有關 *k* 母體之貝氏取樣設計與決策(3/3)

1. Introduction

In many practical occasions, an experimenter often faces with the situation of testing for homogeneity. And when the hypothesis of homogeneity is rejected, the experimenter often needs to rank priority of several categories or treatments under consideration according to his goal. This concerns the multiple comparison of ranking and selection which has been developed in last forty years. Readers are referred to Gupta and Panchaphesan (1979), for instance, among others.

In this area of ranking and selection, most literature are concerned with one criterion, for example, a population is considered as the best if it is associated with some largest (or smallest) parameter in a finite set of populations. In many situations, it may not satisfy the experimenter's demand. For example, in industrial statistics, one needs not only to attain its largest target, but on the other hands, one also needs to keep the variation of product under control. Under this circumstance, a single criterion for selection of potential treatments does not meet our requirement. Recently, Gupta, Liang and Rau (1994) consider selecting the best normal population compared with a control. It involves two criteria for selection, however, they belong to same character and only the location parameter is concerned. For this consideration, most recently, Huang and Lai (1998) consider selecting the best normal population compared with two controls. They consider two main different quantities, i.e. mean and variance for their main concern. Mean parameters are permitted to have some perturbation that they consider some structure on the means. On the other hand, no perturbation is permitted on the quantity of variance. In this paper, some perturbation on the variance is permitted. In a Bayes framework, we develop an empirical Bayes procedure for selecting the best normal population with a normal-gamma prior as its conjugate prior.

In section 2, we formulate the problem and develop some Bayes setup. In section 3, we propose an empirical Bayes procedure. In section 4, we study the large sample behavior of the proposed empirical Bayes rule. It is shown that the proposed empirical Bayes selection rule is asymptotically optimal.

2. Formulation of problem and a Bayes selection rule

In this paper, we utilize the process capability index proposed by Spring (1997) to evaluate the effectiveness of a manufacturing process. This index is defined as follows.

Definition 2.1 Let π be a manufacturing process with mean θ and variance σ^2. T is the target value, and USL and LSL are the upper specification limit and lower specification limit, respectively. Then the process capability index of π is defined as the following

$$C_{pw} = \frac{USL - LSL}{6\sqrt{\sigma^2 + w(\theta - T)^2}},$$

where w is a weight function.

According to process capability index introduced as above, we define the best PCI-qualified manufacturing process as follows.

Definition 2.2 Let $\pi_1, \cdots, \pi_k$ be k manufacturing processes such that π_i has mean θ_i variance σ_i^2 and process capability index $C_{pw}(i)$, $i = 1, \cdots, k$. Let $C_{pw}(0)$ be a control value (prefixed). Define S=$\{\pi_i | C_{pw}(i) \geq C_{pw}(0),\ i = 1, \cdots, k\}$. A manufacturing process π_i is called PCI-qualified, if $\pi_i \in$S. A manufacturing process π_i is considered as the best PCI-qualified, if it simultaneously satisfies the following conditions:

(i) $\pi_i \in$S, and

(ii) $C_{pw}(i) = \max_{\pi_j \in S} C_{pw}(j)$.

Let $\underset{\sim}{\theta} = (\theta_1, \cdots, \theta_k)$, $\underset{\sim}{\sigma} = (\sigma_1, \cdots, \sigma_k)$ and $\Omega = \{(\theta_i, \sigma_i) | -\infty < \theta_i < +\infty, \sigma_i > 0,\ i = 1, \cdots, k\}$ be the parameter space. Let $\underset{\sim}{a} = (a_0, a_1, \cdots, a_k)$ denote an action, when $a_i = 0,1;\ i = 0,1, \cdots, k$, and $\sum_{i=0}^{k} a_i = 1$. If $a_i = 1$, for some $i = 1, \cdots, k$, it means that manufacturing process π_i is selected as the best PCI-qualified. When $a_0 = 1$, it means that no manufacturing process is considered as the best PCI-qualified, i.e. none in k manufacturing processes satisfied the restriction (i) in Definition 2.1. Let $\mathcal{A}$={ $\underset{\sim}{a}$ } denote the action space.

In a decision-theoretic approach, we introduce the following loss function.

In this paper, we consider a Bayes approach for the problem of selecting the best PCI-qualified manufacturing process with normal distribution.

