行政院國家科學委員會專題研究計劃成果報告

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*　　錐體上線性方程的解、哥拉斯—威蘭數與相關問題　　*

*　　Solutions of Linear Equations Over Cones,　　*

*　　Collatz-Wielandt Numbers and Related Problems　　*

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摘要

以 K 表示 $\mathbb{R}^n$ 上的一常態錐體, A 爲一 n 階實矩陣且 $AK \subseteq K$, b 爲 K 的一向量及 λ 爲一已知正整數。在本計劃我們考慮以下兩個錐體上的方程: (I) $(\lambda I_n - A)x = b$, $x \in K$, 及 (II) $(A - \lambda I_n)x = b$, $x \in K$。我們獲得第一個方程有解的一些等價條件。對第二個方程, 當 $\lambda > \rho_b(A)$ 時 (其中 $\rho_b(A)$ 是表示 A 在 b 的局部譜半徑), 我們獲得一充分且必要條件, 當 $\lambda = \rho_b(A)$ 或 $\lambda < \rho(A)$ 但充分靠近 $\rho_b(A)$ 時我們則獲得一必要條件。利用這些結果, 我們導出錐體正變換有關局部譜半徑, 哥拉斯–威蘭集合 (或數) 的一些新結果, 及推廣了一個 M-矩陣的刻劃。

關鍵詞: 錐體、貝龍–佛羅貝紐斯理論、錐體正變換、線性方程、哥拉斯–威蘭集合、哥拉斯–威蘭數、局部譜半徑、非負矩陣。

Abstract

Let K be a proper cone in $\mathbb{R}^n$, and let A be an $n \times n$ real matrix that satisfies $AK \subseteq K$, let b be a given vector of K, and let λ be a given positive real number. The following two linear equations over cones are considered in this project: (i)$(\lambda I_n - A)x = b$, $x \in K$, and (ii) $(A - \lambda I_n)x = b$, $x \in K$. We obtain several equivalent conditions for the solvability of the first equation. For the second equation we give an equivalent condition for its solvability in case when $\lambda > \rho_b(A)$, and we also find a necessary condition when $\lambda = \rho_b(A)$ and also when $\lambda < \rho_b(A)$, sufficiently close to $\rho_b(A)$, where $\rho_b(A)$ denotes the local spectral radius of A at b. Then we derive some new results about local spectral radii and Collatz-Wielandt sets (or numbers) associated with a cone-preserving map, and extend a known characterization of M-matrices among Z-matrices in terms of alternating sequences.

Key words: Cone, Perron-Frobenius theory, cone-preserving map; linear equation, Collatz-Wielandt sets, Collatz-Wielandt number, local spectral radius, nonnegative matrix.

1. Motivation and Aims

Since the end of 1980s, together with Professor Hans Schneider (and, at the beginning, also with my master student S.F. Wu), I embarked on a study of the spectral theory of positive linear operators on a finite-dimensional proper cone. We adopt the cone-theoretic viewpoint or method to study the Perron-Frobenius theory of a nonnegative matrix and its generalizations to cone-preserving maps. This is the sixth of a sequence of papers on this topic; the first five papers in this sequence are [T–W], [Tam 1], [T–S 1], [T–S 2] and [Tam 2]. The main aim of this project is to study the solvability of the following two linear equations over cones:

$$(\lambda I_n - A)x = b, \; x \in K, \tag{1.1}$$

and

$$(A - \lambda I_n)x = b, \; x \in K, \tag{1.2}$$

where K, A, b and λ are given, with K a proper cone, $A \in \pi(K)$, $b \in K$ and $\lambda > 0$. We also want to apply the results we obtain to study Collatz-Wielandt numbers or sets, and other related problems. Here is a list of the problems considered in this project:

Problem 1. Let $A \in \pi(K)$, let b be a nonzero vector of K, and let λ be a given positive real number. Find equivalent conditions for the solvability of the linear equation $(\lambda I_n - A)x = b$, $x \in K$.

Problem 2. Let $A \in \pi(K)$, let b be a nonzero vector of K, and let λ be a given positive real number. Find necessary and/or sufficient conditions for the solvability of the linear equation $(A - \lambda I_n)x = b$, $x \in K$.

Problem 3. Let $A \in \pi(K)$. Find a necessary and sufficient condition for $\rho(A) \in \sum_1$, where $\sum_1$ is the Collatz-Wielandt set defined by: $\sum_1 = \{\sigma \geq 0 : \exists x \in \operatorname{int} K, \sigma x \geq^K Ax\}$.

