行政院國家科學委員會專題研究計畫　期中進度報告

有關 k 母體之貝氏取樣設計與決策(2/3)

計畫類別：個別型計畫
計畫編號：NSC93-2213-E-032-002-
執行期間：93 年 08 月 01 日至 94 年 10 月 31 日
執行單位：淡江大學經營決策學系

計畫主持人：黃文濤

報告類型：精簡報告

處理方式：本計畫可公開查詢

中　華　民　國 94 年 11 月 8 日

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計畫編號 ： 93-2213-E-032-002-

主 持 人 ： 黃 文 濤

E-mail ： 005697@mail.tku.edu.tw

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1. Introduction

To understand and evaluate a system or a process, one of effective methods is to consider some quantitative measure to estimate the performance of the system or process under study. The well-known measure of product quality in industry is the capability index. It is a dimensionless measure based on some parameters and specifications that are involved in the process.

In most literature related to capability index, it is mainly focus on the estimation of its estimators. In many practical applications, instead of estimations of the capability indices of process under study, there occurs a quality related problem that arises in the initial production setting is how to select the most desirable manufacturing process among several available processes. Suppose a new product is under study and development, and suppose there are k processes to produce it. Or, suppose we need to evaluate k systems for its quality. We are interested in identifying one of them as the most desirable process to produce the product.

For selecting the best manufacturing process, Tseng and Wu (1991) considered the selection problem in terms of capability index C_p which is introduced by Kane (1986ab). Since the difference between the upper and lower specification limit is a known quantity, the problem considered in Tseng and Wu (1991) is equivalent to select the process which is corresponding to the smallest variance. There have been several capability indices such as C_{pm} (see Chan, Cheng and Spiring (1988)), C_{pk} (see Gunter (1989)) and C_{pmk} (see Pearn, Kotz and Johnson (1992)). However, mostly C_{pm} and C_{pk} are widely used. Spiring (1997) modified C_{pm} and proposed C_{pw} which included C_p, C_{pm} and C_{pk} as special case. So in this paper, we consider selecting the best process in terms of C_{pw} which is a modified quantity of C_{pm} taking weight between the variance and the square difference between mean and target. Moreover, we consider another criterion so that the capability index of the process selected should be larger than a prefixed value which can be considered as a control.

2. Formulation of problem and a Bayes decision rule

Definition 2.1 Let π be a manufacturing process with mean θ and variance σ^2, T be the target value, and USL and LSL be the upper and lower specification limit, respectively. Then a modified process capability index C_{pw} of π is defined as the following

$$C_{pw} = \frac{USL - LSL}{6\sqrt{\sigma^2 + w(\theta - T)^2}},$$

where w ($0 \le w \le 1$) is a weight.

According to process capability index introduced as above, we define the best $\sigma - qualified$ manufacturing process as follows. The problem of identification of the best among several normal populations under multiple criteria has been first studied by Huang and Lai (1999).

Definition 2.2 Let $\pi_1, \cdots, \pi_k$ be k manufacturing processes such that π_i has mean θ_i variance σ_i^2 and process capability index $C_{pw}(i)$, $i = 1, \cdots, k$. Let $C_{pw}(0)$ and σ_0^2 be two control values (prefixed). Define S= $\{\pi_i | \sigma_i \le \sigma_0, i = 1, \cdots, k\}$. A manufacturing process π_i is called $\sigma - qualified$, if $\pi_i \in$ S. Let $S' = S \cup \{\pi_0\}$, where π_0 is considered as a control manufacturing process with process capability index $C_{pw}(0)$ and variance σ_0^2. A manufacturing process $\pi_i (i \ne 0)$ is considered as the best $\sigma - qualified$, if it simultaneously satisfies the following conditions:

(i) $\pi_i \in$ S, and

(ii) $C_{pw}(i) = \max_{\pi_j \in S} C_{pw}(j)$.

Let $\underset{\sim}{\theta}=(\theta_1,\cdots,\theta_k)$, $\underset{\sim}{\sigma}=(\sigma_1,\cdots,\sigma_k)$ and $\Omega=\{(\theta_i,\sigma_i)| -\infty<\theta_i<+\infty, \sigma_i>0, i=1,\cdots,k\}$ be the parameter space. Let $\underset{\sim}{a}=(a_0,a_1,\cdots,a_k)$ denote an action, where $a_i=0$ or 1; $i=0,1,\cdots,k$, and $\sum_{i=0}^{k} a_i=1$. If $a_i=1$, for some $i=1,\cdots,k$, it means that manufacturing process π_i is selected as the best $\sigma-qualified$. When $a_0=1$, it means that no manufacturing process is considered as the best $\sigma-qualified$, i.e. none in k manufacturing processes satisfied the restriction (i) in **Definition 2.1**. Let $A=\{\underset{\sim}{a}\}$ denote the action space.

