\alpha = 2(1-1/\sqrt{N})\log_2 N$ and $\beta = 2(N-1)/3$ are constants for a given N, and ε_s denotes the average power of the ith equalized symbol [25], [38]. Substituting the assumed average power, 1, in A7) and the ISI-free constraint $\mathbf{G}\bar{\mathbf{H}}_0\mathbf{F} = \mathbf{I}_M$ yields $\varepsilon_s = 1$. Consequently, the expression of $P_{\text{ie(QAM)}}$ is similar to (22) and can be applied to create optimum designs by the same method.

Remarks

The following observations are made.

1) Condition C2) suggests that the channel should be a perfect reconstruction (PR) channel. For TZ systems, recall the proof of Theorem 1 that $\mathbf{U}_2\mathbf{\Psi}_{\text{TZ}}$ of $\mathbf{Z}$ contains $(L+M-P)$ nonzero columns for $P-M < L$. Since the top $(L+M-P)\times(L+M-P)$ submatrix of $\mathbf{U}_2\mathbf{\Psi}_{\text{TZ}}$ is an upper triangular Toeplitz matrix, the PR test matrix can be further reduced from

$$\left[\mathbf{Z} \;\middle|\; \mathbf{U}_0\right] = \left[\mathbf{U}_2\mathbf{\Psi}_{\text{TZ}} \;\middle|\; \mathbf{U}_1 \;\middle|\; \mathbf{U}_0\right]$$

to the lower $(QP-L-M+P)$ rows of

$$\left[\mathbf{U}_1 \;\middle|\; \mathbf{U}_0\right]$$

as observed by Kung *et al.* in [26] (see [26, Ex. 2]). Nevertheless, we address that the min-BER design presented here does not rely on the reduced PR test matrix, but instead relies on the $P \times M$ matrix $\bar{\mathbf{H}}_0$ of (19). Thus, a min-BER design is obtained for a block-based transceiver with redundancy TZ of $L/2$ [11], [27], which was not considered by Ding *et al.* in [10]. Also, as demonstrated in Remark 1

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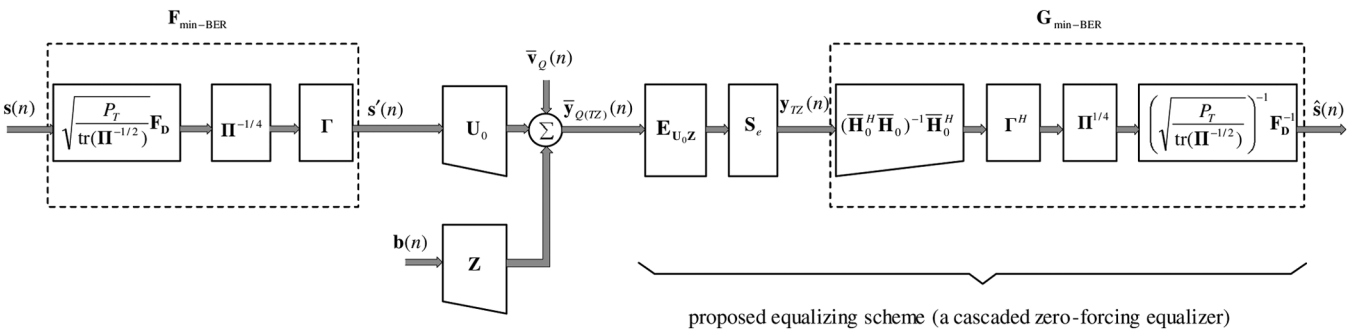

Fig. 2. Transceiver model for TZ systems employing the min-BER block-based precoder and the proposed cascaded ZF equalizer.

of Section III-C, with the results of Theorems 1 and 2, a minimum Q can always be offered for the min-BER design, particularly for insufficient redundancy cases. This finding implies the necessity and significance of Theorems 1 and 2.

