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The performance of bearing estimation using spatial domain forward–backward predictor with higher-order statistics

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The modified forward–backward linear predictor (MFBLP) methods with higher-order statistics, viz., the second-order statistics and the fourth-order cumulants, together with the broadband array structure are developed in this paper for bearing estimation. Here, the desired source signals of interest are narrow band and the additive Gaussian noise sources in the related sensors are assumed to be spatially correlated or uncorrelated with each other and white/colored processes in the temporal domain. In this paper, the MFBLP method with second-order statistics for bearing estimation will be emphasized. Moreover, to extract the principal eigenvalues of presented MFBLP methods a new search algorithm is also proposed. An analytical study of the MFBLP method with second-order statistics is first developed, under the assumption that the additive noise is white Gaussian process. From the observation of analytic results, a modification of the MFBLP method with second-order statistics is suggested. From simulation results, it is shown that the MFBLP methods with higher-order statistics is superior to the conventional MFBLP method with the linear array data directly, in terms of threshold signal-to-noise ratio (SNR). This is especially true when data length becomes relatively larger. Moreover, the performance improvement of using the new search algorithm together with all the MFBLP methods is discussed thoroughly.

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INTRODUCTION

Array processing deals with the methods for processing the data of an array output to achieve a specific goal. In general, two popular linear array structures, viz., the narrow-band and broadband structures, are considered in practical applications. The conventional approach of bearing estimation for narrow-band source signals is to use the narrow-band array structure. The implementation of the broadband array structure for bearing estimation requires extra computation time, but it may have better performance than the one using the narrow-band array structure, even if the desired sources are narrow-band signals. In the broadband array structure, each sensor is realized by a tapped-delay line.[1]

To improve the performance of bearing estimation, recently, the higher-order cumulants methods[2–5] together with the narrow-band array structure were suggested for suppressing the additive Gaussian noise. The additive Gaussian noise sources were assumed to be spatially correlated but are white in temporal domain. To circumvent the problem due to the spatially correlated Gaussian noise, Chiang and Nikias[3] developed a fourth-order ESPRIT algorithm based on the generalized eigenstructure analysis. Porat and Friedlander[4] proposed a MUSIClike algorithm which is based on the eigendecomposition of suitably defined matrix of the fourth-order cumulants. Moreover, by using the asymptotic normality of cross-bispectrum estimate along with the maximum likelihood theory, Forster and Nikias[5] developed an algorithm for bearing estimation.

It is well known that MUSIC,[6] ESPRIT,[7] and MFBLP[8] methods are frequently employed on determining the incident angle (or bearing) of plant wave. It has been shown that, in general, the ESPRIT method has computational advantage,[9] but the variance of bearing estimation error of ESPRIT is larger than the one using the MUSIC[10] method. Moreover, in,[11] Kesler and Shahmirian had shown that the MFBLP method may have better capability in resolving two full correlated source signals than the one using the MUSIC method, in terms of bearing estimation.

In this paper, the MFBLP methods with higher-order statistics, viz., the second-order statistics and the fourth-order cumulants, are developed for bearing estimation using the broadband array structure,[12,13] where the desired source signals are narrow band. The additive Gaussian noise sources in the related sensors are assumed to be spatially correlated and to be colored in temporal domain for each individual sensor. Moreover, the signal-to-noise ratio (SNR) of interest is assumed to be relatively small. The assumptions just described are of interest in practical applications.

It is well known that the MFBLP method is an eigendecomposition approach, the dominant principal eigenvector/eigenvalue pairs of the covariance matrix of the linear array data are used for bearing estimation and are assumed to be known in advance. In fact, the dominant principal eigenvector/eigenvalue pairs are related to the number of desired source signals thus are not available in practical applications. The number of source signals may be estimated by using the existing criterion, e.g., finding the minimum of the MDL function.[14] In this paper, a simple new search algorithm for extracting the principal eigenvalues is suggested and the advantage is also discussed.

In the conventional array signal processing, the second-order statistics of the received signal are used frequently. For bearing estimation, they simply employed the signal- and noise-subspace approaches along with the eigendecomposition of the spatial covariance matrix of linear array data directly. However, in this paper, the approach of the MFBLP method with second-order statistics, is referred to be as the MFBLP-SOS method, is different from the one just described. Here, the autocorrelation functions of linear array data are viewed as an input of the spatial domain predictors. In consequence, the MFBLP-SOS method for bearing estimation can be formulated in the spatial domain to improve the signal-to-noise ratio (SNR).

To investigate the performance of the presented methods, an analytical study based on the second-order statistics with additive white Gaussian noise is first developed. From the analytical result, a modified MFBLP method with second-order statistics will be proposed. Finally, the MFBLP method with fourth-order cumulants is developed. Similarly, we will refer the MFBLP-FOC method as the MFBLP method with fourth-order cumulants. To have fair comparison, the bearing estimation results of the presented methods are compared with the conventional MFBLP method using the linear array data directly and the Cramér–Rao lower bound (CRLB). Moreover, the performance improvement of all methods via the new search algorithm for bearing estimation is also emphasized.

I. THE USE OF SECOND-ORDER STATISTICS FOR BEARING ESTIMATION

In this section, for bearing estimation the MFBLP method based on the second-order statistics of linear array data is first developed. Before that, the problem of bearing estimation is briefly discussed.

A. Statement of bearing estimation

Consider an uniform linear array with M identical omnidirectional sensors spaced apart by a distance d. Each sensor is realized by a tapped-delay line with tap weights, and be summed together to estimate the signal received by the reference sensor, e.g., m_0 $(m=0)$ or m_{M-1} $(m=M-1)$. The received signal $x_m(k)$ of mth sensor at kth time instant with noise can be expressed by

$$x_m(k)=s_m(k)+n_m(k),\quad \text{for}\ \ m=0,1,\ldots,M-1,\quad k=1,2,\ldots,N, \tag{1}$$

where the desired signal component received by the mth sensor, $s_m(k)$, may consist of multiple narrow-band source signals. The schematic diagram of spatial domain forward predictor is depicted in Fig. 1, where the forward prediction error signal $e_f(k)$, at the kth time instant, is given by

$$e_f(k)=x_0(k)-\sum_{m=1}^{M-1}\sum_{l=0}^{L-1} a_{m,l}x_m(k-l)\quad \text{for}\ \ k=L,L+1,\ldots,N. \tag{2}$$

In (2), $x_0(k)$ is the signal received by the reference sensor, the m_0 sensor, and N is the total length of received signal. Also, $a_{m,l}$ denote the tap weights of the mth sensor, $m=1,2,\ldots,M-1$. Similarly, the backward predictor error signal in spatial domain can be represented by

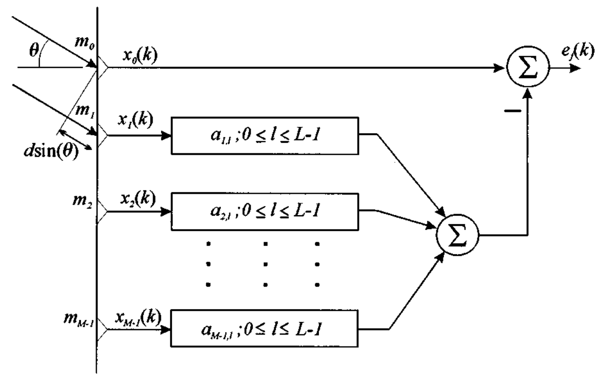

FIG. 1. A block diagram of spatial domain forward predictor.

$$e_b(k)=x_{M-1}(k-L+1)-\sum_{m=0}^{M-2}\sum_{l=0}^{L-1} b_{m,l}x_m(k-l)\quad \text{for}\ \ k=L,L+1,\ldots,N, \tag{3}$$

where $x_{M-1}(k)$ is the signal received by the reference sensor, the m_{M-1} sensor, and $b_{m,l}$ are the lth tap weight in the delay line of the mth sensor.

