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行政院國家科學委員會專題研究計劃成果報告

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* 數值域擁有圓盤對稱或弱對稱性的方陣 *

* On Matrices Whose Numerical Ranges Have Circular *
* or Weak Circular Symmetry *
* *

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計劃類別：☑ 個別計劃 ☐ 整合型計劃

計劃編號：NSC89-2115-M-032-005

執行期間：88 年 8 月 1 日至 89 年 7 月 31 日

個別型計劃：計劃主持人：譚必信
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整合型計劃：總計劃主持人：
子計劃主持人：

註：整合型計劃總報告與子計劃成果報告請分開編印各成一冊，彙整一起繳送國科會。

處理方式：☐可立即對外提供參考

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執行單位：淡江大學數學系
中華民國 89 年 9 月 12 日

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# 摘要

在論文[5]本人曾獲得一複方陣 $A$ 為與某一方塊位移方陣置換相似的數個等價條件，其中一條件為，對每一與 $A$ 擁有相同零型的複方陣 $B$，$B$ 的數值域 $W(B)$ 為以原點為中心的圓盤。在本計劃我們提供一長列新的等價條件；其中有些等價條件須另外假設所考慮的方陣為非負且擁有連通的無向圖。至於數值域的圓盤弱對稱性問題（即 $\exp(2\pi i/m)W(A)=W(A)$ 對某一正整數 $m\geq 2$ 成立）我們亦獲得相對應的結果。此外，我們也解決了何時一非負方陣的數值域為以原點為中心的正規凸多邊形的問題。我們的探討帶來了不少有趣的副產品。特別的，對非負方陣 $A$ 的數值域我們獲得以下兩個意料之外的結果：(i)當 $A$ 的無向圖為連通時，若 $W(A)$ 為以原點為中心的圓盤，則對每一與 $A$ 擁有相同零型的複方陣 $B$，$W(B)$ 也是以原點為中心的圓盤；(ii)當 $A$ 為不可約時，若 $\lambda$ 同時屬 $A$ 的表面譜及 $W(A)$ 的邊界，則 $\lambda$ 為 $W(A)$ 的尖點。我們對不可約非負方陣數值域所獲得的結果是強化或澄清 Issos[2]及 Nylen 與 Tam[4]等人以前在這個題目上的工作。

關鍵詞：數值域、圓盤、圓盤弱對稱性、非負矩陣、連通無向圖、數值半徑、正規凸多邊形、尖點、方塊位移方陣、循數指數、對角相似。

Abstract

In Tam [5], among other equivalent conditions, it is proved that a (square) complex matrix $A$ is permutationally similar to a block-shift matrix if and only if for any complex matrix $B$ with the same zero pattern as $A$, $W(B)$, the numerical range of $B$, is a circular disk centered at the origin. In this project, we add a long list of further new equivalent conditions, some under the additional assumption that the matrix under consideration is a nonnegative matrix with connected undirected graph. The corresponding result for weak circular symmetry of the numerical range (i.e., $\exp(2\pi i/m)W(A) = W(A)$ for some positive integer $m \geq 2$) is also proved. Moreover, we also solve the problem of when the numerical range of a nonnegative matrix is a regular convex polygon with center at the origin. Our investigation also leads to many interesting by-products. In particular, on the numerical range of a nonnegative matrix $A$, the following unexpected results are established: (i) when the undirected graph of $A$ is connected, if $W(A)$ is a circular disk centered at the origin, then so is $W(B)$, for any complex matrix $B$ with the same zero pattern as $A$; (ii) when $A$ is irreducible, if $\lambda$ is an eigenvalue in the peripheral spectrum of $A$ that lies on the boundary of $W(A)$, then $\lambda$ is a sharp point of $W(A)$. We also obtain results on the numerical range of an irreducible nonnegative matrix, which strengthen or clarify the work of Issos [2] and Nylen and Tam [4] on this topic.