For the sake of convenience, corresponding to $C_p(i), \forall i=0,1,\cdots,k$, we define a new quantity $C'_p(i)$ as follows.

Definition 2.3 For a given positive $C^*_{pw}<C_{pw}(0)$ and for $i=0,1,\cdots,k$, define

$$C'_{pw}(i)=C_{pw}(i)I_{\{\sigma_i\le\sigma_0\}}+C^*_{pw}I_{\{\sigma_i>\sigma_0\}}.$$

Accordingly, those manufacturing processes which do not meet the requirement (i) will also fail to meet the requirement (ii) in **Definition 2.2** in terms of the transformed quantity $C'_{pw}(i)$.

In a decision-theoretic approach, we introduce the following loss function.

Definition 2.4 For a control value $C_{pw}(0)$, and parameter vectors $\underset{\sim}{\theta}, \underset{\sim}{\sigma}$, if action $\underset{\sim}{a}$ is taken, a loss $L(\underset{\sim}{\theta},\underset{\sim}{\sigma};\underset{\sim}{a})$ is incurred and which is defined by

$$L(\underset{\sim}{\theta},\underset{\sim}{\sigma};\underset{\sim}{a})=\sum_{i=0}^{k}a_iC'^{-2}_{pw}(i)-C'^{-2}_{pw[k]}, \tag{2.1}$$

where $C'_{pw[k]}=\underset{0\le i\le k}{Max}\,C'_{pw}(i)$.

It is easy to recognize that the loss $L(\underset{\sim}{\theta},\underset{\sim}{\sigma};\underset{\sim}{a})$ defined in (2.1) has reflected the proper penalty for a wrong action. In this paper, we consider a Bayes approach for the problem of selecting the best $\sigma-qualified$ manufacturing process which is normally distributed.

For each $i=1,\cdots,k$, let $X_{i1},\cdots,X_{iM}$ be an independent random sample of size

M from a normally distributed manufacturing process π_i with mean θ_i and variance σ_i^2. The observed value is denoted by $x_{i1},\cdots,x_{iM}$. Let $\tau_i = 1/\sigma_i^2$, $i=1,\cdots,k$. It is assumed that (θ_i,τ_i) is a realization of a random vector (Θ_i,Γ_i) with a normal-gamma prior distribution.

Let $\underset{\sim}{x}=(\underset{\sim}{x}_1,\cdots,\underset{\sim}{x}_k)$ and χ be the sample space generated by $\underset{\sim}{x}$. A selection rule $\underset{\sim}{d}=(d_0,d_1,\cdots,d_k)$ is a mapping defined on the sample space χ into the k+1 product space $[0,1]\times[0,1]\times\cdots\times[0,1]$ such that $\sum_{i=0}^{k} d_i(\underset{\sim}{x})=1$, for all $\underset{\sim}{x}\in\chi$. For every $\underset{\sim}{x}\in\chi$, $d_i(\underset{\sim}{x})$ denotes the probability of selecting manufacturing process π_i as the best $\sigma-qualified$, $i=1,\cdots,k$; and $d_0(\underset{\sim}{x})$ denotes the probability that none is selected as the best $\sigma-qualified$.