For each $i = 1, \cdots, k$, let $X_{i1}, \cdots, X_{iM}$ be an independently random sample of size M from a normally distributed manufacturing process π_i with mean θ_i and variance σ_i^2. The observed value is denoted by $x_{i1}, \cdots, x_{iM}$. Let $\tau_i = 1/\sigma_i^2,\ i = 1, \cdots, k$. It is assumed that (θ_i, τ_i) is a realization of a random vector (Θ_i, T_i) with a normal-gamma prior distribution. For convenience, for $i=1,\cdots,k$, we denote $\underset{\sim}{x}_i=(x_{i1},\cdots,x_{iM})$ and $x_i=\frac{1}{M}\sum_{j=1}^{M}x_{ij}$, $S_i^2=\frac{1}{M-1}\sum_{j=1}^{M}(x_{ij}-x_i)^2$. It can be shown that the conditional posterior distribution of Θ_i given $\underset{\sim}{x}_i$ and τ_i is a normal distribution

$$N(\varphi_i(x_i),[(2\alpha_i+M-1)\tau_i]) \quad (1)$$

and the marginal posterior distribution of T_i given $\underset{\sim}{x}_i$ is a gamma distribution $G(\alpha_i',\eta_i)$, where

$\alpha_i'=2\alpha_i+\frac{M}{2}$, and

$$\eta_i=\beta_i+\frac{(M-1)S_i^2}{2}+\frac{(2\alpha_i+3)M(x_i+\mu_i)^2}{2(2\alpha_i+M+2)}. \quad (2)$$

The random vectors $(\Theta_1,\mathrm{T}_1),\cdots,(\Theta_k,\mathrm{T}_k)$ are assumed to be mutually independent.

Let $\underset{\sim}{x}=(\underset{\sim}{x}_1,\cdots,\underset{\sim}{x}_k)$ and χ be the sample space generated by $\underset{\sim}{x}$. A selection rule $\underset{\sim}{d}=(d_0,d_1,\cdots,d_k)$ is a mapping defined on the sample space χ into the k+1 product space $[0,1]\times[0,1]\times\cdots\times[0,1]$ such that $\sum_{i=0}^{k}d_i(\underset{\sim}{x})=1$, for all $\underset{\sim}{x}\in\chi$. For every $\underset{\sim}{x}\in\chi$, $d_i(\underset{\sim}{x})$ denotes the probability of selecting manufacturing process π_i as the best PCI-qualified, $i=1,\cdots,k$; and $d_0(\underset{\sim}{x})$ denotes the probability that none is selected as the best PCI-qualified.

For ease of notation, let $\underset{\sim}{\tau}=(\tau_1,\cdots,\tau_k)$, $\underset{\sim}{\mu}=(\mu_1,\cdots,\mu_k)$, $\underset{\sim}{\alpha}=(\alpha_1,\cdots,\alpha_k)$, $\underset{\sim}{\beta}=(\beta_1,\cdots,\beta_k)$, $\underset{\sim}{\Theta}=(\Theta_1,\cdots,\Theta_k)$ and $\underset{\sim}{\mathrm{T}}=(\mathrm{T}_1,\cdots,\mathrm{T}_k)$. Let $h_i(\theta_i|\underset{\sim}{x}_i,\tau_i;\mu_i,\alpha_i)$ and $g_i(\tau_i|\underset{\sim}{x}_i;\alpha_i,\beta_i)$ be the marginally conditional probability density function of Θ_i and T_i, respectively. Under the preceding formulation, the Bayes risk of a selection rule $\underset{\sim}{d}$, denoted by $r(\underset{\sim}{d})$, is given by

$$r(\underset{\sim}{d})=\mathrm{E}_{\underset{\sim}{\tau}}\mathrm{E}_{\underset{\sim}{\theta}}\mathrm{E}_{\underset{\sim}{x}}L(\underset{\sim}{\theta},\underset{\sim}{\tau};\underset{\sim}{d})$$

$$=\iint_{\Omega}\int_{\chi}\sum_{i=0}^{k}d_i(\underset{\sim}{x})C_{pw}(i)f(\underset{\sim}{x}|\underset{\sim}{\theta},\underset{\sim}{\tau})\cdot h(\underset{\sim}{\theta}|\underset{\sim}{\mu},\underset{\sim}{\tau})g(\underset{\sim}{\tau};\underset{\sim}{\alpha},\underset{\sim}{\beta})d\underset{\sim}{x}\,d\underset{\sim}{\theta}\,d\underset{\sim}{\tau}$$

$$-\iint_{\Omega}\int_{\chi}f(\underset{\sim}{x}|\underset{\sim}{\theta},\underset{\sim}{\tau})h(\underset{\sim}{\theta}|\underset{\sim}{\mu},\underset{\sim}{\tau})\cdot g(\underset{\sim}{\tau};\underset{\sim}{\alpha},\underset{\sim}{\beta})d\underset{\sim}{x}\,d\underset{\sim}{\theta}\,d\underset{\sim}{\tau}$$

$=I_1-I_2$, say.