Key words: Numerical range, circular disk, weak circular symmetry, nonnegative matrix, connected undirected graph, numerical radius, regular convex polygon, sharp point, block-shift matrix, cyclic index, diagonal similarity.

## 1. Motivation and Aims

The question of when the (classical) numerical range of a complex (square) matrix is a circular disk has been studied by several people. In 1987 Marcus and Pesce [3] first gave a necessary and sufficient condition for a $3 \times 3$ or $4 \times 4$ real strictly upper triangular matrix to have a circular disk centered at the origin as its numerical range. In 1994 Mao-Ting Chien and this author [1] extended the result to the setting of a $3 \times 3$ complex or $4 \times 4$ real upper triangular matrix. In the same paper we also introduced graph-theoretic ideas into the study of circularity of numerical ranges. The following natural question was posed: Given a complex matrix $A$, when is it true that for any complex matrix $B$ with the same digraph as $A$, $W(B)$ (the numerical range of $B$) is a circular disk centered at the origin ? In the same year in [5] this author proved several equivalent conditions for $A$ to have this property. One equivalent condition is that, $A$ is permutationally similar to a block-shift matrix. In early 1999, Professor Shangjun Yang of Anhui University (Mainland China) told this author an interesting characterization of a nonnegative matrix that has a circular disk centered at the origin as its numerical range. His original result was formulated in terms of the digraph of the matrix under consideration. The meanings of his conditions, however, were rather obscure, and moreover his proofs were involved and contained gaps. After a careful study, this author found that the proof could be saved and simplified, if one applied Wielandt's lemma and also used the concept of closed chains. The following unexpected result was obtained: *Suppose $A$ is a nonnegative matrix with connected undirected graph. If $W(A)$ is a circular disk centered at the origin, then so is every matrix $B$ that has the same digraph as $A$.* On the other hand, in a recent study of matrices with cyclic structure (see [6]), this author also obtained the following new equivalent condition for $A$ to be permutationally similar to a block-shift matrix: For each nonzero complex number $\lambda$, $A$ is diagonally similar to $\lambda A$. Moreover, this author also found that if $A$ and $e^{i\theta}A$ are diagonally similar (where $\theta$ is a real number), then they must be unitarily diagonally similar, and as a consequence we have $e^{i\theta}W(A) = W(A)$, and furthermore in this case we also have, $H(A)$ is unitarily diagonally similar to $H(e^{i\theta}A)$, where $H(A)$ denotes the hermitian part of $A$, i.e., $(A + A^*)/2$. The above are concerned with the circularity of numerical ranges; the weak circularity of numerical ranges have not yet been studied. Also, there have been only scanty studies on the numerical range of a nonnegative matrix. Even the simple question of when the numerical range of a doubly stochastic matrix is a regular convex polygon is still unresolved (see [4]). Concerning the circularity or weak circularity of numerical ranges, it seems that there is much room for investigations. This prompts the author to work on the present project.

Below is a list of problems considered in this project:

**Problem 1.** Let $A \in \mathcal{M}_n$. Investigate the logical relations between the following conditions. What can be said if in addition $A$ is nonnegative (or, $A$ is nonnegative and the undirected graph of $A$ is connected) ?

(a) $A$ is permutationally similar to a block-shift matrix.

(b) $G(A)$ is linearly partite.

(c) All cycles of $G(A)$ have zero signed length.

(d) $W(A)$ is a circular disk centered at the origin for all $B \in \mathcal{M}_n$ with the same zero pattern (or ray pattern) as $A$.

(e) $A$ is diagonally similar to $\alpha A$ for all nonzero complex scalars $\alpha$.

(f) For any real number $\theta$, $H(e^{i\theta}A)$ is diagonally similar to $H(A)$.

(g) $W(A)$ is a circular disk centered at the origin.

(h) $e^{i\varphi}r(A) \in W(A)$ for some real number $\varphi$ which is an irrational multiple of $\pi$.

(i) $\lambda_{\max}(H(e^{i\varphi}A)) = \lambda_{\max}(H(A))$ for some real number $\varphi$ which is an irrational multiple of $\pi$.