Problem 4. Let $A \in \pi(K)$ and let $0 \neq x \in K$. Determine when we have (i) $\lim_{i\to\infty} r_A(A^i x) = \rho_x(A)$; and (ii) $\lim_{i\to\infty} R_A(A^i x) = \rho_x(A)$.

Problem 5. Prove the following

Conjecture: Let $A \in \pi(K)$, and let λ be a given real number. Then $\lambda < \rho(A)$ if and only if there exists a vector $x \in K$ such that $(A - \lambda I_n)^j x \in K\backslash\{0\}$ for all positive integers j.

Problem 6. Let $A \in \pi(K)$, and let λ be a given real number. Are the following two conditions equivalent?

(a) $\lambda \geq \rho(A)$.

(b) For any $x \in \mathbb{R}^n$, if $(\lambda I_n - A)x \geq^K 0$ then there exist a vector u of K and a generalized eigenvector v of A corresponding to $\rho(A)$ such that $x = u - v$.

If not, what about for a particular class of proper cones K, such as the class of polyhedral cones ?

Below is the historical background for the above problems.

Equation (1.1) arose in the study of nuclear physics. As early as 1963, Carlson [Car] has studied equation (1.1) for the special case when $K = \mathbb{R}^n_+$ and $\lambda = \rho(A)$. Carlson's work was followed by Nelson [Nel 1,2], Friedland and Schneider [F–S], Victory [Vic 1,2], and Jang and Victory [J–V 1,2]. Indeed, most of their works were done in the infinite-dimensional settings, in particular, in the setting when A is an eventually compact positive linear operator on a Banach lattice. Here, focusing on the finite-dimensional case, we want to provide a more complete set of equivalent conditions for the solvability of equation (1.1) and to give simpler proofs.

In [T–W], together with my student S.F. Wu, I have examined equation (1.2) for the special case when $\lambda = \rho(A)$ and $K = \mathbb{R}^n_+$; at that time we applied graph-theoretic arguments. As far as I know, no other people have ever studied this equation.

Collatz-Wielandt numbers or sets are fundamental objects in the Perron-Frobenius theory. In [T–W] we have already determined the supremum or infimum of the Collatz-Wielandt sets Σ, Σ_1, Ω and Ω_1. In particular, we have $\inf \Sigma_1 = \rho(A)$. However, when $\rho(A) \in \Sigma_1$ occurs is still an open problem. This is Problem 3 of the present project.

For the nonnegative matrix case, Friedland and Schneider [F–S, Theorem 6.8] have answered the following question: if $A \in \pi(K)$ and $O \neq x \in K$, when do we have $\lim_{i\to\infty} r_A(A^i x) = \rho(A) = \lim_{i\to\infty} R_A(A^i x)$. Förster and Nagy [F–N 2, Theorem 6] extended the result to the case when $\rho(A)$ is replaced by $\rho_x(A)$. In [T–W, Theorem 5.2] we solved the problem for the case when K is a general proper cone and A is K-irreducible. In Problem 4 we consider the most general case.

In [H–R–S, Corollary 3.5 and Theorem 4.1] Hershkowitz, Rothblum and Schneider gave two characterization of M-matrices among Z-matrices. In Problems 5 and 6 we hope to extend their results.

2. Results and Discussions

Below is an answer to Problem 1:

THEOREM 1. *Let $A \in \pi(K)$, let $0 \neq b \in K$, and let λ be a positive real number. The following conditions are equivalent:*

(a) *There exists a vector $x \in K$ such that $(\lambda I_n - A)x = b$.*

(b) $\rho_b(A) < \lambda$.

(c) $\lim_{m\to\infty} \sum_{j=0}^{m} \lambda^{-j} A^j b$ *exists.*

(d) $\lim_{m\to\infty} (\lambda^{-1}A)^m b = 0$.

(e) $\langle z, b\rangle = 0$ *for each generalized eigenvector z of A^T corresponding to an eigenvalue with modulus greater than or equal to λ.*

(f) $\langle z, b\rangle = 0$ *for each generalized eigenvector z of A^T corresponding to a distinguished eigenvalue of A for K which is greater than or equal to λ.*

When the equivalent conditions are satisfied, the vector $x^0 = \sum_{j=0}^{\infty} \lambda^{-j-1} A^j b$ is a solution of the equation $(\lambda I_n - A)x = b, x \in K$. Furthermore, if λ is a distinguished eigenvalue of A, then the solution set of the equation consists of precisely all vectors of the form $x^0 + u$, where u is either the zero vector or is a distinguished eigenvector of A corresponding to λ; otherwise, x^0 is the unique solution of the equation.