Hence, for some constant *C*,

$$r(\underset{\sim}{d})=\int_{\chi}\sum_{i=0}^{k}d_i(\underset{\sim}{x})\phi_i(\underset{\sim}{x}_i)f(\underset{\sim}{x})d\underset{\sim}{x}-C. \quad (3)$$

For each $\underset{\sim}{x}\in\chi$, let

$$Q(\underset{\sim}{x})=\{i|\phi_i(\underset{\sim}{x}_i)=\underset{0\le j\le k}{Min}\,\phi_j(\underset{\sim}{x}_j),\ i=0,1,\cdots,k\}. \quad (4)$$

Then, define

$$i^* = i^*(\underset{\sim}{x}) = \begin{cases} 0 & if\ Q(\underset{\sim}{x}) = \{0\}, \\ Min\{i \mid i \in Q(\underset{\sim}{x}),\ i \neq 0\} & otherwise. \end{cases} \quad (5)$$

Then, according to (2.6)~(2.8), it can be derived that a Bayes selection rule $\underset{\sim}{d}^B = (d_0^B, d_1^B, \cdots, d_k^B)$ is given by follows

$$\begin{cases} d_{i^*}^B(\underset{\sim}{x}) = 1, \\ d_j^B(\underset{\sim}{x}) = 0, \quad for\ \ j \neq i^*. \end{cases} \quad (6)$$

3. The empirical Bayes selection rule

In the problem formulated in sec.2, we consider that $\alpha_1, \cdots, \alpha_k$ are known, and $\alpha_i > 1$, for any $i = 1, \cdots, k$. Since $\phi_i(\underset{\sim}{x_i})$ still involves the unknown parameters μ_i, β_i, $i = 1, \cdots, k$, hence, the proposed Bayes selection $\underset{\sim}{d}^B$ is not applicable. However, based on the past data, these unknown parameters can be estimated and a decision can be made if one more observation is taken. For $i = 1, \cdots, k$, let X_{ijt} denote a sample of size M from π_i with a normal distribution $N(\theta_{it}, \tau_{it}^{-1})$ at time t ($t = 1, \cdots, n$), $j = 1, \cdots, M$ and (θ_{it}, τ_{it}) is a realization of a random vector $(\Theta_{it}, \mathrm{T}_{it})$ which is an independent copy of (Θ_i, T_i) with a normal-gamma distribution described in preceding section. It is assumed that $(\Theta_{it}, \mathrm{T}_{it})$, $i = 1, \cdots, k$, $t = 1, \cdots, n$, are mutually independent. For our convenience, we denote the current random sample X_{ijn+1} by X_{ij}, for $j = 1, \cdots, M$, $i = 1, \cdots, k$.

For each π_i, $i = 1, \cdots, k$, we estimate the unknown parameters μ_i and β_i based on the past data X_{ijt}, $j = 1, \cdots, M$, $t = 1, \cdots, n$. We denote

For ease of notation, we define μ_{in} and β_{in} as estimators of μ_i and β_i, respectively, by the following

$$\begin{cases} \mu_{in} = X_i(n), \\ \beta_{in} = (\alpha_i - 1) W_i^2(n). \end{cases} \quad (7)$$

Also, for $i = 1, \cdots, k$, we define

$$\phi_{in}(\underset{\sim}{x_i}) = \frac{36}{(USL - LSL)^2} \cdot \{[w + (2\alpha_i + M - 1)][(\alpha_i' - 1)\eta_{in}]^{-1}, \quad (8)$$

where

$$\eta_{in} = \beta_{in} + \frac{(M+1)S_i^2}{2} + \frac{(2\alpha_i - 1)M(x_i + \mu_{in})^2}{2(2\alpha_i + M)}, \quad (9)$$

and

$$\varphi_{in}(x_i) = \frac{(2\alpha_i + 1)\mu_{in} + Mx_i}{2\alpha_i + M}.$$

Note that $\phi_{0n}(\underset{\sim}{x_0}) = C_{pw}^{-2}(0)$. We consider $\phi_{in}(\underset{\sim}{x_i})$ to be an estimator of $\phi_i(\underset{\sim}{x_i})$. The properties of the estimators proposed above will be discussed in the following section.