**Problem 2.** Let $A \in \mathcal{M}_n$, and let $m$ be a given positive integer, $2 \leq m \leq n$. Investigate the logical relations between the following conditions. What can be said if $A$ is nonnegative (and the undirected graph of $A$ is connected)?

(a) $A$ is $m$-cyclic.

(b) $A$ is diagonally similar to $e^{2\pi i/m}A$.

(c) All cycles of $G(A)$ are of signed length an integral multiple of $m$.

(d) For any $B \in \mathcal{M}_n$ with the same zero pattern (or ray pattern) as $A$, $e^{2\pi i/m}W(B) = W(B)$.

(e) For any $B \in \mathcal{M}_n$ with the same zero pattern (or ray pattern) as $A$, $H(B)$ is diagonally similar to $H(e^{2\pi i/m}B)$.

(f) $e^{2\pi i/m}W(A) = W(A)$.

(g) $e^{2\pi i/m}r(A) \in W(A)$.

(h) $\lambda_{\max}(H(e^{2\pi i/m}A)) = \lambda_{\max}(H(A))$.

**Problem 3.** Find a necessary and sufficient condition on a (irreducible) nonnegative matrix $A$ so that $W(A)$ is a regular polygon with center at the origin.

## 2. Results and Discussions

Below is an answer to Problem 1:

**THEOREM 1.** *Let $A \in \mathcal{M}_n, n \geq 2$. The following conditions are equivalent:*

(a) *$A$ is permutationally similar to a block-shift matrix.*

(b) *The digraph of $A$ is linearly partite.*

(c) *All cycles of the digraph of $A$ have zero signed length.*

(d) *$W(B)$ is a circular disk centered at the origin for all $B \in \mathcal{M}_n$ with the same zero pattern (or ray pattern) as $A$.*

(e) *$W(B)$ is a circular disk for all $B \in \mathcal{M}_n$ with the same zero pattern (or ray pattern) as $A$.*

(f) *There is a real number $\varphi$ which is an irrational multiple of $\pi$ or is a rational multiple of the form $2\pi p/q$, where $p, q$ are relatively prime integers with $q > n$ such that $e^{i\varphi}W(B) = W(B)$ for all $B \in \mathcal{M}_n$ with the same zero pattern (or ray pattern) as $A$.*

(g) *$A$ is diagonally similar to $\lambda A$ for all nonzero complex numbers $\lambda$.*

(h) *$A$ is diagonally similar to $e^{i\varphi}A$ for some real number $\varphi$ which is an irrational multiple of $\pi$ or is a rational multiple of the form $2\pi p/q$, where $p, q$ are relatively prime integers with $q > n$.*

(i) *$A$ is diagonally similar to $\lambda A$ for some nonzero complex number $\lambda$ which is not a root of unity.*

(j) *For any $B \in \mathcal{M}_n$ with the same zero pattern (or ray pattern) as $A$, $H(e^{i\theta}B)$ is diagonally similar to $H(B)$ for all real numbers $\theta$.*

(k) *For any real number $\theta$, $H(e^{i\theta}A)$ is diagonally similar to $H(A)$.*

(l) *For any $B \in \mathcal{M}_n$ with the same zero pattern (or ray pattern) as $A$, $H(e^{i\theta}B)$ has the same characteristic polynomial for all real numbers $\theta$.*

(m) *For any $B \in \mathcal{M}_n$ with the same zero pattern (or ray pattern) as $A$, $e^{i\theta}\mathcal{U}(B) = \mathcal{U}(B)$ for all real numbers $\theta$.*

(n) *For any $B \in \mathcal{M}_n$ with the same zero pattern (or ray pattern) as $A$, $e^{i\theta}W_C(B) = W_C(B)$ for all $C \in \mathcal{M}_n$ and all real numbers $\theta$.*