For convenience, the weight vectors, $\mathbf{a}$ and $\mathbf{b}$, are designated by $\mathbf{a}=[a_{1,0}\ \ a_{1,1}\ldots a_{1,L-1}\ \ a_{2,0}\ldots a_{M-1,L-1}]^T$ and $\mathbf{b}=[b_{0,0}\ b_{0,1}\ldots b_{0,L-1}\ b_{1,0}\ldots b_{M-2,L-1}]^T$, respected. The optimum weight vectors $\mathbf{a}^o$ and $\mathbf{b}^o$ can be obtained by simply minimizing $E[(e_f(k))^2]$ and $E[(e_b(k))^2]$,[13] respectively. This follows the derivation as in Ref. 15 which may also show that $\mathbf{a}^o=\mathbf{b}^{oB*}$, where superscript "^{B}" denotes reversing the sequence. This means that we may modify a backward predictor into a forward predictor by reversing the sequence in which its tap weights are positioned and also complex conjugating them. Thus we may define the optimum weight vector as

$$\mathbf{g}=\mathbf{a}^o=\mathbf{b}^{oB*}, \tag{4}$$

or the optimum weights,

$$g_{m,l}=a^o_{m,l}=b^{o*}_{M-m-1,L-l-1}\,,\quad \text{for}\ \ m=1,2,\ldots,M-1,\quad l=0,1,\ldots,L-1. \tag{5}$$

For bearing estimation, we simply apply the weight vector to the following formula:[16]

$$G(f_c,\theta)=\frac{1}{|1-H(f_c,\theta)|^2}, \tag{6}$$

where

$$H(f_c,\theta)=\sum_{m=1}^{M-1}\sum_{l=0}^{L-1} g_{m,l}e^{-j(2\pi l f_c+m\pi\sin\theta)}. \tag{7}$$

Based on (6), we can determine the incident angles as $\theta=\theta_k$, for $k=1,2,\ldots,K$, at which the spatial spectrum, $G(f_c,\theta)$, has sharp peaks.

B. The MFBLP method with second-order statistics

To derive the MFBLP method based on the second-order statistics of linear array data, first, we multiply $x_0^*(k-\tau)$ on both sides of (2) and taking the expectation, after some manipulation, we have

$$E[x_0(k)x_0^*(k-\tau)]=E\left[\sum_{m=1}^{M-1}\sum_{l=0}^{L-1}a_{m,l}x_m(k-l)x_0^*(k-\tau)\right]+E[e_f(k)x_0^*(k-\tau)],\quad \text{for}\ \ k=L,L+1,\ldots,N. \tag{8}$$

By the principle of orthogonality, the estimation error $e_f(k)$ will be orthogonal to the input samples at time k, if $a_{m,l}$ reach the optimal weights, $a_{m,l}^o$, i.e.,

$$E[e_f^o(k)x_0^*(k-\tau)]=0,\quad \text{for}\ \ k=L,L+1,\ldots,N, \tag{9}$$

where $e_f^o(k)$ denotes the estimation error with the spatial filter operating in its optimum condition. For convenience, let us define $R_{x_ix_j}(\tau)=E[x_i(k)x_j^*(k-\tau)]$ to be the correlation function between $x_i(k)$ and $x_j(k)$, such that (8) with the optimal weights may be represented by

$$R_{x_0x_0}(\tau)=\sum_{m=1}^{M-1}\sum_{l=0}^{L-1}g_{m,l}R_{x_mx_0}(\tau-l)\quad \text{for}\ \ \tau=L-N,L-N+1,\ldots,N-1. \tag{10}$$

Similarly, if we multiply $x_{M-1}^*(k+\tau)$ on (3) and then take expectations and complex conjugating and using the fact that $E[e_b^o(k)x_{M-1}^*(k+\tau)]=0$, we have

$$R_{x_{M-1}x_{M-1}}^*(1-\tau-L)=\sum_{m=1}^{M-1}\sum_{l=0}^{L-1}g_{m,l}R_{x_{M-m-1}x_{M-1}}^*\times(1-\tau-L+l)\quad \text{for}\ \ \tau=-N+1,-N+2,\ldots,N-L, \tag{11}$$

for the backward predictor. It is noticed that $e_b^o(k)$ denotes the estimation error with the filter operating in its optimum condition. Based on Eqs. (10) and (11), the MFBLP-SOS method can be developed.

The configuration of the spatial domain forward predictor with second-order statistics of received signal described in (10) is depicted in Fig. 2. In Fig. 2, we can see that the auto- and cross-correlation functions of the received signals from sensors are first estimated. Then, we feed these correlation functions into spatial domain tapped-delay line structure and solve the weight vector. If we compare Fig. 1 with Fig. 2, we can find that in the MFBLP-SOS method, $R_{x_ix_j}(\tau)$ are used to instead of $x_m(k)$ in the conventional MFBLP method using the linear array data directly. The configuration of the spatial domain backward predictor with second-order statistics described in (11) is similar to the one of the forward predictor. That is, in Fig. 2, the reference sensor m_0 $(n=0)$ is replaced by m_{M-1} $(m=M-1)$, in consequence, the $\hat{R}_{x_{M-1}x_{M-1}}(\tau)$ is used to instead of $\hat{R}_{x_0x_0}(\tau)$.

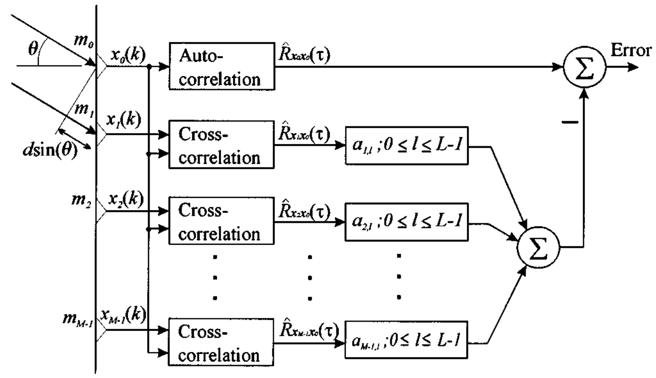

FIG. 2. A block diagram of spatial domain forward predictor with second-order statistics.

Similarly, the $\hat{R}_{x_mx_0}(\tau-l)$ are replaced by $\hat{R}_{x_{M-m-1}x_{M-1}}^*(1-\tau-L+l)$, for $m=1,2,\ldots,M-1$, and so on.

To see how the MFBLP-SOS method can improve the SNR over the conventional approach, the following discussion will be useful. Theoretically, for white Gaussian noise, the correlation functions will be

$$R_{n_in_j}(\tau)=\begin{cases}\sigma_n^2, & \tau=0\ \ \text{and}\ \ i=j,\\ 0, & \text{otherwise.}\end{cases} \tag{12}$$

Thus only when $\tau=0$ and $i=j$, $R_{x_ix_j}(\tau)$ will be affected by the additive noise. Therefore, the use of the MFBLP method with second-order statistics can expect to have better performance over the conventional MFBLP method with linear array data directly.

In the following, we would like to formulate the MFBLP-SOS method for bearing estimation. In the MFBLP methods with higher-order statistics, the accuracy of estimating different order statistics of signal is significant. In practice, when the lag, τ, of correlation function becomes relatively large, the estimation of correlation function, $\hat{R}_{x_ix_j}(\tau)$, due to insufficient data, will be inaccurate. Moreover, for data length being N and having L tap weights in each sensor, the total number of equations in (10) and (11) will be $2N-L$. However, due to insufficient data available, as far as the accuracy of the estimated correlation function is concerned, only Q equations located at the medial of the $2N-L$ are selected for further processing to have desired result. To satisfy an overdetermined system, i.e., the number of equations should be larger than unknowns, the range of $2Q$, total number of equations, will be

$$(M-1)L\leqslant 2Q\leqslant 2(2N-L), \tag{13}$$

where $(M-1)L$ is the total number of tap weights in the linear array. In practice, $2Q$ is chosen to be $1.6(M-1)L$ and, for convenience, we simply let $L+Q$ be an even number, e.g., $S=(L+Q)/2$.