2) For TZ systems with $P = M + L$, the results presented in Remark 2 of Section III-C can be used to fulfill the conditions C0)–C2). Hence, $\bar{\mathbf{H}}_0 = \mathbf{U}_0 = \mathbf{H}_0\mathbf{F}_{\mathrm{TZ}}$ when $Q = 1$, and $\mathbf{P}_\mathbf{Z}^\perp$ is an identity matrix. Substituting $\mathbf{P}_\mathbf{Z}^\perp = \mathbf{I}_P$ and $\mathbf{U}_0 = \bar{\mathbf{H}}_0$ into (33) yields $\bar{\mathbf{H}}_0^H\bar{\mathbf{H}}_0 = \mathbf{\Gamma\Pi\Gamma}^H$. With this result, we can simplify (34) and obtain $\mathbf{F}_{\max-\mathrm{SNR}} = \mathbf{\Gamma\Pi}^{-(1/2)}$ and $\mathbf{G}_{\max-\mathrm{SNR}} = \mathbf{\Pi}^{(-1/2)}\mathbf{\Gamma}^H\bar{\mathbf{H}}_0^H$, the same as the results in [2] (see [2, (57)], where the constant scale of (57) is considered unity here). Similarly, we obtain $\mathbf{F}_{\min-\mathrm{BER}} = \sqrt{P_T/\mathrm{tr}(\mathbf{\Pi}^{-1/2})}\mathbf{\Gamma\Pi}^{-1/4}\mathbf{F}_\mathbf{D}$ and $\mathbf{G}_{\min-\mathrm{BER}} = \sqrt{\mathrm{tr}(\mathbf{\Pi}^{-1/2})/P_T}\mathbf{F}_\mathbf{D}^{-1}\mathbf{\Pi}^{-3/4}\mathbf{\Gamma}^H\bar{\mathbf{H}}_0^H$ by simplifying (40), identical to the results in [10] (see [10, (13) and (36)]).

3) As indicated in (20), the proposed equalizing scheme offers a matrix degree-of-freedom, i.e., $\mathbf{G}$. Therefore, the MMSE criteria studied by Palomar *et al.* in [25] and [30] for sufficient redundancy can also be applied to $\mathbf{G}$. Interestingly, if CSI is unavailable at the transmitter and $\mathbf{G}$ is employed with $\mathbf{G} = \mathbf{F}^{-1}\bar{\mathbf{H}}_0^+$, then an alternative implementation of the ZF equalizer in Theorems 1 or 2 is obtained. This fact has been shown in [6] and [21] for CP systems (e.g., see [6, pt. 3.3]) and can be similarly extended to TZ systems. Notably, the concept of the proposed equalizing scheme resembles the notion proposed in [29] for physical MIMO channels. This study begins from the exploitation of a cascaded equalizer, while the authors [29] started by utilizing dynamical coordinate transformations of the PR channels, i.e., the decomposition of PR channels.

V. Computer Simulations

This section conducted various simulation examples to verify the proposed design in terms of BER. The BER is sketched as a function of E_b/N_0 (dB), where E_b is the average energy per bit, i.e., $E_b = (1/M)\mathrm{tr}\left(\mathbf{F}_0\mathbf{F}_0^H\right)$, and N_0 is the noise power spectral density. Except for the equalizers in Theorems 1 and 2, our min-BER design, i.e., (40), was implemented with a cascaded equalizer, as depicted in Fig. 2. Similarly, the max-SNR design can be obtained by replacing the min-BER design inside the dashed-line box of Fig. 2 with the design of (34). For CP systems, we simply substitute $\bar{\mathbf{y}}_{Q(\mathrm{CP})}(n)$, $\mathbf{\Psi}_{\mathrm{CP}}$, and $\mathbf{F}_{\mathrm{CP}}$ of CP systems for $\bar{\mathbf{y}}_{Q(\mathrm{TZ})}(n)$, $\mathbf{\Psi}_{\mathrm{TZ}}$, and $\mathbf{F}_{\mathrm{TZ}}$ of TZ systems, as mentioned in Sections IV-A and IV-B. Furthermore, when the OFDM precoder is employed for comparison, we use the DFT basis in (17). The BER was computed according to (21) and (23).