(o) *For any $B \in \mathcal{M}_n$ with the same zero pattern (or ray pattern) as $A$, $e^{i\theta}W_{B^*}(B) = W_{B^*}(B)$ for all real numbers $\theta$.*

(p) *For any $B \in \mathcal{M}_n$ with the same zero pattern (or ray pattern) as $A$, $W_C(B)$ is symmetric about the real axis (or any axis passing through the origin) on the complex plane for all $C \in \mathcal{M}_n$.*

*When $A$ is nonnegative, the following is another equivalent condition:*

(q) *The characteristic polynomial of $H(e^{i\theta}A)$ is the same for all real numbers $\theta$.*

*When $A$ is nonnegative and the undirected graph of $A$ is connected, to the above list of equivalent conditions we can add the following:*

(r) *$W(A)$ is a circular disk centered at the origin.*

(s) *$e^{i\varphi}r(A) \in W(A)$ for some real number $\varphi$ which is an irrational multiple of $\pi$ or is a rational multiple of the form $2\pi p/q$, where $p, q$ are relatively prime integers with $q > n$.*

(t) *$\lambda_{\max}(H(e^{i\varphi}A)) = \lambda_{\max}(H(A))$ for some real number $\varphi$ which is an irrational multiple of $\pi$ or is a rational multiple of the form $2\pi p/q$, where $p, q$ are relatively prime integers with $q > n$.*

(u) *$\rho(H(e^{i\varphi}A)) = \rho(H(A))$ for some real number $\varphi$ which is an irrational multiple of $\pi$ or is a rational multiple of the form $2\pi p/q$, where $p, q$ are relatively prime integers with $q > n$ and $q \not\equiv 2 \pmod 4$.*

As for Problem 2, we have the following answer:

THEOREM 2. *Let $m, n$ be positive integers, $2 \le m \le n$. For any $A \in \mathcal{M}_n$, the following conditions are equivalent:*

(a) *$A$ is diagonally similar to $e^{2\pi i/m}A$.*

(b) *All cycles of the digraph of $A$ are of signed length an integral multiple of $m$.*

(c) *For any $B \in \mathcal{M}_n$ with the same ray pattern as $A$, $e^{2\pi i/m}W(B) = W(B)$.*

(d) *For any $B \in \mathcal{M}_n$ with the same zero pattern as $A$, $H(B)$ is diagonally similar to $H(e^{2\pi i/m}B)$.*

(e) *For any $B \in \mathcal{M}_n$ with the same zero pattern as $A$, $H(B)$ and $H(e^{2\pi i/m}B)$ have the same characteristic polynomial.*

*When the digraph of $A$ has at least one cycle with nonzero signed length, the following are each an additional equivalent condition:*

(f) *For any $B \in \mathcal{M}_n$ with the same zero pattern as $A$, $e^{2\pi i/m}\mathcal{U}(B) = \mathcal{U}(B)$.*

(g) *For any $B \in \mathcal{M}_n$ with the same zero pattern as $A$, $e^{2\pi i/m}W_C(B) = W_C(B)$ for all $C \in \mathcal{M}_n$.*

(h) *For any $B \in \mathcal{M}_n$ with the same zero pattern as $A$, $e^{2\pi i/m}W_{B^*}(B) = W_{B^*}(B)$.*

(i) *$A$ is $m$-cyclic.*

*When $A$ is nonnegative and the undirected graph of $A$ is connected, each of the following is equivalent to conditions* (a)-(e):

(j) $\lambda_{\max}(H(e^{2\pi i/m}A)) = \lambda_{\max}(H(A))$.

(k) $e^{2\pi i/m}W(A) = W(A)$.

(l) $e^{2\pi i/m}r(A) \in W(A)$.

(m) *$H(e^{2\pi i/m}A)$ is diagonally similar to $H(A)$.*

*If, in addition, $m \not\equiv 2 \pmod 4$, the following is another equivalent condition:*

(n) $\rho(H(e^{2\pi i/m}A)) = \rho(H(A))$.