To develop the MFBLP-SOS method for bearing estimation, we may rewrite both Eqs. (10) and (11) into a matrix form

$$\mathbf{Ag}=\mathbf{h} \tag{14}$$

with

$$\mathbf{A}=\begin{bmatrix} \mathbf{F}_1 & \mathbf{F}_2 & \cdots & \mathbf{F}_{M-1} \\ \mathbf{B}_{M-2} & \mathbf{B}_{M-3} & \cdots & \mathbf{B}_0 \end{bmatrix}_{2Q\times(M-1)L}, \tag{15}$$

$$\mathbf{g}=[\mathbf{g}_1^T \quad \mathbf{g}_2^T \quad \cdots \quad \mathbf{g}_{M-1}^T]^T \tag{16}$$

and

$$\mathbf{h}=\begin{bmatrix} \mathbf{h}_0 \\ \mathbf{h}_{M-1} \end{bmatrix}_{2Q\times 1}, \tag{17}$$

where the parameters in (15) and (17) are defined in the following:

$$\mathbf{F}_m=\begin{bmatrix} R_{x_m x_0}(L-S) & R_{x_m x_0}(L-1-S) & \cdots & R_{x_m x_0}(1-S) \\ R_{x_m x_0}(L+1-S) & R_{x_m x_0}(L-S) & \cdots & R_{x_m x_0}(2-S) \\ \vdots & \vdots & \ddots & \vdots \\ R_{x_m x_0}(S-1) & R_{x_m x_0}(S-2) & \cdots & R_{x_m x_0}(S-L) \end{bmatrix}_{Q\times L}, \tag{18}$$

$$\mathbf{B}_m=\begin{bmatrix} R^*_{x_m x_{M-1}}(S-L) & R^*_{x_m x_{M-1}}(S-L+1) & \cdots & R^*_{x_m x_{M-1}}(S-1) \\ R^*_{x_m x_{M-1}}(S-L\text{-}1) & R^*_{x_m x_{M-1}}(S-L) & \cdots & R^*_{x_m x_{M-1}}(S-2) \\ \vdots & \vdots & \ddots & \vdots \\ R^*_{x_m x_{M-1}}(1-S) & R^*_{x_m x_{M-1}}(2-S) & \cdots & R^*_{x_m x_{M-1}}(L-S) \end{bmatrix}_{Q\times L}, \tag{19}$$

$$\mathbf{g}_m=[g_{m,0} \quad g_{m,1} \quad \cdots \quad g_{m,L-1}]^T, \tag{20}$$

and

$$\mathbf{h}_m=[R_{x_m x_m}(L-S) \quad R_{x_m x_m}(L+1-S) \quad \cdots \quad R_{x_m x_m}(S-1)]^T. \tag{21}$$

To solve the equation $\mathbf{Ag}=\mathbf{h}$ via the so-called principal component approach, the number of source signals is assumed to be known in advance. However, this is not the case in the practical application. In the next section, we will suggest a new search algorithm for estimating and determining the number of source signals.

C. Search algorithm for estimating the number of source signals

For extracting the principal eigenvalues here, a new search procedure is proposed. The solution of **g** in (14) using the principal component approach can be expressed as

$$\hat{\mathbf{g}}=\sum_{i=1}^{K}\left(\frac{1}{\lambda_i}\right)\mathbf{u}_i\mathbf{u}_i^H\mathbf{A}^H\mathbf{h}, \tag{22}$$

where the eigenvalues, λ_i, of $\mathbf{A}^H\mathbf{A}$ are arranged in the decreasing order and $\mathbf{u}_i$ are the corresponding eigenvectors. In fact, (22) can be viewed as a special case of solving (14) by the singular value decomposition directly when no noise is present, in which

$$\lambda_i=0, \quad \text{for } i=K+1,K+2,\ldots,(M-1)L. \tag{23}$$

It is noted that, here, λ_i, for $i=K+1,\ldots,(M-1)L$, corresponding to the mean-squared values of the difference between $R_{x_ix_j}$ and $R_{s_is_j}$, and, for convenience, we refer to it as the averaged error power and it is denoted as σ_A^2. However, if the noisy case is considered, ideally, the eigenvalues shown in (23) will equal to σ_A^2, instead of null. Thus theoretically, the following relationship will hold:

$$[(M-1)L-K]\sigma_A^2=\frac{1}{2Q}\sum_{i=K+1}^{(M-1)L}\lambda_i. \tag{24}$$

In practice, the averaged error power σ_A^2 is not available, but it can be estimated via adaptive filter shown in Fig. 3. Thus if we feed the correlation functions, $\hat{R}_{x_ix_j}(\tau)$, into the adaptive filter, we may obtain the estimate of the averaged error power from the filtered output, i.e., $\hat{\sigma}^2_{A_{i,j}} = E[\hat{\epsilon}_{ij}(\tau)]$. From (15), we see that the matrix **A** contains different cross-correlation functions described above. Such that, the estimate of the averaged error power can be represented by

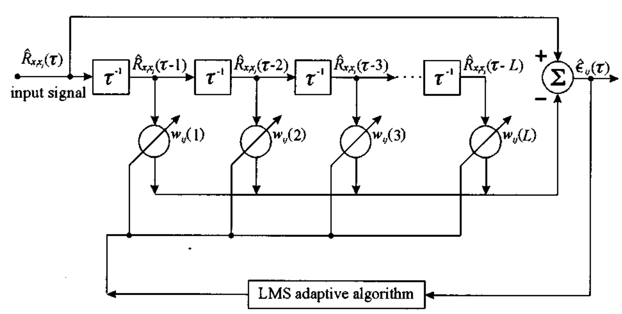

FIG. 3. The block diagram of adaptive filter for estimating the average error power $\hat{\sigma}_A^2$.

$$\hat{\sigma}_A^2 = \frac{1}{2(M-1)} \left(\sum_{m=1}^{M-1} \hat{\sigma}_{A_{m,0}}^2 + \sum_{m=0}^{M-2} \hat{\sigma}_{A_{m,M-1}}^2 \right). \tag{25}$$

After obtaining $\hat{\sigma}_A^2$, the new search algorithm for estimating the number of source signals can be developed. To see how it works, we assume that two principal eigenvalues are to be estimated. As can be seen from (24), for $K=0$, λ_1 and λ_2 [on the right side of (24)] should correspond to the eigenvalues of desired signal components plus extra error power terms. On the other hand, the left-hand side of (24) will introduce two more σ_A^2. However, in general, λ_1 and λ_2 are greater than σ_A^2, such that the equality of (24) may not be hold and the value on the left-hand side of (24) will be less than the value on the right side. For $K=1$, the same situation will have occurred. Now, for $K=2$ (since we assumed that we have two principal eigenvalues), ideally, both sides of (24) will have the same value. As discussed earlier, since σ_A^2 is not available, we have to use the estimated value, $\hat{\sigma}_A^2$ instead. Thus in this situation ($K=2$), the left side of (24) may be greater or less than the right side (and is undetermined). However, for $K>2$, in this case, the number of source signals will be overestimated. We recalled that the eigenvalues are arranged in the decreasing order and $\hat{\sigma}_A^2$ is a fixed value, thus the values of the term on the right-hand side of (24) should be decreased faster than the term on the left. So, we may conclude that for $K>2$, the following inequality holds, i.e.,

$$[(M-1)L-\hat{K}]\hat{\sigma}_A^2 \geqslant \frac{1}{2Q} \sum_{i=\hat{K}+1}^{(M-1)L} \lambda_i, \tag{26}$$

therefore, the minimum value of $\hat{K}$ which satisfied inequality (26) can be viewed as the estimated number of source signals. From the above description, in general, more eigenvalues than the principal eigenvalues might be selected. The accuracy and the advantage of using the presented search algorithm to extract the principal eigenvalues will be examined in Sec. IV by computer simulations.

Again, for bearing estimation, we may simply apply the weights obtained from the MFBLP-SOS method to (6) and (7). It is noticed that the entries of matrix **A** and **h** in Eqs. (15) and (17) are the correlation functions. Since under the condition that the desired source signal and the noise signal are uncorrelated with each other, the use of correlation functions has the capability to reduce the effect due to the noise signal. Moreover, if the noise signal is white or less correlated, the use of the MFBLP-SOS method of linear array data can expect to have better performance over the conventional MFBLP method with linear array data directly. Based on this observation, in next section, the modification of the MFBLP-SOS method can be developed.