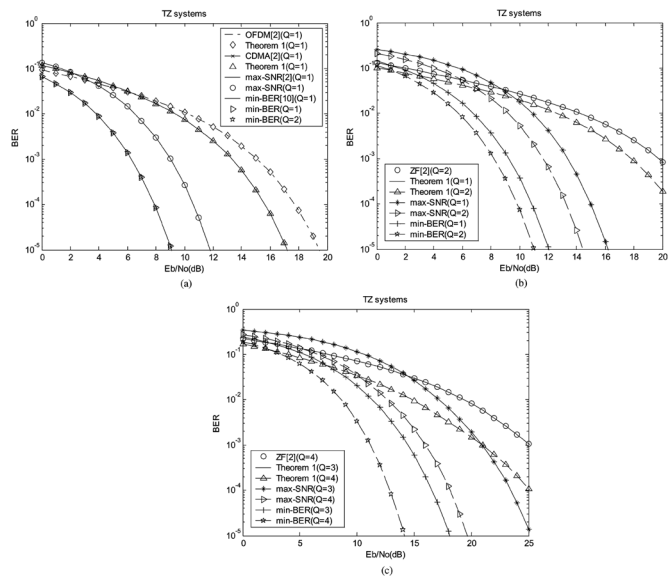

Fig. 3. Comparison of BER performance for TZ systems with the proposed designs. (a) Sufficient redundancy case: $(P, M, L) = (20, 16, 4)$ with channel $\mathbf{h}_1(n)$. (b) Insufficient redundancy case: $(P, M, L) = (34, 32, 4)$ with channel $\mathbf{h}_1(n)$. (c) Minimum redundancy case: $(P, M, L) = (17, 16, 4)$ with 100 randomly generated channels.

Example 1—TZ Systems

First, to check the validity of our designs in the case of sufficient redundancy (i.e., $P - M \geq L$), channel and parameters from [2, Fig. 8] were used. Specifically, we considered $P = M + L$ with $P = 20$, and $M = 16$ for a channel $\mathbf{h}_1(n)$ of order $L = 4$ with zeros at 1, $0.9j$, $-0.9j$, $1.3e^{j5\pi/8}$. Fig. 3(a) compared BER performances for the OFDM precoder and our equalizer in Theorem 1, the CDMA precoder using Hadamard basis and the equalizer in Theorem 1, our max-SNR design, and our min-BER design. Additionally, the BER performances of the equalizers using [2, (44)] for the OFDM and CDMA

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precoders are plotted. We also compared our designs with the min-BER design in [10], [10, (36)] and with the max-SNR design in [2], [2, (57)]. As illustrated in Fig. 3(a), all of the proposed designs performed as well as the corresponding ones in [2] and in [10] for $Q = 1$. Furthermore, as examined in [10], designs with min-BER criterion demonstrated the best performance. Notably, the performance of the proposed min-BER design did not improve further as Q was increased from 1 to 2. This finding is expected since the IBI is completely isolated if sufficient trailing zeros are used. Thus, increasing Q is no longer useful for accurately estimating symbols.

Next, to further verify the validity of our design in the case of insufficient redundancy, i.e., $P - M < L$, we implemented a system with $M = 32$ and $P = 34$ for the channel $\mathbf{h}_1(n)$ of $L = 4$, as defined previously. Since the max-SNR design and min-BER design were not discussed in [2] and [10], the BER comparisons of Fig. 3(b) were given for the OFDM precoder and the equalizer using [2, (29)], the OFDM precoder and our equalizer in Theorem 1, our max-SNR design, and our min-BER design. In this case, i.e., $P - M = L/2$, a minimum $Q = 1$ (i.e., block-based) was required for our equalizer in Theorem 1 and the optimum designs, while a minimum $Q = 2$ was required in [2]. Indeed, the proposed framework provides a block-based min-BER transceiver design for $P - M = L/2$, but the results presented in [10] failed to consider this case. Fig. 3(b) shows that the proposed min-BER design achieved the best performance throughout most of the E_b/N_0 range for $Q = 2$ and $Q = 1$, respectively. Interestingly, we notice that the min-BER design can further improve the BER performance near 1 dB by increasing Q from 1 to 2 in the previous case. Moreover, the equalizer with $Q = 1$ in Theorem 1 was shown to perform as well as that with $Q = 2$ in [2]. In fact, this result can be proven mathematically, as in the Appendix. Finally, we considered the case of minimum redundancy, i.e., $P = M + 1$. The system parameters are $P = 17$, $M = 16$, and $L = 4$, and 100 randomly generated FIR channels, each with a unit 2-norm, were utilized to calculate the average BER. This system investigated all of the designs compared in Fig. 3(b). Again, the average BER curves of Fig. 3(c) exhibit similar results to those of Fig. 3(b). As shown in Fig. 3(c), our equalizer with minimum $Q = 3$ in Theorem 1 and that with its minimum $Q = 4$ in [2] showed identical BER performance. However, by increasing Q from 3 to 4, all of the proposed designs could achieve improved performance. Furthermore, the proposed min-BER design performed substantially better than all other designs for $Q = 3$ and $Q = 4$, respectively.