The proofs of Theorems 1 and 2 depend on the following results which have interest of their own.

We denote the *numerical radius* of $A$ by $r(A)$, i.e., $r(A) = \max\{|w| : w \in W(A)\}$.

LEMMA 1. *Let $A$ be an $n \times n$ nonnegative matrix with connected undirected graph. Let $\varphi$ be a real number such that $e^{i\varphi} \neq 1$, and suppose $e^{i\varphi}r(A) \in W(A)$.*

(i) *If $\varphi$ is an irrational multiple of $\pi$, then $A$ is permutationally similar to a block-shift matrix.*

(ii) *If $\varphi$ is a rational multiple of $\pi$, say $\varphi = 2\pi p/q$, where $p, q$ are relatively prime integers, $q$ being positive, then all cycles of $G(A)$ are of signed length an integral multiple of $q$.*
*In any case, $A$ is diagonally similar to $e^{i\varphi}A$.*

LEMMA 2. *Let $A = (a_{rs}) \in \mathcal{M}_n$, and let $\varphi$ be a given real number. Suppose $a_{rs}a_{sr} = 0$ for all $r, s \in \langle n \rangle$. If $H(A)$ is diagonally similar to $H(e^{i\varphi}A)$, then $A$ is diagonally similar to $e^{i\varphi}A$.*

LEMMA 3. *For a nonnegative matrix $A$ and any given real number $\varphi$, we have $\lambda_{\max}(H(e^{i\varphi}A)) = \lambda_{\max}(H(A))$ if and only if $e^{-i\varphi}r(A) \in W(A)$ if and only if $e^{i\varphi}r(A) \in W(A)$. When $A$ is a nonnegative matrix with connected undirected graph, each of the following is an additional equivalent condition:*

(a) *$A$ is diagonally similar to $e^{i\varphi}A$.*

(b) *$e^{i\varphi}W(A) = W(A)$.*

*If, in addition, $\varphi$ is an irrational multiple of $\pi$ or is a rational multiple of the form $2\pi p/q$, where $p, q$ are relatively prime integers, $q$ being positive such that $q \not\equiv 2 \pmod 4$, then the following is another equivalent condition:*

(c) *$\rho(H(e^{i\varphi}A)) = \rho(H(A))$.*

LEMMA 4. *Let $\varphi$ be a real number such that $e^{i\varphi} \neq 1$. For any $A \in \mathcal{M}_n$, $n \geq 2$, the following conditions are equivalent:*

(a) *$A$ is diagonally similar to $e^{i\varphi}A$.*

(b) *For any $B \in \mathcal{M}_n$ with the same zero pattern (or ray pattern) as $A$, $e^{i\varphi}W(B) = W(B)$.*

(c) *For any $B \in \mathcal{M}_n$ with the same zero pattern as $A$, $H(B)$ is diagonally similar to $H(e^{i\varphi}B)$.*

(d) *For any $B \in \mathcal{M}_n$ with the same zero pattern as $A$, $H(B)$ and $H(e^{i\varphi}B)$ have the same characteristic polynomial.*

By the *index of imprimitivity* of an irreducible nonnegative matrix $A$, we mean the number of eigenvalues of $A$ of maximal modulus. By Lemma 3 we readily obtain, as a by-product, the following related result which strengthens the work of Nylen and Tam [14, Corollaries 1.5 and 1.6] on the numerical range of an irreducible doubly stochastic matrix.

COROLLARY 1. *Let $A$ be a nonnegative matrix with connected undirected graph. If the digraph of $A$ has at least one cycle with nonzero signed length, then the cyclic index of $A$ is equal to the largest positive integer $m$ that satisfies $e^{2\pi i/m}W(A) = W(A)$. If, in addition, $A$ is irreducible, then this common value is also equal to the index of imprimitivity of $A$.*

We also obtain the following reformulataed unpublished main result of Issos [2, Theorem 7].