II. MSE OF THE CORRELATION ESTIMATOR AND THE MODIFIED SECOND-ORDER METHOD

In practice, the correlation functions $R_{x_ix_j}(\tau)$ are not available, thus the estimators of correlation functions, $\hat{R}_{x_ix_j}(\tau)$, have to be used. Since the accuracy of the estimator of the correlation function directly affects the performance of bearing estimation. Recall that the received signal is composed of the desired source and the noise components. Thus we may define the mean-squared error (MSE) between the estimators of correlation functions with respect to the received signal and the desired source signal as the performance index. Thus the MSE just described can be viewed as an averaged noise power with respect to the desired correlation function, $\hat{R}_{s_is_j}(\tau)$. In consequence, the analysis and evaluation of the MSE between $\hat{R}_{x_ix_j}(\tau)$ and $\hat{R}_{s_is_j}(\tau)$ will be significant.

To derive the MSE between the estimators of correlation functions with respect to the received signal and the desired source signal, we assume the signal model is such that K narrow-band signals impinge on the linear array sensors with center frequency f_c from directions θ_i and the distance between neighboring sensors is one-half of the wavelength. The narrow-band signal received at the mth sensor is given by

$$s_m(k) = \sum_{i=1}^{K} A_i e^{j(2\pi k f_c - m\pi \sin\theta_i + \phi_i)}, \tag{27}$$

where A_i and ϕ_i are the amplitude and initial phase of the ith source signal, respectively. Moreover, for convenience, we assume that the source signals are stationary and uncorrelated with the sensor noise, $n_m(k)$, which is assumed to be white (or colored) Gaussian random process with zero mean.

Now, if the MSE between the estimators of correlation functions described above is defined as $\xi_{i,j}(\tau)$, from the Appendix, we have

$$\xi_{i,j}(\tau) = E[(\hat{R}_{x_ix_j}(\tau) - \hat{R}_{s_is_j}(\tau))^2],$$
$$= \begin{cases} \dfrac{\sigma_n^2}{N}[P_{s_i}+P_{s_j}+\sigma_n^2]+\sigma_n^4, & \tau=0 \text{ and } i=j, \\ \dfrac{\sigma_n^2}{N-|\tau|}[P_{s_i}+P_{s_j}+\sigma_n^2], & \text{otherwise}, \end{cases} \tag{28}$$

where σ_n^2 is the power of white Gaussian noises and P_{s_m} is defined as the averaged power of desired source signals received at the mth sensor.

Now, based on expression in (28), an explicit expression for the averaged MSE of the entries in matrix **A** can be derived and may be viewed as the performance index of bearing estimation. This is because that each block matrix of **A** contains the elements of $R_{x_ix_j}(\tau)$. To do so, apply (28) to the entries of the matrix **A**, and define the averaged MSE of the entries in matrix **A** to be E_p:

$$E_p = \frac{1}{2Q}\frac{1}{L}\frac{1}{M-1}\left\{ \sum_{i=1}^{M-1}\sum_{j=0}^{Q-1}\sum_{k=0}^{L-1} [\xi_{i,0}(L-S+j-k) + \xi_{M-1,M-1-i}(L-S+j-k)] \right\}. \tag{29}$$

Next, applying (28) to (29), we obtain

$$E_p=\frac{\sigma_n^2}{2QL(M-1)}\left\{\sum_{i=1}^{M-1}\sum_{j=0}^{Q-1}\sum_{k=0}^{L-1}\left(\frac{P_{s_0}+P_{s_{M-1}}+P_{s_i}+P_{s_{M-1-i}}+2\sigma_n^2}{N-|L-S+j-k|}\right)\right\}$$
$$=\frac{\sigma_n^2}{2(M-1)}\left\{M(P_{s_0}+P_{s_{M-1}})+2(M-1)\sigma_n^2+2\sum_{i=1}^{M-2}P_{s_i}\right\}E_t, \qquad (30)$$

where

$$E_t=\frac{1}{QL}\sum_{j=0}^{Q-1}\sum_{k=0}^{L-1}\left(\frac{1}{N-|L-S+j-k|}\right)=\begin{cases}\frac{1}{QL}\left[\frac{L}{N}+2\sum_{r=1}^{\frac{Q-L}{2}}\frac{1}{N-r}+\sum_{t=1}^{L-1}\left(\frac{t}{N-S+t}\right)\right], & \text{for } Q\geqslant L\\ \frac{1}{QL}\left[\frac{Q}{N}+2\sum_{r=1}^{\frac{L-Q}{2}}\frac{1}{N-r}+\sum_{t=1}^{Q-1}\left(\frac{t}{N-S+t}\right)\right], & \text{for } Q\leqslant L.\end{cases} \qquad (31)$$

Using the result of (30), E_p can be evaluated.

As can be seen from (14), e.g., $\mathbf{Ag}=\mathbf{h}$, to solve the weight vector the accuracy of the vector $\mathbf{h}$ also affects the performance of the proposed method. Therefore, it will be necessary to investigate the averaged error of the correlation estimators as we did for the matrix $\mathbf{A}$. To do so, we similarly, apply (28) to the entries of vector in (17), the averaged correlation errors, E_{ph}, correspond to vector $\mathbf{h}$ and can also be derived:

$$E_{ph}=\frac{1}{2Q}\left\{\sigma_n^2\sum_{i=0}^{Q-1}\frac{2P_{s_0}+\sigma_n^2}{N-|L-S+i|}+\sigma_n^2\sum_{i=0}^{Q-1}\frac{2P_{s_{M-1}}+\sigma_n^2}{N-|L-S+i|}+2\sigma_n^4\right\}$$
$$=\frac{\sigma_n^2(P_{s_0}+P_{s_{M-1}}+\sigma_n^2)}{Q}\sum_{i=0}^{Q-1}\frac{1}{N-|L-S+i|}+\frac{\sigma_n^4}{Q}. \qquad (32)$$

In the following, we would like to see the relationship between the length of the received signal N and E_p. From (31), we see that the value of E_t is negligible when data length N is unbounded. That is $E_p \simeq 0$ when N is relatively large. But, under the same condition, from (32), we have $E_{ph}=\sigma_n^4/Q$ even if N is infinitude. From (28), we know that the term, σ_n^4/Q, is introduced from $\xi_{m,m}(0)$. At this point, it is interesting to point out that to improve the performance of the the MFBLP-SOS method, we may suggest crossing out the terms $R_{x_m x_m}(0)$. This is equivalent to removing the equations with $\tau=0$ in (10) and $\tau=1-L$ in (11), respectively. That is, the MFBLP-SOS method can be modified to have better performance when SNR is relatively low. In consequence, the dimension of the matrix $\mathbf{A}$ is reduced to $2(Q-1)$-by-$L(M-1)$ and accordingly the dimension of $\mathbf{h}$ is $2(Q-1)$-by-1. In this case, since it can be shown that E_p and E_{ph} will approximate to zero when the length of received data N become very large. Thus we can expect that the modified version of the MFBLP-SOS method may perform superior to the one without crossing out the terms, $R_{x_m x_m}(0)$.

Now, when the length of received data N become very large, from (31), we have

$$E_t=\frac{1}{QL}\sum_{j=0}^{Q-1}\sum_{k=0}^{L-1}\left(\frac{1}{N-|L-S+j-k|}\right)\simeq\frac{1}{N},$$
$$\text{if } N\gg|L-S+j-k|. \qquad (33)$$

Substitute (31) into (33), accordingly, we have E_p to be

$$E_p\simeq\frac{\sigma_n^2}{N}\frac{1}{2(M-1)}\left\{M(P_{s_0}+P_{s_{M-1}})+2(M-1)\sigma_n^2+2\sum_{i=1}^{M-2}P_{s_i}\right\}. \qquad (34)$$

Similarly, when the modified MFBLP-SOS method is employed and N becomes very large, we have the approximated value of E_{ph}:

$$E_{ph}\simeq\frac{\sigma_n^2}{N}(P_{s_0}+P_{s_{M-1}}+\sigma_n^2), \quad \text{if } N\gg|L-S+i|. \qquad (35)$$

It is interesting to note here that from (34) and (35), we learn that if σ_n^2/N is constant for the same array system, the values of E_p and E_{ph} will be unchanged. That is, when data length increases ten times, SNR will reduce 10 dB, and the performance will be similar.