For each $\underset{\sim}{x} \in \chi$, let

$$Q_n(\underset{\sim}{x}) = \{i \,|\, \phi_{in}(\underset{\sim}{x_i}) = \underset{0 \le j \le k}{Min}\, \phi_{jn}(\underset{\sim}{x_j}),\ i = 0,1,\cdots,k\}. \quad (10)$$

Again, define

$$i_n^* = i_n^*(\underset{\sim}{x}) = \begin{cases} 0 & if\ Q_n(\underset{\sim}{x}) = \{0\}, \\ Min\{i \,|\, i \in Q_n(\underset{\sim}{x}), i \ne 0\} & otherwise. \end{cases} \quad (11)$$

Then, according to (3.3), (3.6) and (3.7), we have a empirical Bayes selection rule $d^{*n} = (d_0^{*n}, d_1^{*n}, \cdots, d_k^{*n})$ as follows

$$\begin{cases} d_{i_n^*}^{*n}(\underset{\sim}{x}) = 1, \\ d_j^{*n}(\underset{\sim}{x}) = 0, \quad for\ j \ne i_n^*. \end{cases} \quad (12)$$

4. Some large sample properties

In this section, we study the asymptotic optimality of the proposed empirical Bayes selection rule. Before we start to investigate the asymptotic property, we discuss the consistency of the estimators defined in (7)-(9) in the case $M \ge 2$. We just prove Lemma 4.2 here, the others can be proved analogously.

Lemma 4.1 $X_i(n)$ defined in (3.1), is a consistent estimator of μ_i, $i = 1,\cdots,k$.

Lemma 4.2 $W_i^2(n)$ defined in (3.1),is a consistent estimator of $\frac{\beta_i}{\alpha_i - 1}$, $i = 1,\cdots,k$.

Proof: (1) At time *t*, it is well-known that $\frac{(M-1)W_{i.t}^2}{\tau_{it}^{-1}}$ has a chi-square distribution $\chi^2(M-1)$. It implies

$$\mathrm{E}[W_{i.t}^2] = \mathrm{E}\mathrm{E}[W_{i.t}^2 | \Theta_{it}, \mathrm{T}_{it}] = \frac{\beta_i}{\alpha_i - 1}.$$

Hence,

$$\mathrm{E}[W_i^2(n)] = \frac{1}{n}\sum_{i=1}^{n} \mathrm{E}[W_{i.t}^2] = \frac{\beta_i}{\alpha_i - 1}.$$

(2)

$$\mathrm{E}[W_{i.t}^2]^2 = \mathrm{E}\ \mathrm{E}[(W_{i.t}^2)^2 | \Theta_{it}, \mathrm{T}_{it}]$$

$$= \mathrm{E}[\frac{(M+1)\mathrm{T}_{it}^{-2}}{M-1}]$$

$$= \frac{(M+1)\beta_i^2}{(M-1)(\alpha_i - 1)(\alpha_i - 2)}.$$

Hence,

$$Var[W_{i.t}^2] = \mathrm{E}[W_{i.t}^2]^2 - \mathrm{E}^2[W_{i.t}^2]$$

$$= \frac{(2\alpha_i + M - 1)\beta_i^2}{(M-1)(\alpha_i - 1)^2(\alpha_i - 2)} .$$

Therefore,

$$Var[W_i^2(n)] = \frac{1}{n^2}\sum_{i=1}^{n} Var[W_{i.t}^2]$$

$$= \frac{(2\alpha_i + M - 1)\beta_i^2}{n(M-1)(\alpha_i - 1)^2(\alpha_i - 2)} .$$

Hence,

$$\lim_{n\to+\infty} Var[W_i^2(n)] = 0 .$$

We complete the proof. ◇

From above Lemma 4.1 and Lemma 4.2, the following lemmas are immediate.

Lemma 4.3 μ_{in} and β_{in} defined in (7) are consistent estimators of μ_i and β_i, respectively, $i = 1, \cdots, k$.

Definition 4.1 A sequence of empirical Bayes selection rule $\{\underset{\sim}{d}^n\}_{n=1}^{\infty}$ is said to be asymptotically optimal, if $\lim_{n\to\infty}[\mathrm{E}_n[r(\underset{\sim}{d}^n)] - r(\underset{\sim}{d}^B)] = 0$.

Theorem 4.1 The empirical Bayes selection rule $\underset{\sim}{d}^{*n}(\underset{\sim}{x})$, defined in (10)-(12), is asymptotically optimal.

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