行政院國家科學委員會補助專題研究計畫成果報告

※※※※※※※※※※※※※※※※※※※※※※※※※

※　　　　　　　　　　　　　　　　　　　　　　　※

※　　不等變異數時檢定排序對立的全距檢定　　※

※　　　　　　　　　　　　　　　　　　　　　　　※

※※※※※※※※※※※※※※※※※※※※※※※※※

計畫類別：☒個別型計畫　　☐整合型計畫

計畫編號：NSC　89－2118－M－032－014－

執行期間：89 年 8 月 1 日　至　90 年 7 月 31 日

計畫主持人：陳順益

共同主持人：

研究助理：莊麗足 游振汎 高豐渝

本成果報告包括以下應繳交之附件：

- ☐赴國外出差或研習心得報告一份
- ☐赴大陸地區出差或研習心得報告一份
- ☒出席國際學術會議心得報告及發表之論文各一份
- ☐國際合作研究計畫國外研究報告書一份

執行單位：淡江大學數學系

中 華 民 國　90　年　10　月　30　日

ONE-SIDED RANGE TEST FOR TESTING AGAINST AN ORDERED ALTERNATIVE UNDER HETEROSCEDASTICITY

Shun-Yi Chen
Department of Mathematics, Tamkang University, Tamsui,Taiwan 251

Key Words: One-stage; two-stage; power; critical points.

ABSTRACT

In a one-way fixed effects analysis of variance model, when normal variances are unknown and possibly unequal, a one-sided range test for testing the null hypothesis $H_0 : \mu_1 = \ldots = \mu_k$ against an ordered alternative $H_a : \mu_1 \leq \ldots \leq \mu_k$ by a single-stage and a two-stage procedure, respectively, is proposed. The critical values under H_0 and the power under a specific alternative are calculated. Relation between the one-stage and the two-stage test procedures is discussed. A numerical example to illustrate these procedures is given.

1 Introduction

Suppose that $k(\geq 2)$ independent populations $\pi_1, \ldots, \pi_k$ are available where observations taken from population π_i are normally distributed with mean μ_i and variance $\sigma_i^2 (1 \leq i \leq k)$. When variances, $\sigma_1^2, \ldots, \sigma_k^2$, are equal (known or unknown), Hayter (1990) and Hayter and Liu (1996) proposed a one-sided studentized range test for testing the equality of means, $H_0 : \mu_1 = \ldots = \mu_k$, against an ordered alternative, $H_a : \mu_1 \leq \ldots \leq \mu_k$, with at least one strict inequality. In this paper, we propose a new range test for the same purpose under the situations where the variances are unknown and possibly unequal using a general two-stage and a one-stage sampling procedure.

The procedures of testing the equality of means in the conventional analysis of variance (ANOVA) are based on the assumptions of normality, independence, and equality of the error variances. Studies have shown that the distribution of Tukey's studentized range test depends heavily on the unknown variances and is not robust under the violation of equal error variances, especially if the sample sizes are not equal. (See Bishop, 1979). As pointed out by Bishop and Dudewicz (1981) that "in practice the assumption of equal error variances is often unjustified, and at the same time there is no exact theory to handle the case of heteroscedasticity. Historically transformations of the data, for example, logarithm or arcsine transformations, have been used ..., they are only approximate in terms of equal variances, normality, and model specification. These approximations, inherent in the transformation technique, lead to test in the ANOVA setting, have only approximate level and power. Finally, if the errors associated with the original observations are normal, the errors associated with the transformed observations will not be normal." Furthermore, the transformed data may lose its practical meaning, and sometimes, the acceptance of the equality of means using transformed data does not automatically imply the acceptance of the equality of means in the original scale. When the variances are unknown and unequal, Bishop and Dudewicz (1978) developed an exact analysis of variance for testing the equality of the means of k independent normal populations by using a two-stage procedure. The two-stage procedure is a design-oriented procedure which requires additional samples in order to meet the power requirement, and the sample sizes can be large at the second stage. It may not be practical in some real problems when the time and/or budget are limited in an experiment. When working with statistical data analysis one often has only one single sample available. Chen and Lam (1989), Wen and Chen (1994) and Chen and Chen (1998) developed a one-stage method for some statistical inference problems. The one-stage procedure also has an exact distribution for its test statistic and it is a data analysis-oriented method. The one-stage procedure provides a feasible alternative to the two-stage procedure when the experiment is terminated earlier and its required sample sizes are not met due to budget shortage, time limitation, or cost factors. We first propose the two-stage procedure and then the one-stage procedure. Finally, their relationship is discussed. Statistical tables of the critical values and the power-related design constants to implement these procedures are given in Tables 1-2. A numerical example is also given to illustrate the use of the range test.