Example 2—CP Systems

In this example, all compared designs employed CP except those which are specified to stated, have used TZ. For the case of sufficient redundancy, we considered $L = 5$, $M = 32$, and $P = M + L = 37$. The randomly generated FIR channel was $\mathbf{h}_2(n) = [-0.54 - 0.17j\ 0.37 + 0.08j\ 0.43 - 0.09j\ 0.30 + 0.06j\ -0.40 - 0.05j\ 0.23 + 0.08j]^T$. In Fig. 4(a), BER comparisons were given for the OFDM precoder and the equalizer in Theorem 2, the OFDM precoder and the standard OFDM receiver (use [2, (15)]), the min-BER design (use [10, (43)]),

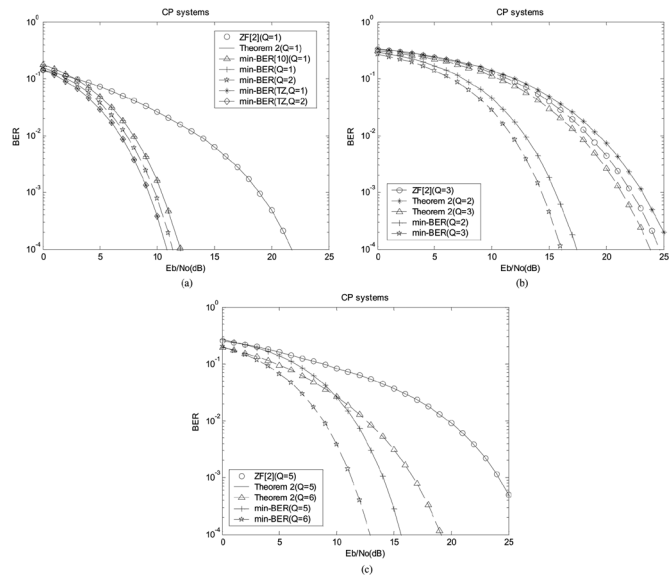

Fig. 4. Comparison of BER performance for CP systems with the proposed designs. (a) Sufficient redundancy case: $(P, M, L) = (37, 32, 5)$ with channel $\mathbf{h}_2(n)$. (b) Insufficient redundancy case: $(P, M, L) = (13, 8, 11)$ with 100 randomly generated channels. (c) Minimum redundancy case: $(P, M, L) = (17, 16, 5)$ with channel $\mathbf{h}_2(n)$.

and the proposed min-BER design. Moreover, the BER performances of the proposed TZ min-BER designs were also sketched for comparison. For the case of $Q = 1$, Fig. 4(a) indicated that the proposed equalizer in Theorem 2 and the proposed min-BER design, respectively, achieved identical BER performance to the corresponding design in [2] and [10]. These observation results confirmed Remark 3 of Section III-C and the validity of our min-BER design for the case of sufficient redundancy. Again, the results shown in Fig. 4(a) reveal that the proposed TZ min-BER designs with $Q = 1$ and $Q = 2$ exhibited identical BER performance. Additionally, they achieved a better BER performance than the proposed min-BER designs in the previous case. Notably, the proposed min-BER design with $Q = 2$ could improve BER performance by 0.8 dB as Q was increased from 1 to 2 (or compared with the min-BER design in [10]). Interestingly, for CP systems, even in the case of sufficient redundancy, the BER performance of the min-BER design in [10] could be further improved by using the oblique projection framework with a larger Q. This finding differs significantly from the results in Fig. 3(a) for TZ systems.

Furthermore, to verify the validity of the proposed designs in the case of insufficient redundancy, a CP system with $P - M < L$ and $M < L$ was especially implemented. The system parameters were $M = 8$, $P = 13$, and $L = 11$, and 100 randomly generated FIR channels, each with a unit 2-norm, were utilized to calculate the average BER. In Fig. 4(b), the curves compared the average BER performance for the OFDM precoder and its equalizer using [2, (29)], the OFDM precoder and our equalizer in Theorem 2, and also our min-BER design. In this case, a minimum $Q = 2$ is required for our equalizer in Theorem 2 and the min-BER design, while $Q = 3$ is required for the equalizer in [2], as discussed in Remark 1 of Section III-C. According to Fig. 4(b), the proposed equalizer with $Q = 3$ in Theorem 2 had an improved BER performance despite the fact that the