As a corollary, we have

COROLLARY 1. *Let P be an $n \times n$ nonnegative matrix, let $b \in \mathbb{R}^n_+$, and let λ be a positive real number. To the list of equivalent conditions of Theorem 1 (but with A and K replaced respectively by P and $\mathbb{R}^n_+$) we can add the following:*

(g) *For any class α of P having access to* supp(b), $\rho(P_{\alpha\alpha}) < \lambda$.

(h) *For each distinguished class* α *of* P *for which* $\rho(P_{\alpha\alpha}) \geq \lambda$, *we have* $b_\beta = 0$ *whenever* β *is a class that has access from* α.

(i) $\langle |z|, b\rangle = 0$ *for each generalized eigenvector* z *of* P^T *corresponding to an eigenvalue with modulus greater than or equal to* λ.

(j) $\langle |z|, b\rangle = 0$ *for each generalized eigenvector* z *of* P^T *corresponding to a distinguished eigenvalue of* P *for* $\mathbb{R}^n_+$ *which is greater than or equal to* λ.

When the equivalent conditions are satisfied, the vector $x^0 = \sum_{j=0}^{\infty} \lambda^{-j-1}P^j b$ *is a solution for the given equation and is also the unique solution with the property that its support is included in (in fact, equal to) the union of all classes of* P *having access to* supp(b). *In this case, if* λ *is not a distinguished eigenvalue of* P, *then* x^0 *is the unique solution, and if* λ *is a distinguished eigenvalue, then the solutions of the equation are precisely all the vectors of the form* $x^0 + u$, *where* u *is the zero vector or is a distinguished eigenvector of* P *corresponding to* λ.

REMARK 1. Let $A \in \pi(K)$, and let $\lambda > 0$ be given. The set $(\lambda I_n - A)K \bigcap K$, which consists of all vectors $b \in K$ for which equation (1.1) has a solution, is equal to the set $\{y \in K : \rho_y(A) < \lambda\}$. The latter set is, in fact, an A-invariant face of K.

In contrast with Problem 1, Problem 2 is more delicate. In general, the set $(A - \lambda I_n)K \bigcap K$, which consists of all vectors $b \in K$ for which equation (1.2) has a solution, is an A-invariant subcone of K, but it need not be a face of K. The answer to Problem 2 depends on whether λ is greater than, equal to, or less than $\rho_b(A)$.

THEOREM 2. *Let* $A \in \pi(K)$, *let* $0 \neq b \in K$, *and let* λ *be a given positive real number such that* $\lambda > \rho_b(A)$. *Then the equation* (1.2) *is solvable if and only if* λ *is a distinguished eigenvalue of* A *for* K *and* $b \in \Phi(\mathcal{N}(\lambda I_n - A) \bigcap K)$. *In this case, for any solution* x *of* (1.2) *we have* $\mathrm{sp}_A(x) = (\lambda, 1)$.

COROLLARY 2. *Let* P *be an* $n \times n$ *nonnegative matrix, let* $0 \neq b \in \mathbb{R}^n_+$, *and let* λ *be a positive real number such that* $\lambda > \rho_b(P)$. *Then the equation*

$$(P - \lambda I_n)x = b, \quad x \geq 0$$

is solvable if and only if λ *is a distinguished eigenvalue of* P *such that for any class* α *of* P, *if* $\alpha \bigcap$ supp$(b) \neq \emptyset$, *then* α *has access to a distinguished class of* P *associated with* λ.

THEOREM 3. *Let $A \in \pi(K)$, and let $0 \neq b \in K$. If the linear equation*

$$(A - \rho_b(A)I_n)x = b, \quad x \in K$$

is solvable, then $b \in (A - \rho_b(A)I_n)\Phi(\mathcal{N}((\rho_b(A)I_n - A)^n) \bigcap K)$.