2 Two-Stage Range Test

Consider the one-way fixed effects analysis of variance (ANOVA) model

$$X_{ij} = \mu_i + \epsilon_{ij}, \quad 1 \le j \le n_i, \quad 1 \le i \le k, \quad (1)$$

where the ϵ_{ij} are independent and normally distributed random variables with mean zero and variance σ_i^2, denoted by $N(0, \sigma_i^2)$, $i = 1, \ldots, k$, and μ_i is the mean of the ith treatment. Both μ_i and σ_i^2 are unknown parameters. The considerable interest is to test the null hypothesis of equality of means,

$$H_0 : \mu_1 = \ldots = \mu_k$$

against the ordered alternative hypothesis

$$H_a : \mu_1 \le \ldots \le \mu_k$$

with at least one strict inequality.

The two-stage sampling procedure ($\boldsymbol{P_2}$) (see Bishop and Dudewicz (1978)) for testing H_0 against H_a by a new range statistic is given as follows:

$\boldsymbol{P_2}$: Choose a number $z > 0$ (z is determined by the power of the test), and take an initial sample size n_0 (at least 2, but 10 or more will give better results) from each of the k populations. For the ith population let S_i^2 be the usual unbiased estimate of σ_i^2 based on the initial n_0 observations, and define

$$N_i = \max\left\{ n_0 + 1, \left[\frac{S_i^2}{z}\right] + 1 \right\} \quad (2)$$

where $[x]$ denotes the greatest integer less than or equal to x. Then, take $N_i - n_0$ additional observations from the ith population so we have a total of N_i observations denoted by $X_{i1}, \ldots, X_{in_0}, \ldots, X_{iN_i}$. For each i, set the coefficients $a_{i1}, \ldots, a_{in_0}, \ldots, a_{iN_i}$, so that

$$a_{i1} = \ldots = a_{in_0} = \frac{1 - (N_i - n_0)b_i}{n_0} = a_i,$$

$$a_{i,n_0+1} = \ldots = a_{iN_i} = \frac{1}{N_i}\left[1 + \sqrt{\frac{n_0(N_i z - S_i^2)}{(N_i - n_0)S_i^2}}\right] = b_i,$$

and then compute the weighted mean

$$\tilde{X}_i = a_i \sum_{j=1}^{n_0} X_{ij} + b_i \sum_{j=n_0+1}^{N_i} X_{ij}$$

which is a linear combination of the first-stage data $(X_{i1}, \ldots, X_{in_0})$ and the second-stage data $(X_{in_0+1}, \ldots X_{iN_i})$. The motivation of the choice of N_i in (2) is based on the following rationale: The above coefficients a_{ij}'s are so chosen in such a way that

$$(a) \sum_{j=1}^{N_i} a_{ij} = 1, (b) a_{i1} = \ldots = a_{in_0}, (c) S_i^2 \sum_{j=1}^{N_i} a_{ij}^2 = z.$$

This is surely possible since

$$\min \sum_{j=1}^{N_i} a_{ij}^2 = \frac{1}{N_i} \le z/S_i^2 \quad \text{by (2)},$$

the minimum being taken subject to the conditions (a) and (b). Furthermore, condition (a) is to ensure the unbiasedness of $\tilde{X}_i$ for μ_i, (b) guarantees that the sample mean $\bar{X}_i$ and sample variance S_i^2 based on the first stage n_0 observations are independent, and (c) is the variance estimate of $\tilde{X}_i$ controlled at a power-specified value z which guarantees that the r.v.'s $T_i = (\tilde{X}_i - \mu_i)/\sqrt{z}$, $i = 1, \ldots, k$, have i.i.d. t distributions with $\nu = n_0 - 1$ degrees of freedom (df), free of the unknown variances. (See, e.g., Dudewicz and Dalal, 1975).

A one-sided range test for testing the null hypothesis H_0 against the ordered alternative H_a is proposed below. Consider the test statistic

$$T = \max_{1 \le i < j \le k} (\tilde{X}_j - \tilde{X}_i)/\sqrt{z}, \quad (3)$$

which rejects the null hypothesis H_0 iff

$$T > c_{\alpha,k,\nu},$$

where $c_{\alpha,k,\nu}$ is the critical value such that this test has size exactly equal to α under H_0.