COROLLARY 2. *Let $A \in \mathcal{M}_n$, $n \geq 2$, be an irreducible nonnegative matrix with index of imprimitivity $h$. For any complex number $\lambda \in W(A)$, $|\lambda| = r(A)$ if and only if $\lambda$ is one of the $h$ numbers $r(A)$, $r(A)e^{2\pi i/h}, \ldots,$ $r(A)e^{2\pi(h-1)i/h}$.*

As for Problem 3, we have the following answer:

THEOREM 3. *Let $A \in \mathcal{M}_n$ be a nonnegative matrix, $n \geq 2$. Let $P$ be a permutation matrix such that $P^T AP = A_1 \oplus \cdots \oplus A_k$, where $A_1, \ldots, A_k$ are square matrices each with a connected undirected graph. Then $W(A)$ is a regular convex polygon with center at the origin if and only if there exists $j$, $1 \leq j \leq k$, such that $W(A_j)$ is a regular convex polygon with center at the origin and $W(A_j)$ includes $W(A_l)$ for all $l \neq j$, $1 \leq l \leq k$.*

In view of Theorem 3, a relevant problem is when the numerical range of a nonnegative matrix with connected undirected graph is a regular convex polygon with center at the origin.

To answer this, we obtain first the following interesting Lemma 5 and then Theorem 4.

LEMMA 5. *Let $A$ be an irreducible nonnegative matrix. If $\lambda$ is an eigenvalue in the peripheral spectrum of $A$ that lies on the boundary of $W(A)$, then $\lambda$ is a sharp point of $W(A)$.*

THEOREM 4. *Let $A \in \mathcal{M}_n$ be a nonnegative matrix, $n \geq 2$. Let $P$ be a permutation matrix such that $P^T AP = A_1 \oplus \cdots \oplus A_k$, where $A_1, \ldots, A_k$ are square matrices each with a connected undirected graph. A necessary and* *sufficient condition for $r(A)$ to be a sharp point of $W(A)$ is that, for any $j$, $1 \leq j \leq k$, we have*

(i) *If $\rho(A_j) = \rho(A)$, then $A_j$ is a maximal irreducible principal submatrix of $A$ and $\rho(A_j) = r(A_j)$; and*

(ii) *If $\rho(A_j) < \rho(A)$, then $r(A_j) < \rho(A)$.*

By Theorem 4 we have the following answer to the question we posed above.

**Remark 1.** If $A$ is a nonnegative matrix with connected undirected graph such that $W(A)$ is a regular convex polygon with center at the origin, then $A$ is necessarily an irreducible matrix.

Now we have

THEOREM 5. *Let $A$ be an irreducible nonnegative matrix with index of imprimitivity $m$. In order that $A$ is a regular convex polygon with center at the origin, it is necessary and sufficient that the following conditions are both satisfied:*

(a) $\rho(A) = \rho(H(A))$.

(b) *For* $t = 1, \ldots, m,\ \lambda_{\max}(H(e^{-(2t+1)\pi i/m}A)) = \rho(A)\cos(\pi/m)$.

Based on Theorem 3, Remark 1, Theorem 5 and the following Remark 2, one can readily construct an algorithm to determine when the numerical range of a nonnegative matrix is a regular convex polygon with center at the origin.

**Remark 2.** Let $A$ be a square matrix. Let $\rho$ be positive real number and let $m \geq 2$ be a positive integer. In order that

$$W(A) \subseteq \operatorname{conv}\{\rho e^{2\pi t i/m} : t = 1, \ldots, m\},$$

it is necessary and sufficient that for $t = 1, \ldots, m$, we have

$$\lambda_{\max}(H(e^{-(2t+1)\pi i/m}A)) \leq \rho\cos(\pi/m).$$

## 3. Self-evaluation of Performance

The project has been carried out with a great success. Not only are all the problems solved, but also we have obtained many interesting unexpected by-products. The solutions of Problems 1 and 3 have already appeared in [7], a joint paper of this author with S. Yang. The solutions of Problem 3 and other related results will appear in a future paper.

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