COROLLARY 3. *Let P be an $n \times n$ nonnegative matrix. Let I denote the union of all classes α of P such that $\alpha >- \beta$ for some basic class β of P. Then $\Phi((P - \rho(P)I_n)\mathbb{R}^n_+ \bigcap \mathbb{R}^n_+)$ is equal to the P-invariant face F_I of $\mathbb{R}^n_+$.*

THEOREM 4. *Let $A \in \pi(K)$. Let r denote the largest real eigenvalue of A less than $\rho(A)$. (If no such eigenvalues exist, take $r = -\infty$.) Then for any λ, $r < \lambda < \rho(A)$, we have*

$$\Phi((A - \lambda I_n)K \bigcap K) = \Phi(\mathcal{N}((\rho(A)I_n - A)^n) \bigcap K).$$

COROLLARY 4. *Let $A \in \pi(K)$. If A has no distinguished generalized eigenvectors, other than distinguished eigenvectors, corresponding to $\rho(A)$, and if A has no eigenvectors in $\Phi(\mathcal{N}((\rho(A)I_n - A)^n) \bigcap K)$ corresponding to an eigenvalue other than $\rho(A)$, then $(A - \rho(A)I_n)K \bigcap K = \{0\}$. The converse also holds if the cone K is polyhedral.*

COROLLARY 5. *Let P be an $n \times n$ nonnegative matrix. Let I denote the union of all classes of P that have access to some basic class. Then for any λ, $r < \lambda < \rho(P)$, where r denotes the largest real eigenvalue of P less than $\rho(P)$ (and equals $-\infty$ if there is no such eigenvalue), we have $\Phi((P - \lambda I_n)\mathbb{R}^n_+ \bigcap \mathbb{R}^n_+) = F_I$.*

As for Problem 3, we have the following solution:

THEOREM 5. *Let $A \in \pi(K)$ with $\rho(A) > 0$. Let C denote the set $\{x \in K : \rho_x(A) < \rho(A)\}$. Then $\rho(A) \in \sum_1$ if and only if $\Phi((\mathcal{N}(\rho(A)I_n - A) \bigcap K) \bigcup C) = K$.*

COROLLARY 6. *Let P be an $n \times n$ nonnegative matrix. A necessary and sufficient condition for $\rho(P) \in \sum_1$ is that every basic class of P is final.*

By definition, $\rho(A) \in \sum_1$ if and only if there exists $x \in \operatorname{int} K$ such that $\rho(A)x - Ax \in K$. For $x \in \operatorname{int} K$, we always have $\rho_x(A) = \rho(A)$. So a related question is the following: Given $A \in \pi(K)$ and $0 \neq x \in K$, when do we have $(\rho_x(A)I_n - A)x \in K$? For the latter question, we have the following answer:

THEOREM 6. *Let $A \in \pi(K)$, and let $0 \neq x \in K$. The following conditions are equivalent:*

(a) $R_A(x) = \rho_x(A)$.

(b) $(\rho_x(A)I_n - A)x \in K$.

(c) *x can be written as $x_1 + x_2$, where $x_1, x_2 \in K$ such that x_1 is an eigenvector of A corresponding to $\rho_x(A)$ and x_2 satisfies $\rho_{x_2}(A) < \rho_x(A)$ and $R_A(x_2) \leq \rho_x(A)$.*

As for Problem 5, we have the following result, from which we deduce a positive answer to Problem 5.

THEOREM 7. *Let $A \in \pi(K)$, let $0 \neq x \in K$, and let $x = x_1 + \cdots + x_k$ be the representation of x as a sum of generalized eigenvectors of A, where $\lambda_1, \ldots, \lambda_k$ are the corresponding distinct eigenvalues. Let Γ denote the set $\{j \in \langle k \rangle : |\lambda_j| = \rho_x(A)$ and $\lambda_j \neq \rho_x(A)\}$. Let m be a positive integer and suppose that $(A - \rho_x(A)I_n)^m x \in K$ and $0 \neq (A - \rho_x(A)I_n)^j x \in K$ for $j = 0, \ldots, m-1$. If $\Gamma = \emptyset$, then $m \leq \operatorname{ord}_A(x)$. If $\Gamma \neq \emptyset$, then $m \leq \operatorname{ord}_A(x) - \max_{j \in \Gamma} \operatorname{ord}_A(x_j)$.*

COROLLARY 7. *Let $A \in \pi(K)$, and let λ be a real number. Then $\lambda < \rho(A)$ if and only if there exists a vector $x \in K$ such that $0 \neq (A - \lambda I_n)^j x \in K$ for all positive integers j.*

3. Self-evaluation of Performance

The project has been carried out pretty well. Except for Problems 4 and 6, which we have not enough time to tackle, we solve all problems in the project, and indeed we have obtained more results than we expected. The solutions of these problems are included in a joint paper with Hans Schneider [T–S 3], a long paper of 56 pages. The paper is being considered for publication in Linear Algebra and Its Applications.

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