The test statistic T can be rewritten as

$$\begin{aligned} T &= \max_{1 \le i < j \le k} \left(\frac{\tilde{X}_j - \mu_j}{\sqrt{z}} - \frac{\tilde{X}_i - \mu_i}{\sqrt{z}} + \frac{\mu_j - \mu_i}{\sqrt{z}} \right) \\ &= \max_{1 \le i < j \le k} (T_j - T_i + (\mu_j - \mu_i)/\sqrt{z}) \quad (4) \end{aligned}$$

where $T_1, \ldots, T_k$ are i.i.d. t r.v.'s with ν df. Then the critical values $c_{\alpha,k,\nu}$ can be evaluated under H_0 using the equation

$$Pr\{\max_{1 \le i < j \le k} (T_j - T_i) < c_{\alpha,k,\nu}\} = 1 - \alpha. \quad (5)$$

The range test (3) can be inverted to produce the following $(1-\alpha)$-level simultaneous one-sided confidence intervals for the ordered pairwise differences of the treatment means $\mu_j - \mu_i (1 \le i < j \le k)$:

$$Pr\{\mu_j - \mu_i \ge \tilde{X}_j - \tilde{X}_i - \sqrt{z} c_{\alpha,k,\nu}; 1 \le i < j \le k\} = 1 - \alpha,$$

which are often of the most interest to an experimenter.

The critical values $c_{\alpha,k,\nu}$ of the null distribution of T in (5) can be obtained from a short SAS (Version 6.12, 1990) computer simulation program given by Chen and Chen (1999). The program can be run on a Pentium II personal computer with a SAS PC software. For selected number of populations k and df ν, k independent t random variates, $t_1, \dots, t_k$ were generated by the formula $t = Y/\sqrt{U/\nu}$, where Y is the standard normal random variate generated from the random number generator RANNOR and U is the independent chi-squared random variate with ν degrees of freedom generated by the gamma random number generator RANGAM in SAS 6.12 (SAS Institute Inc., 1990). Then the value of $Q = \max_{1 \le i < j \le k}(t_j - t_i)$ was computed for each simulation run. After 10,000 simulation runs, all of the Q values were ranked in ascending order. The $90th, 95th$ and $99th$ percentiles were used to estimate the upper α percentage points $10\%, 5\%$ and 1%, respectively. This process was replicated 16 times. The average values of the 16 critical points and their corresponding standard errors (in the parentheses) are reported in Table 1 for $k = 3, 4, 6, 10$, and df $\nu = 4, 6, 9, 14, 19, 24, 29, 59, \infty$. The extended tables were given by Chen and Chen (1999).

The power is calculated by the expression

$$\beta = Pr\{T > c_{\alpha,k,\nu} | \boldsymbol{\theta}_0\}$$

for given values of k, α, ν and the ratio $\delta/\sqrt{z}$, where $\boldsymbol{\theta}_0 = (-\delta/2\sqrt{z}, 0, \dots, 0,$
$\delta/2\sqrt{z})$. (Since this is the least favorable configuration of means for the power of the range test subject to the restrictions $\mu_k - \mu_1 > \delta$ and $\mu_1 \le \dots \le \mu_k$ for normal distribution, see Hayter (1990), and hence the asymptotically least favorable configuration of means for the power of the range test for the t distribution.) For each k, ν, α and the ratio $\delta/\sqrt{z}$, k independent t random variates $t_1, \dots, t_k$ were generated as described above. The statistic of T was calculated as in (4) at $\boldsymbol{\theta}_0$. This process was repeated 20,000 times and the power was estimated by

$$\beta \cong \frac{\text{No. of times } (T > c_{\alpha,k,\nu})}{20,000}. \qquad (6)$$

The values of the ratio $\delta/\sqrt{z}$ such that a size α test has the power β are given in Table 2 for $k = 3(1)6(2)10, 15, 20$. $\nu = 4, 9, 14, 24$, $\alpha = .10, .05, .01$, and power=.80, .90, .95. An example of how to use these Tables is illustrated as follows: If one has $k = 5$ treatments in the experiment, and the initial sample available is $n_0 = 15$ observations (df $\nu = 14$), at the price of $\alpha = 10\%$ risk; he/she would like to detect a difference of at least $\delta = 3.0$ with a required power of .95. From Table 2, the ratio $\delta/\sqrt{z} = 5.58$ can be found corresponding to the required power .95. Then, the design constant is found to be $z = (\delta/5.58)^2$ or $z = .289$ which will be employed in (2) to determine the total sample size N_i in the experiment. Simulation study shows that linear interpolation in $\delta/\sqrt{z}$ would give satisfactory results for values of power being not tabulated. The extended tables for the power of the test were given by Chen and Chen (1999).