III. THE USE OF FOURTH-ORDER CUMULANTS FOR BEARING ESTIMATION

As discussed in the previous section, theoretically, when noises in the related sensors are white, Gaussian processes and the data length of received signals are unbounded, the result of bearing estimation using the modified MFBLP-SOS method can be made without error. Unfortunately, for colored Gaussian noises, the correlation functions of the entries of the matrix $\mathbf{A}$, will be affected by the colored noises significantly. That is, the components respect to the correlation functions of noises, may not be null, e.g., $R_{n_i n_j}(\tau)\neq 0$, for all τ. In consequence, the MSE of the correlation estimators, $\xi_{i,j}(\tau)$, could not be ignored, even if data length is unbounded. In this situation, the MFBLP-SOS method may not perform as well as in the case of white Gaussian noises. Thus the method with higher-order statistics (or cumulants) is required to further improve the resolution of bearing estimation.

As addressed before, the advantage of using higher-order statistics, e.g., fourth-order cumulants, arises from the fact that if noises in the related sensors are Gaussian processes, white or colored, then all the statistics of noise components with an order higher than two will be identical to zero.[2] Therefore, the use of higher-order cumulants of received signals can suppress the effect due to the noise components, yields improvement in the SNR, accordingly.

To develop the MFBLP-FOC method, we may multiply $x_0^*(k-\tau)x_0(k-\tau)x_0^*(k-\tau)$ on both sides of (2), and after taking the expectation, we get

$$E[x_0(k)x_0^*(k-\tau)x_0(k-\tau)x_0^*(k-\tau)] = E\left[\sum_{m=1}^{M-1}\sum_{l=0}^{L-1} a_{m,l}x_m(k-l)x_0^*(k-\tau)x_0(k-\tau)x_0^*(k-\tau)\right] + E[e_f(k)x_0^*(k-\tau)x_0(k-\tau)x_0^*(k-\tau)], \quad \text{for } k=L,L+1,\ldots,N. \tag{36}$$

For convenience to proceed, we define the fourth-order moment, $R_{x_ix_j}^{(4)}(\tau)$, by

$$R_{x_ix_j}^{(4)}(\tau)=E[x_i(k)x_j^*(k-\tau)x_j(k-\tau)x_j^*(k-\tau)]. \tag{37}$$

Under the assumption that the desired source components are deterministic and the colored noises are circular complex Gaussian processes[17] the last term of (36) will be zero, provided that the filter is operating in its optimum condition. In consequence, the weight coefficients $a_{m,l}$ of forward predictor are equal to the optimal weights $g_{m,l}$. Similarly, the forward predictor based on the fourth-order moment is given by

$$R_{x_0x_0}^{(4)}(\tau)=\sum_{m=1}^{M-1}\sum_{l=0}^{L-1} g_{m,l}R_{x_mx_0}^{(4)}(\tau-l) \quad \text{for } \tau=L-N,L-N+1,\ldots,N-1. \tag{38}$$

Since by definition, the fourth-order cumulant, $C_{x_ix_j}^{(4)}(\tau)$, can be written by

$$C_{x_ix_j}^{(4)}(\tau)=R_{x_ix_j}^{(4)}(\tau)-2R_{x_ix_j}(\tau)R_{x_jx_j}(0)-E[x_i(k)x_j(k-\tau)]E[x_j^*(k-\tau)x_j^*(k-\tau)]. \tag{39}$$

Again, for circular complex Gaussian processes with zero mean, the last term on the right side of (39) can be shown to be zero.[17] Recalled from the signal model in (27), the signal in the last term on (39) can be also shown to be zero, therefore, the fourth-order cumulants will reduce to

$$C_{x_ix_j}^{(4)}(\tau)=R_{x_ix_j}^{(4)}(\tau)-2R_{x_ix_j}(\tau)P_{s_j}. \tag{40}$$

Now, based on (38), (40), and (10), the forward predictor based on the fourth-order cumulants having the form similar to (38) can be developed. First, we multiply (10) by $2P_{s_0}$ and subtract it from (38) and after some manipulations, we get

$$C_{x_0x_0}^{(4)}(\tau)=\sum_{m=1}^{M-1}\sum_{l=0}^{L-1} g_{m,l}C_{x_mx_0}^{(4)}(\tau-l), \quad \text{for } \tau=L-N,L-N+1,\ldots,N-1. \tag{41}$$

Follow a similar procedure, we have the backward predictor based on the fourth-order cumulants:

$$C_{x_{M-1}x_{M-1}}^{(4)*}(1-\tau-L) = \sum_{m=1}^{M-1}\sum_{l=0}^{L-1} g_{m,l}C_{x_{M-m-1}x_{M-1}}^{(4)*}(1-\tau-L+l) \quad \tau=-N+1,-N+2,\ldots,N-L. \tag{42}$$

From (41) and (42), the MFBLP-FOC method for bearing estimation can be similarly derived when the matrix form expression is similar to the MFBLP-SOS method, but the entries of the matrix in (18), (19), and (21) are $C_{x_ix_j}^{(4)}(\tau)$ instead of $R_{x_ix_j}(\tau)$. Again, after using the principal component approach and the search algorithm suggested earlier, finally, we may obtain the weight vector $\hat{\mathbf{g}}$ which can be applied to determine the incident angles.

In the MFBLP-FOC method, since the colored Gaussian noises can be completely removed, no further processing, as in the modified MFBLP-SOS method, is required. Therefore, we can expect that the use of MFBLP-FOC method will have better performance than the MFBLP methods with lower-order statistics of received signals.

IV. COMPUTER SIMULATIONS

In this section, to document the advantage of the MFBLP methods with higher-order statistics for bearing estimation, computer simulations are carried out. In the simulation, we consider two narrow-band source signals burried in additive Gaussian noise with incident angles of θ_1 and θ_2. The amplitudes of narrow-band sources are set to unity with

TABLE I. Theoretical and simulation errors ($N=202$, $\theta_1=5°$, $\theta_2=-3°$).

SNR	Theoretical E_p	Simulation E_p	Theoretical E_{ph}	Simulation E_{ph}
−5 dB	1.15073×10^{-1}	1.13243×10^{-1}	1.25803×10^{-1}	1.31675×10^{-1}
0 dB	2.53204×10^{-2}	2.55543×10^{-2}	2.87149×10^{-2}	2.84885×10^{-2}
5 dB	6.90012×10^{-3}	6.84974×10^{-3}	7.97370×10^{-3}	7.53374×10^{-3}
10 dB	2.07132×10^{-3}	2.13097×10^{-3}	2.41083×10^{-3}	2.43099×10^{-3}
15 dB	6.43942×10^{-4}	6.40928×10^{-4}	7.51304×10^{-4}	7.55579×10^{-4}
20 dB	2.02524×10^{-4}	2.10292×10^{-4}	2.36476×10^{-4}	2.28593×10^{-4}
25 dB	6.39334×10^{-5}	6.33623×10^{-5}	7.46697×10^{-5}	7.49970×10^{-5}
30 dB	2.02065×10^{-5}	2.09492×10^{-5}	2.36015×10^{-5}	2.33383×10^{-5}

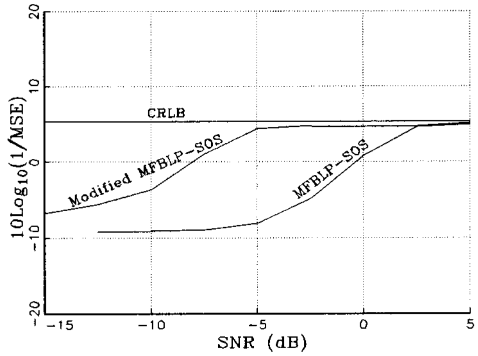

FIG. 4. The MSE of the modified MFBLP-SOS and MFBLP-SOS methods. Incident angle $\theta_1=3°$, $\theta_2=-3°$, $N=50\times10^{5-\mathrm{SNR}/10}$.

the digital normalized frequency being 0.3, where, the maximum digital normalized frequency is 0.5. The noises in all sensors are assumed to be circular complex Gaussian processes with zero mean and the variance to be σ_n^2. Here, the cases that the additive noise in both spatial and temporal domains are white and the additive noise to be spatially correlated and colored in temporal domain are considered. For convenience, the signal-to-noise ratio (SNR) is defined by $\mathrm{SNR}=10\log_{10}(1/\sigma_n^2)$. Moreover, ten sensors ($M=10$) are considered with each sensor of the linear array having four tap weights ($L=4$). In consequence, 30 equations ($Q=30$) are selected in the simulation for the MFBLP method with higher-order statistics.