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proposed equalizer with $Q = 2$ failed to perform as well as the equalizer with $Q = 3$ in [2]. Moreover, for $Q = 2$ and 3, respectively, the proposed min-BER design achieved the best performance. Similar to the results for the TZ systems, the BER performance of our min-BER design can be further improved given a larger Q. Finally, this study examines the case of minimum redundancy. Specifically, a CP system with $L = 5$, $M = 16$, and $P = M + 1 = 17$ was employed for the channel $\mathbf{h}_2(n)$, as defined before. The BER curves compared among the designs used in Fig. 4(b) are sketched in Fig. 4(c). Since $M > L$ holds, it was observed that the equalizers in Theorem 2 and in [2] not only had the same minimum $Q = 5$, but also shared identical BER performance, as indicated in Fig. 4(c). Once again, the proposed min-BER designs achieved the best performance throughout most of the E_b/N_0 range, for $Q = 5$ and $Q = 6$, respectively. Particularly, with increasing Q from 5 to 6, almost 3 dB E_b/N_0 was gained at a BER of around 10^{-4} in the case considered here.

VI. Conclusion

In this paper, we have designed a minimum BER block-based precoder and devised a cascaded ZF equalizer for block transmission systems with sufficient or insufficient redundancy. First, Theorems 1 and 2 can be used to determine the received signal model and the minimum $(Q-1)$th-order ZF equalizer of the TZ and CP systems, respectively, based on knowledge of parameters P, M, and L. Then, for a PR channel, by exploiting the property of oblique projection, we proposed a new equalizing scheme, i.e., a cascaded equalizer, in which the IBI is completely eliminated by the oblique projection and subsequently followed by a matrix degree-of-freedom for ISI equalization, i.e., the matrix $\mathbf{G}$. Following imposing an ISI-free constraint on $\mathbf{G}$, the cascaded equalizer leads to a cascaded ZF equalizer. Consequently, the use of this cascaded ZF equalizer enabled results such as the max-SNR and min-BER designs in [2] and [10], respectively, to be extended to the case of insufficient redundancy, as shown in Section IV. Simulations have also demonstrated the validity of the proposed min-BER design for CP and TZ systems with insufficient redundancy. Moreover, the theoretical derivations and simulation results have shown that the design framework can retain identical BER performance to previous designs, such as those presented in [2] and [10] for cases with sufficient redundancy. This study also demonstrated that even in the case of sufficient redundancy, the proposed framework can trade the complexity of the cascaded ZF equalizer with improved BER performance for CP systems, as indicated in Fig. 4(a). Similarly, this tradeoff can be utilized with the proposed framework in cases with insufficient redundancy. Since the proposed cascaded equalizer offers a matrix degree-of-freedom for ISI equalization, whether for sufficient or insufficient redundancy, designs incorporating other criteria may represent a potential future research direction.

Appendix

This Appendix demonstrates the equivalency of the BER performance between the $(Q-2)$th-order ZF equalizer in Theorem 1 and the $(Q-1)$th-order ZF equalizer in [2], as indicated in Fig. 3(b) and (c). Given $\mathbf{C}$ and $\mathbf{F}_Q$, defined in Section II, the $(Q-1)$th-order ZF equalizer in [2] can be expressed as $\mathbf{G}_{\mathrm{TZ}} = [\,\mathbf{O}_{M\times L} \quad \bar{\mathcal{G}}_{\mathrm{TZ}}]$, where $\bar{\mathcal{G}}_{\mathrm{TZ}}[\,\mathbf{O}_{M\times(Q-1)M} \quad \mathbf{I}_M\,]\cdot(CF_Q)^+$ (see (22)–(29) in [2] and assume that $\mathbf{C}\mathcal{F}_Q$ has full column rank). Let $\mathbf{C}\mathcal{F}_Q$ be decomposed into four block submatrices, $\mathbf{T}_1$, $\mathbf{T}_2$, $\mathbf{T}_3$, and $\mathbf{T}_4$, which are the upper left $M\times M$ submatrix, the upper right $M\times(Q-1)M$ submatrix, the lower left $(QP-M-L)\times M$ submatrix and the lower right $(QP-M-L)\times(Q-1)M$ submatrix, respectively. Consequently, based on the previous, $(\mathbf{C}\mathcal{F}_Q)^+$ can be simplified by applying the definition of pseudo-inverse [3] and the inversion formula for 2×2 block matrices [35]