For ease of notation, let $\underset{\sim}{\tau}=(\tau_1,\cdots,\tau_k)$, $\underset{\sim}{\mu}=(\mu_1,\cdots,\mu_k)$, $\underset{\sim}{\alpha}=(\alpha_1,\cdots,\alpha_k)$, $\underset{\sim}{\beta}=(\beta_1,\cdots,\beta_k)$, $\underset{\sim}{\Theta}=(\Theta_1,\cdots,\Theta_k)$ and $\underset{\sim}{\Gamma}=(\Gamma_1,\cdots,\Gamma_k)$. Let $h(\underset{\sim}{\theta}\,|\,\underset{\sim}{x},\underset{\sim}{\tau};\underset{\sim}{\mu},\underset{\sim}{\alpha})$ be the joint conditional posterior probability density function of $\underset{\sim}{\Theta}$ given $\underset{\sim}{x}$ and $\underset{\sim}{\tau}$, and $g(\underset{\sim}{\tau}\,|\,\underset{\sim}{x};\underset{\sim}{\alpha},\underset{\sim}{\beta})$ be the joint conditional posterior probability density function of $\underset{\sim}{\Gamma}$ given $\underset{\sim}{x}$. Let $h_i(\theta_i\,|\,\underset{\sim}{x}_i,\tau_i;\mu_i,\alpha_i)$ and $g_i(\tau_i\,|\,\underset{\sim}{x}_i;\alpha_i,\beta_i)$ be the conditional posterior probability density function of Θ_i and Γ_i, respectively. Let $f(\underset{\sim}{x})$ be the marginal probability density function of $\underset{\sim}{x}$. Under the previous formulation, the Bayes risk of a selection rule $\underset{\sim}{d}$, denoted by $r(\underset{\sim}{d})$, is given by

$$\begin{aligned}
r(\underset{\sim}{d}) &= \mathrm{E}_{\underset{\sim}{\tau}}\mathrm{E}_{\underset{\sim}{\theta}}\mathrm{E}_{\underset{\sim}{x}}L(\underset{\sim}{\theta},\underset{\sim}{\tau};\underset{\sim}{d}) \\
&= \iint_{\Omega}\int_{\chi}\sum_{i=0}^{k} d_i(\underset{\sim}{x})C_p'^{-2}(i)f(\underset{\sim}{x}\,|\,\underset{\sim}{\theta},\underset{\sim}{\tau})h(\underset{\sim}{\theta}\,|\,\underset{\sim}{\mu},\underset{\sim}{\tau})\,g(\underset{\sim}{\tau};\underset{\sim}{\alpha},\underset{\sim}{\beta})d\,\underset{\sim}{x}\,d\,\underset{\sim}{\theta}\,d\,\underset{\sim}{\tau} \\
&\quad -\iint_{\Omega}\int_{\chi} C_{p[k]}'^{-2}f(\underset{\sim}{x}\,|\,\underset{\sim}{\theta},\underset{\sim}{\tau})h(\underset{\sim}{\theta}\,|\,\underset{\sim}{\mu},\underset{\sim}{\tau})\,g(\underset{\sim}{\tau};\underset{\sim}{\alpha},\underset{\sim}{\beta})d\,\underset{\sim}{x}\,d\,\underset{\sim}{\theta}\,d\,\underset{\sim}{\tau} \\
&\equiv I_1 - I_2, \text{ say.}
\end{aligned}$$

$$r(\underset{\sim}{d}) = \int_{\chi}\sum_{i=0}^{k} d_i(\underset{\sim}{x})\phi_i(\underset{\sim}{x}_i)\,f(\underset{\sim}{x})d\,\underset{\sim}{x}-C.$$

where

$$\phi_i(\underset{\sim}{x_i}) = \frac{36}{(USL - LSL)^2}\{[1 + w(2\alpha_i + M - 1)^{-1}][(\alpha_i' - 1)\eta_i]^{-1}$$

$$+ w(\varphi_i(x_i) - T)^2\}, \qquad (2.2)$$

For convenience of notation, we define $\phi_0(\underset{\sim}{x_0}) = C_{pw}^{-2}(0)$.

Hence, for some constant C,

$$r(\underset{\sim}{d}) = \int_{\chi} \sum_{i=0}^{k} d_i(\underset{\sim}{x})\phi_i(\underset{\sim}{x_i}) f(\underset{\sim}{x}) d\underset{\sim}{x} - C. \qquad (2.3)$$

For each $\underset{\sim}{x} \in \chi$, let

$$Q(\underset{\sim}{x}) = \{i \,|\, \phi_i(\underset{\sim}{x_i}) = \underset{0 \le j \le k}{Min}\, \phi_j(\underset{\sim}{x_j}),\ i = 0,1,\cdots,k\}. \qquad (2.4)$$

Then, define

$$i^* = i^*(\underset{\sim}{x}) = \begin{cases} 0 & if\ \ Q(\underset{\sim}{x}) = \{0\}, \\ Min\{i \,|\, i \in Q(\underset{\sim}{x}),\ i \ne 0\} & otherwise. \end{cases} \qquad (2.5)$$