A. The additive noise sources in both spatial and temporal domains are white

As discussed in Sec. II, to evaluate the performance of the MFBLP-SOS method, the average MSE of the entries in matrix **A** and vector **h** can be viewed as an performance index for bearing estimation. First, we will verify the accuracy of the theoretical values of E_p and E_{ph}. For convenience, we assume that the incident angles of two narrowband sources are $\theta_1=5°$ and $\theta_2=-3°$ and the additive Gaussian noise is white. The statistical values of E_p and E_{ph} are evaluated by averaging 500 independent runs and the length of received signal is $N=202$. The theoretical values of E_p and E_{ph} are calculated based on the expression of (30)

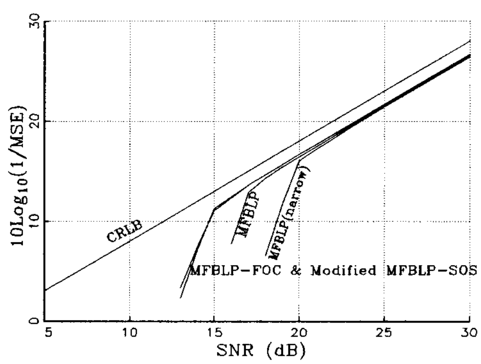

FIG. 5. The MSE of different order statistics methods and the CRLB with white Gaussian noise for $N=64$.

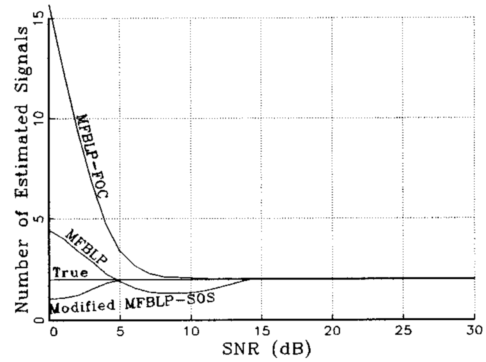

FIG. 6. Average number of estimated signals with white Gaussian noise for $N=64$.

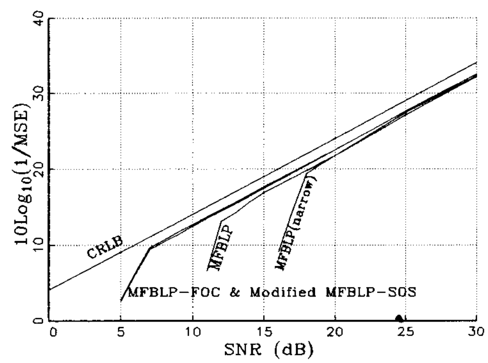

FIG. 7. The MSE of different order statistics methods and the CRLB with white Gaussian noise for $N=256$.

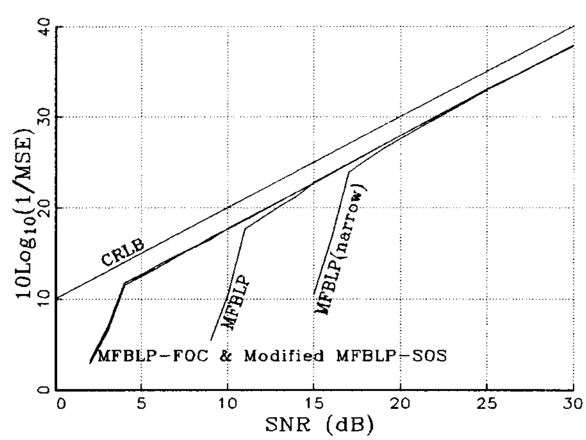

FIG. 8. The MSE of different order statistics methods and the CRLB with white Gaussian noise for $N=1024$.

TABLE II. Threshold SNR improvement.

Date length	Threshold SNR of MFBLP method with higher-order statistics	Threshold SNR of MFBLP method with received data directly	Threshold SNR improvement
64	15 dB	17 dB	2 dB
128	11 dB	14 dB	3 dB
256	7 dB	12 dB	5 dB
512	6 dB	12 dB	6 dB
1024	4 dB	11 dB	7 dB

and (32), respectively, and the results are listed in Table I. From Table I, we learn that the theoretical results agree quite well with the simulation results.

Recall from (33), when the length of received data become very large, we have an approximated value of $E_t \simeq 1/N$. In consequence, if we set N/σ_n^2 to be a constant, then for the same array system with a different set of parameters, we will have the same values of E_p and E_{ph}. Therefore, for $N/\sigma_n^2 = 50\sqrt{10}$, i.e., $N=50\times 10^{(5-\text{SNR})/10}$, the results of the MFBLP-SOS and the modified MFBLP-SOS methods for bearing estimation are shown in Fig. 4 together with the CRLB. From Fig. 4, we see that for SNR to be greater than 3 dB, the performance of bearing estimation for both methods is quite similar. The reason this can be easily seen from (32) is that, in this situation, the value of σ_n^4/Q [the last term of the right side of (32)], will be very small. However, this term is the only difference between both MFBLP-SOS and modified MFBLP-SOS methods. This is true, because for relatively large SNR, the effect due to the noise is not significant. On the other hand, for SNR to be less than 3 dB, the modified MFBLP-SOS method is superior to the MFBLP-SOS method. This is also true, because in this case, the value of σ_n^4/Q will become larger when the SNR is decreased. Therefore, to remove the terms of $R_{x_m x_m}(0)$ from (15) and (17) will result in having better performance. It is noticed that in Fig. 4, under the same condition, from simulation results we found that the CRLB will approximate to a constant value. Moreover, for SNR to be relatively large both methods will converge to CRLB.

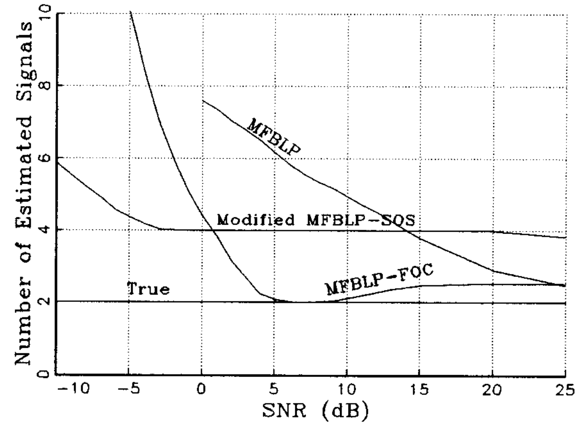

FIG. 9. Average number of estimated signals with colored Gaussian noise for $N=2048$.

Now, let us investigate the performance of MFBLP method with different order statistics of received signals, in terms of the MSE of bearing estimation results. Here, we consider the case that the received signal consists of two source signals burried in the additive white Gaussian noise with incident angles, e.g., $\theta_1=14°$ and $\theta_2=17°$. The other parameters of the linear array will be the same as before.