$$
\begin{aligned}
(\mathcal{C}\mathcal{F}_Q)^+ &= \left(\left[\begin{array}{c|c}\mathbf{T}_1 & \mathbf{T}_2\\ \hline \mathbf{T}_3 & \mathbf{T}_4\end{array}\right]^H \left[\begin{array}{c|c}\mathbf{T}_1 & \mathbf{T}_2\\ \hline \mathbf{T}_3 & \mathbf{T}_4\end{array}\right]\right)^{-1} \\
&\quad\times \left[\begin{array}{c|c}\mathbf{T}_1 & \mathbf{T}_2\\ \hline \mathbf{T}_3 & \mathbf{T}_4\end{array}\right]^H \\
&= \left[\begin{array}{c|c}\mathbf{T}_1^{-1} & -\mathbf{T}_1^{-1}\mathbf{T}_2\left(\mathbf{T}_4^H\mathbf{T}_4\right)^{-1}\mathbf{T}_4^H\\ \hline \mathbf{O}_{(Q-1)M\times M} & \left(\mathbf{T}_4^H\mathbf{T}_4\right)^{-1}\mathbf{T}_4^H\end{array}\right]
\end{aligned}
\tag{41}
$$

where (41) is derived using the fact that matrix $\mathbf{T}_3$ is a null matrix. Substituting (41) into the previous $\bar{\mathcal{G}}_{\mathrm{TZ}}$, after some mathematical manipulation, yields

$$
\mathcal{G}_{\mathrm{TZ}} = [\,\mathbf{O}_{M\times P} \quad \mathcal{G}'_{\mathrm{TZ}}\,] \tag{42}
$$

where

$$
\mathcal{G}'_{\mathrm{TZ}} = \left[\mathbf{O}_{M\times(L+M-P)} \;\middle|\; [\,\mathbf{O}_{M\times(Q-2)M} \quad \mathbf{I}_M\,]\cdot\mathbf{T}_4^+\right].
$$

The zero matrix $\mathbf{O}_{M\times P}$ of $\mathcal{G}_{\mathrm{TZ}}$ in (42) implies that the $(Q-1)$th-order ZF equalizer in [2] is implemented effectively using the $(Q-2)$th-order equalizing matrix, $\mathcal{G}'_{\mathrm{TZ}}$, if $Q \geq 2$. As shown in the following, $\mathcal{G}'_{\mathrm{TZ}}$ equals the $(Q-2)$th-order ZF equalizer in Theorem 1. For $P - M \leq L$, the $(Q-1)P\times 1$ vector $\bar{\mathbf{y}}_{Q-1(\mathrm{TZ})}$ can be constructed based on Theorem 1. Since the top $(L+M-P)\times(L+M-P)$ submatrix of $\mathbf{U}_2\mathbf{\Psi}_{\mathrm{TZ}}$ is an upper triangular Toeplitz matrix, the $\mathbf{O}_{M\times(L+M-P)}$ of $\mathcal{G}'_{\mathrm{TZ}}$ can just be employed to reduce the matrix $\left[\mathbf{Z} \;\middle|\; \mathbf{U}_0\right]$ of $\bar{\mathbf{y}}_{Q-1(\mathrm{TZ})}$ to the lower $((Q-1)P-L-M+P)$ rows of $\left[\mathbf{U}_1 \;\middle|\; \mathbf{U}_0\right]$. Subsequently, $[\,\mathbf{O}_{M\times(Q-2)M} \quad \mathbf{I}_M\,]\cdot\mathbf{T}_4^+$ of $\mathcal{G}'_{\mathrm{TZ}}$ equalizes the lower $((Q-1)P-L-M+P)\times 1$ vector of $\bar{\mathbf{y}}_{Q-1(\mathrm{TZ})}$ to obtain $\mathbf{s}(n)$. Accordingly, based on the previous fact and the uniqueness property of the ZF equalizer derived from Theorem 1, $\mathcal{G}'_{\mathrm{TZ}}$ must be identical to the $(Q-2)$th-order $\mathcal{G}_{\mathrm{ob(TZ)}}$ for retrieving $\mathbf{s}(n)$ in $\bar{\mathbf{y}}_{Q-1(\mathrm{TZ})}$.

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