Then, according to (2.3), (2.4) and (2.5), it can be derived that a Bayes selection rule $\underset{\sim}{d}^B = (d_0^B, d_1^B, \cdots, d_k^B)$ is given as follows

$$\begin{cases} d_{i^*}^B(\underset{\sim}{x}) = 1, \\ d_j^B(\underset{\sim}{x}) = 0, \quad for\ \ j \ne i^*. \end{cases} \qquad (2.6)$$

3. The empirical Bayes selection rule

In the problem formulated in section 2, we consider that $\alpha_1, \cdots, \alpha_k$ are all known with $\alpha_i > 1$. Since $\phi_i(\underset{\sim}{x_i})$ still involves the unknown parameters μ_i, β_i, $i = 1, \cdots, k$, hence, the proposed Bayes selection rule $\underset{\sim}{d}^B$ is not applicable.

For each π_i, $i = 1, \cdots, k$, we estimate the unknown parameters μ_i and β_i based on the past data X_{ijt}, $j = 1, \cdots, M$, $t = 1, \cdots, n$. We denote

$$\begin{cases} X_{i.t} = \frac{1}{M}\sum_{j=1}^{M} X_{ijt}, \quad X_i(n) = \frac{1}{n}\sum_{t=1}^{n} X_{i.t}, \\ W_{i.t}^2 = \frac{1}{M-1}\sum_{j=1}^{M}(X_{ijt} - X_{i.t})^2, \quad W_i^2(n) = \frac{1}{n}\sum_{t=1}^{n} W_{i.t}^2. \end{cases} \tag{3.1}$$

For ease of notation, we define μ_{in} and β_{in} as estimators of μ_i and β_i, respectively, by the following

$$\begin{cases} \mu_{in} = X_i(n), \\ \beta_{in} = (\alpha_i - 1)W_i^2(n). \end{cases} \tag{3.2}$$

Also, for $i = 1, \cdots, k$, we define

$$\phi_{in}(\underset{\sim}{x_i}) = \frac{36}{(USL - LSL)^2}\Big\{\frac{(w + (2\alpha_i + M - 1))(1 - G_{in}(\tau_0 \,|\, \underset{\sim}{x_i}, \alpha_i' - 1, \eta_{in}))}{(2\alpha_i + M - 1)(\alpha_i' - 1)\eta_{in}}$$

$$+ w(\varphi_{in}(x_i) - T)^2(1 - G_{in}(\tau_0 \,|\, \underset{\sim}{x_i}, \alpha_i', \eta_{in}))\Big\} + C_{pw}^{*-2} G_{in}(\tau_0 \,|\, \underset{\sim}{x_i}, \alpha_i', \eta_{in}). \tag{3.3}$$

where

$$\eta_{in} = \beta_{in} + \frac{(M-1)S_i^2}{2} + \frac{(2\alpha_i - 1)M(x_i - \mu_{in})^2}{2(2\alpha_i + M - 1)}, \tag{3.4}$$

and

$$\varphi_{in}(x_i) = \frac{(2\alpha_i - 1)\mu_{in} + Mx_i}{2\alpha_i + M - 1} \tag{3.5}$$

For convenience of notation, we define $\phi_{0n}(\underset{\sim}{x_0}) = C_{pw}^{-2}(0)$. We consider $\phi_{in}(\underset{\sim}{x_i})$ to be an estimator of $\phi_i(\underset{\sim}{x_i})$. The properties of the estimators proposed above will be discussed in the following section.

For each $\underset{\sim}{x} \in \chi$, let

$$Q_n(\underset{\sim}{x}) = \{ i \,|\, \phi_{in}(\underset{\sim}{x_i}) = \underset{0 \le j \le k}{Min}\, \phi_{jn}(\underset{\sim}{x_j}), i = 0, 1, \cdots, k \}. \tag{3.6}$$

Again, define

$$i_n^* = i_n^*(\underset{\sim}{x}) = \begin{cases} 0 & if\ Q_n(\underset{\sim}{x}) = \{0\}, \\ Min\{i \,|\, i \in Q_n(\underset{\sim}{x}), i \ne 0\} & otherwise. \end{cases} \tag{3.7}$$

Then, according to (3.3), (3.6) and (3.7), we have a empirical Bayes selection rule $d^{*n} = (d_0^{*n}, d_1^{*n}, \cdots, d_k^{*n})$ as follows

$$\begin{cases} d^{*n}_{i^*_n}(\underset{\sim}{x}) = 1, \\ d^{*n}_{j}(\underset{\sim}{x}) = 0, \quad for\ j \neq i^*_n. \end{cases} \tag{3.8}$$

Definition 3.1 A sequence of empirical Bayes selection rule $\{\underset{\sim}{d}^n\}_{n=1}^{\infty}$ is said to be asymptotically optimal, if $\lim_{n\to\infty}\left\{ \mathrm{E}_n[r(\underset{\sim}{d}^n)] - r(\underset{\sim}{d}^B)\right\} = 0$.