The curves of the MSE of bearing estimation results versus SNR are shown in Fig. 5 together with the CRLB for $N=64$. In Fig. 5, where "MFBLP" stands for the result of using the MFBLP method with linear array data directly and "MFBLP (narrow)" denotes the results when the narrowband array structure is employed, we can see that the performance of MFBLP with linear data directly using the narrowband array structure is degeneration. Moreover, from Fig. 5, we learn that the MSE of the modified MFBLP-SOS and the MFBLP-FOC methods are very close and the threshold SNR occurs at about 15 dB. By the threshold SNR we mean that the value of SNR at which the accuracy of the bearing estimates begins to depart very rapidly from the CRLB as SNR is lower. On the other hand, the threshold SNR of MFBLP method with linear array data directly occurs at about 17 dB. Thus the improvement of threshold SNR is about 2 dB.

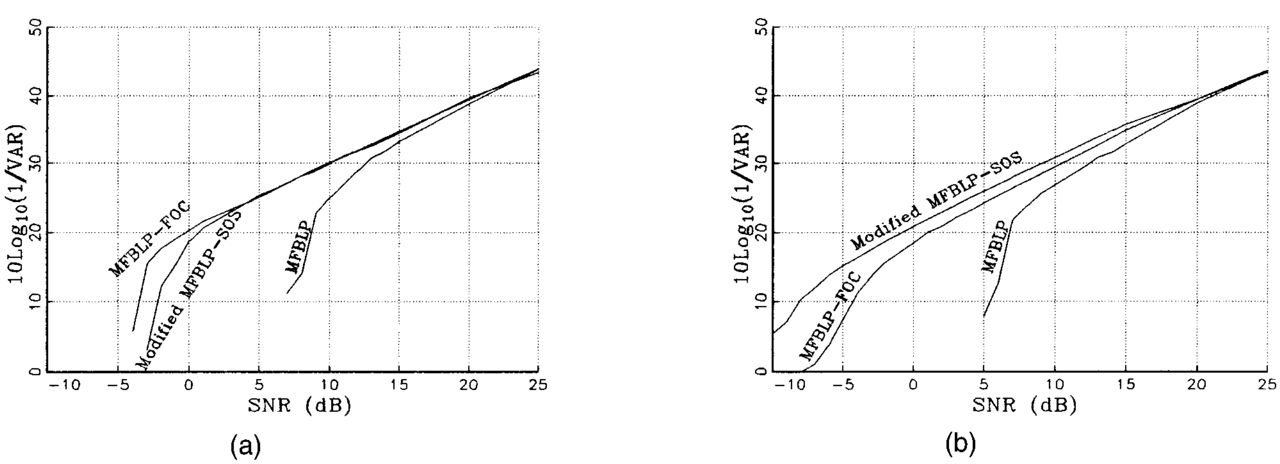

FIG. 10. The variance bearing estimation error for different order statistics methods with colored Gaussian noise. (a) Number of signals is known. (b) Number of signals is estimated with search algorithm.

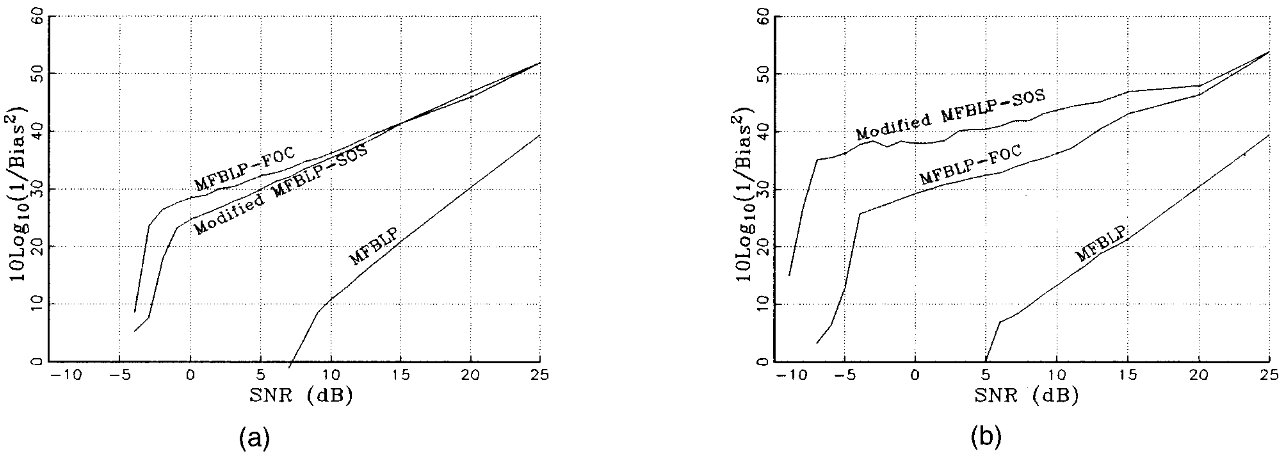

FIG. 11. The bias bearing estimation error for different order statistics methods with colored Gaussian noise (a) Number of signals is known. (b) Number of signals is estimated with search algorithm.

Moreover, the average number of source signals estimated by the search algorithm suggested in this paper is given in Fig. 6. We learn that in Fig. 6, the number of source signals being estimated is incorrect when SNR is lower. But it can be estimated correctly when SNR is higher, this is also true even if the SNR is slightly lower than the threshold SNR. Therefore, in this situation, the new search algorithm proposed in this paper may results in having the identical performance as the one using the exact number of source signals. Again, for $N=256$ and $N=1024$, the MSE of bearing estimation results are shown in Fig. 7 and Fig. 8, respectively. The threshold SNR of both methods is listed in Table II. From these results, we observed that when data length N becomes larger, the threshold SNR can be further improved.

B. The additive noise sources are spatially correlated and colored in temporal domain

Next, we would like to investigate the performance of the MFBLP methods with the higher-order statistics when the additive noise is spatially correlated and colored in temporal domain. The additive colored Gaussian noise is generated by passing a circular complex white Gaussian process through a second-order IIR filter, the transfer function of the system is

$$H(z)=\frac{1}{1-2r(\cos\,\Omega)z^{-1}+r^2z^{-2}}. \tag{43}$$

TABLE III. The CPU time of different methods and data length (Unit: second).

Data length	Conventional MFBLP	Modified MFBLP-SOS	MFBLP-FOC
64	2.703	3.741	3.948
128	3.520	4.233	5.810
256	5.376	5.271	9.328
512	9.030	7.142	17.283
1024	17.202	11.021	30.102
2048	34.689	19.250	60.457

That is, the additive noise sources, $n_m(k)$, which are spatially correlated and colored in temporal domain can be generated using the following equations:

$$u_m(k)=2r_1(\cos\,\Omega_1)u_m(k-1)-r_1^2u_m(k-2)+w_m(k),$$
$$\text{for}\ \ m=0,1,\ldots,M-1,\quad k=1,2,\ldots,N, \tag{44}$$

and

$$n_m(k)=2r_2(\cos\,\Omega_2)n_{m-1}(k)-r_2^2n_{m-2}(k)+u_m(k),$$
$$\text{for}\ \ m=0,1,\ldots,M-1,\quad k=1,2,\ldots,N, \tag{45}$$

where $w_m(k)$ is a white Gaussian sequence in both spatial and temporal domains and the coefficients are chosen to be $r_1=r_2=0.95$ and $\Omega_1=\Omega_2=0.25$, respectively. Again, we consider two source signals with incident angles, e.g., $\theta_1=23°$ and $\theta_2=27°$ and the data length is chosen to be $N=2048$.

The search algorithm is, again, employed in this case and the results are shown in Fig. 9. Figures 10(a) and 11(a) are the variance and bias of bearing estimation error when the number of source signals is known exactly. On the other hand, Figs. 10(b) and 11(b) are the results obtained via the proposed search algorithm. In this case, due to the colored noises, we found that the MFBLP method with the linear array data directly could not work satisfactorily. Moreover, the bias is larger compared with the MFBLP method with higher-order statistics. From Fig. 9, we found that the number of signals so estimated via the new search algorithm is greater than the exact number of source signals. As can be seen from Fig. 10(a), the MFBLP-FOC method performed slightly better than the modified MFBLP-SOS method. However, from Fig. 10(b), we see that the modified MFBLP-SOS method outperformed the MFBLP-FOC method, when the new search algorithm is employed in both methods. Although the number of source signals is over estimated by the search algorithm, from Figs. 10 and 11, we learn that all methods via the new search algorithm may perform superior to the methods using the exact number of source signals.