Theorem 3.1 The empirical Bayes selection rule $\underset{\sim}{d}^{*n}(\underset{\sim}{x})$, defined by (3.5), (3.6) and (3.7), is asymptotically optimal.

The proof is omitted.

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Table 1. Behavior of empirical Bayes rules with respect to various sample sizes (*w*=1)

n	f_n	$\overline{D}_n$	$n\overline{D}_n$	$SE(\overline{D}_n)$
10	0.8556	1.7125E-02	1.7125E-01	3.3902E-03
20	0.8917	9.4061E-03	1.8812E-01	1.3359E-03
30	0.9110	5.9942E-03	1.7983E-01	6.4524E-04
40	0.9231	4.8998E-03	1.9599E-01	5.0000E-04
50	0.9339	3.6897E-03	1.8449E-01	3.4546E-04
60	0.9431	2.8443E-03	1.7066E-01	2.3656E-04
70	0.9435	2.5919E-03	1.8144E-01	2.0210E-04
80	0.9461	2.3654E-03	1.8923E-01	1.6538E-04
90	0.9522	1.9827E-03	1.7845E-01	1.4753E-04
100	0.9491	1.9496E-03	1.9496E-01	1.2354E-04
200	0.9674	9.4790E-04	1.8958E-01	4.5927E-05
300	0.9753	5.6293E-04	1.6888E-01	2.1498E-05
400	0.9775	4.4768E-04	1.7907E-01	1.4806E-05
500	0.9808	3.0333E-04	1.5166E-01	8.2669E-06
600	0.9788	3.4662E-04	2.0797E-01	9.3995E-06
700	0.9802	2.7777E-04	1.9444E-01	6.6305E-06
800	0.9830	2.0734E-04	1.6587E-01	4.0981E-06
900	0.9818	2.2765E-04	2.0488E-01	4.8548E-06
1000	0.9849	1.6283E-04	1.6283E-01	3.3842E-06

Table 2 The frequency of the process selected as the best
under various weights for Group 2 (*n*=100)

Weight	CD	Process					
		0	1	2	3	4	5
0.0	9914	0 (0) [0]	0 (0) [0]	0 (0) [0]	1609 (1695) [1609]	38 (37) [37]	8353 (8268) [8268]
0.1	9454	0 (0) [0]	8 (8) [7]	2714 (2772) [2471]	7268 (7208) [6966]	10 (12) [10]	0 (0) [0]
0.2	9437	0 (0) [0]	11 (13) [8]	2641 (2695) [2390]	7334 (7275) [7027]	14 (17) [12]	0 (0) [0]
0.3	9414	0 (0) [0]	8 (9) [8]	2687 (2746) [2424]	7294 (7232) [6972]	11 (13) [10]	0 (0) [0]
0.4	9442	23 (28) [19]	9 (8) [6]	2708 (2789) [2475]	7255 (7170) [6939]	5 (5) [3]	0 (0) [0]
0.5	9250	1204 (1233) [1014]	0 (0) [0]	1662 (1699) [1425]	7133 (7067) [6810]	1 (1) [1]	0 (0) [0]
0.6	9364	3324 (3362) [3049]	0 (0) [0]	219 (235) [182]	6457 (6403) [6133]	0 (0) [0]	0 (0) [0]
0.7	9286	4394 (4362) [4023]	0 (0) [0]	32 (34) [26]	5574 (5603) [5237]	0 (1) [0]	0 (0) [0]
0.8	9283	5056 (5095) [4718]	0 (0) [0]	12 (12) [11]	4932 (4893) [4554]	0 (0) [0]	0 (0) [0]
0.9	9323	5615 (5606) [5272]	0 (0) [0]	2 (2) [2]	4383 (4392) [4049]	0 (0) [0]	0 (0) [0]
1.0	9326	6073 (6053) [5726]	0 (0) [0]	2 (2) [2]	3925 (3945) [3598]	0 (0) [0]	0 (0) [0]