Since the implementation of the higher-order statistics methods involve more computation time, it is of interest to evaluate the computation effort of the higher-order statistics MFBLP methods and compare it to the conventional MFBLP method. This is addressed in the following. For convenience, again, let us consider the case that the linear array has ten sensors ($M=10$), with each sensor having four tap weights ($L=4$). In consequence, 30 equations ($Q=30$) are chosen for higher-order statistics methods. Under the the same condition, correspondingly, the number of equations in the conventional MFBLP method, will equal to $N-L+1$.

Now, for different data length, we would like to see the computational requirement, in terms of CPU time when DECstation 5000/25 is executing, for implementing each individual method. The results are listed in Table III. From Table III, for N larger (equal) than 256, we see that the use of second-order statistics method has the least computation time among all methods. This is because in this case the dimension of matrix **A** in the conventional MFBLP method will be increased to be $2(N-L+1)$-by-$L(M-1)$.

From (40), it is clear that the computational requirement of the MFBLP method with fourth-order cumulants will be much larger than other methods for bearing estimation. But the performance improvement is as much as the modified MFBLP-SOS method when additive noise is white Gaussian process. On the other hand, when the search algorithm is employed and the background noises are colored, the modified MFBLP-SOS method has shown to have the least bias, variance and computation time for larger data length (say $N \geqslant 256$).

V. CONCLUDING REMARKS

In this paper, the MFBLP methods with higher-order statistics have been developed for bearing estimation. The advantage of the proposed methods are proved from the results of computer simulation.

The statistical properties, in terms of bias, variance and the mean-squared value of bearing estimation error, were investigated and compared to the CRLB. From the simulation results we found that if the Gaussian noises are white, all methods can perform well when SNR is relatively large. The threshold SNR improvement of MFBLP with higher-order statistics is in the range of 2–7 dB, when the received data lengths are changed from 64 to 1024. Under this circumstance, the performance of the modified MFBLP-SOS method and MFBLP-FOC method is quite similar.

Next, if the background noise sources are spatially correlated and colored in temporal domain, the MFBLP method with linear array data could not work properly even if the noise power is relatively small. But, the modified MFBLP method with higher-order statistics can still perform well when SNR is lower. Moreover, all methods via the new search algorithm outperformed the methods using the exact number of source signals.

APPENDIX

In this Appendix, the MSE of both correlation estimators, $\hat{R}_{x_i x_j}(\tau)$ and $\hat{R}_{s_i s_j}(\tau)$, will be derived. For white Gaussian noise, the correlation function will be

$$R_{n_i n_j}(\tau)=\begin{cases}\sigma_n^2, & \tau=0 \text{ and } i=j\\ 0, & \text{otherwise.}\end{cases} \tag{A1}$$

Moreover, the source signals are assumed to be uncorrelated with the noises, i.e., $R_{s_i n_j}(\tau)=0$. The MSE of both correlation function estimators, $\hat{R}_{x_i x_j}(\tau)$ and $\hat{R}_{s_i s_j}(\tau)$, is defined as

$$\begin{aligned}\xi_{i,j}(\tau)&=E[(\hat{R}_{x_i x_j}(\tau)-\hat{R}_{s_i s_j}(\tau))^2]\\ &=E[(\hat{R}_{s_i n_j}(\tau)+\hat{R}_{s_j n_i}(\tau)+\hat{R}_{n_i n_j}(\tau))^2],\end{aligned} \tag{A2}$$

and can be expanded as

$$\begin{aligned}\xi_{i,j}(\tau)=&E[\hat{R}_{s_i n_j}(\tau)\hat{R}^*_{s_i n_j}(\tau)]+E[\hat{R}_{s_i n_j}(\tau)\hat{R}^*_{n_i s_j}(\tau)]\\ &+E[\hat{R}_{s_i n_j}(\tau)\hat{R}^*_{n_i n_j}(\tau)]+E[\hat{R}_{n_i s_j}(\tau)\hat{R}^*_{s_i n_j}(\tau)]\\ &+E[\hat{R}_{n_i s_j}(\tau)\hat{R}^*_{n_i s_j}(\tau)]+E[\hat{R}_{n_i s_j}(\tau)\hat{R}^*_{n_i n_j}(\tau)]\\ &+E[\hat{R}_{n_i n_j}(\tau)\hat{R}^*_{s_i n_j}(\tau)]+E[\hat{R}_{n_i n_j}(\tau)\hat{R}^*_{n_i s_j}(\tau)]\\ &+E[\hat{R}_{n_i n_j}(\tau)\hat{R}^*_{n_i n_j}(\tau)].\end{aligned} \tag{A3}$$

From (A2), we see that $\xi_{i,j}(\tau)$ can be viewed as the error power with respect to the desired correlation function, $\hat{R}_{s_i s_j}(\tau)$. Since the ensemble average of two correlation estimators, $\hat{R}_{ij}(\tau_1)$ and $\hat{R}_{kl}(\tau_2)$, can be shown[18]

$$\begin{aligned}E[\hat{R}_{ij}(\tau)\hat{R}^*_{kl}(\tau)]=&\frac{1}{(N-|\tau|)^2}\sum_{r=-(N-\tau-1)}^{N-\tau-1}(N-\tau-|r|)\\ &\times\{R_{ij}(\tau)R^*_{kl}(\tau)+\bar{R}_{il}(r+\tau)\bar{R}^*_{jk}(r\\ &-\tau)+R_{ik}(r)R^*_{jl}(r)+\mathrm{cum}(r,\tau,\tau),\end{aligned} \tag{A4}$$

where cum(r,τ,τ) is the fourth-order joint cumulants. Thus applying (A4) into (A3), and recognizing that the term cum(r,τ,τ) in (A4) is identical to zero,[2] the first term on the right side of (A3) is

$$E[\hat{R}_{s_i n_j}(\tau)\hat{R}^*_{s_i n_j}(\tau)]=\frac{1}{N-|\tau|}\,\sigma_n^2 P_{s_i}, \tag{A5}$$

where P_{s_i} denotes the signal power. Similarly, the 5th term, $E[\hat{R}_{n_i s_j}(\tau)\hat{R}^*_{n_i s_j}(\tau)]$, can be expressed as $[(1-|\tau|)/N]\sigma_n^2 P_{s_j}$, and the last term on the right side of (A3) is easily shown to be

$$E[\hat{R}_{n_i n_j}(\tau)\hat{R}^*_{n_i n_j}(\tau)]=\begin{cases}\left(1+\dfrac{1}{N}\right)\sigma_n^4, & \tau=0 \text{ and } i=j,\\ \dfrac{1}{N-|\tau|}\,\sigma_n^4, & \text{otherwise.}\end{cases} \tag{A6}$$

Moreover, the 2nd and the 4th terms on the right side of (A3) (for complex random processes) are shown to be zero. (Note, the 2nd and the 4th terms are shown to be $[(N-|2\tau|)/(N-|\tau|)^2]R_{s_i s_i}(2\tau)$ and $[(N-|2\tau|)/(N-|\tau|)^2]R_{s_j s_j}(2\tau)$, respected, when the signals are real valued and $N>2|\tau|$.) Furthermore, the other terms left on the right side of (A3) are equal to zero, provided that the desired signal and the noise are uncorrelated with each other.

Now by substituting (A5) and (A6) into (A2), the MSE of correlation estimators can be computed by the following expression

$$\xi_{i,j}(\tau)=\begin{cases}\dfrac{\sigma_n^2}{N}[P_{s_i}+P_{s_j}+\sigma_n^2]+\sigma_n^4, & \tau=0 \text{ and } i=j,\\ \dfrac{\sigma_n^2}{N-|\tau|}[P_{s_i}+P_{s_j}+\sigma_n^2], & \text{otherwise.}\end{cases} \quad \text{(A7)}$$

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