3 One-Stage Range Test

When the two-stage procedure cannot meet the required sample sizes in (2) due to some shortfalls, the one-stage procedure provides a feasible solution by recalculating the weighted sample means using new weights. We now describe the general one-stage sampling procedure ($\boldsymbol{P}_1$) in the context of the one-way layout (1).

$\boldsymbol{P}_1$: Let $X_{ij} (j = 1, \dots, n_i)$ be an independent random sample of size $n_i (\ge 3)$ on hand where X_{ij} have normal population $\pi_i (i = 1, \dots, k)$ with unknown mean μ_i and unknown and unequal variance σ_i^2. Employ the first (or randomly chosen) $n_0 (2 \le n_0 < n_i)$ observations to calculate the usual sample mean and sample variance, respectively, by

$$\bar{X}_i = \sum_{j=1}^{n_0} X_{ij}/n_0$$

and

$$S_i^2 = \sum_{j=1}^{n_0} (X_{ij} - \bar{X}_i)^2/(n_0 - 1).$$

Let the weights of the observations be

$$U_i = \frac{1}{n_i} + \frac{1}{n_i}\sqrt{\frac{n_i - n_0}{n_0}(n_i z^*/S_i^2 - 1)}$$
$$V_i = \frac{1}{n_i} - \frac{1}{n_i}\sqrt{\frac{n_0}{n_i - n_0}(n_i z^*/S_i^2 - 1)}$$

where z^* is the maximum of $\{S_i^2/n_1, \dots, S_k^2/n_k\}$. Let the final weighted sample mean using all observations be defined by

$$\tilde{X}_i = \sum_{j=1}^{n_i} W_{ij} X_{ij} \qquad (7)$$

where

$$W_{ij} = \begin{cases} U_i & \text{for } 1 \le j \le n_0 \\ V_i & \text{for } n_0 < j \le n_i \end{cases} \qquad (8)$$

and U_i and V_i satisfy the following conditions

$$n_0 U_i + (n_i - n_0) V_i = 1,$$

$$[n_0 U_i^2 + (n_i - n_0) V_i^2] S_i^2 = z^*.$$

When dealing with a statistical data analysis where the data are already available on hand, Chen and Lam (1989) suggested taking n_0 to be $n_i - 1$, because such a choice can push the weights U_i and V_i to be as close to $1/n_i$ as possible where $1/n_i$ in traditional way is a desirable coefficient for calculating the sample estimate of population mean. Another reason of choosing $n_0 = n_i - 1$ is that it is optimal in the sense that the Student's t distribution has the smallest critical value (among $n_0 \le n_i - 1$) for a fixed level of probability. In situations where the one-stage procedure is used to repair the two-stage procedure when the later one cannot complete its required sampling process (see Section 4), one may choose n_0 to be between 10 and 15 as the two-stage procedure does at the initial stage (Bishop and Dudewicz (1978)).

Given the sample variances S_i^2, $i = 1, \ldots, k$, the weighted sample mean $\tilde{X}_i$ has a conditional normal distribution with mean μ_i and variance $\Sigma_j W_{ij}^2 \sigma_i^2$. Furthermore, the transformation

$$T_i = \frac{\tilde{X}_i - \mu_i}{\sqrt{S_i^2 \sum_{j=1}^{n_i} W_{ij}^2}}$$

has a conditional normal distribution with mean zero and variance σ_i^2 / S_i^2.

It is easy to see that the conditional normal distributions of T_i given S_i^2, $i = 1, \ldots, k$, are unconditional and independent Student's t distributions each with $\nu = n_0 - 1$ df using a similar derivation by Chen and Chen (1998). That is, the random variables

$$T_i = \frac{\tilde{X}_i - \mu_i}{\sqrt{z^*}}, \quad i = 1, \ldots, k$$

are distributed as independent Student's t with ν df.

Assume that for $i = 1, \ldots, k$, the one-stage sampling procedure has been conducted and that the final weighted sample means $\tilde{X}_i$ have been computed as in (7). A new range test statistic for testing H_0 against the ordered alternative H_a is given by

$$T = \max_{1 \le i < j \le k} (\tilde{X}_j - \tilde{X}_i) / \sqrt{z^*}. \qquad (9)$$

The null hypothesis H_0 is rejected in favor of H_a iff

$$T \ge c_{\alpha,k,\nu}$$

where $c_{\alpha,k,\nu}$ is the critical value of size α satisfying

$$Pr(\max_{1 \le i < j \le k} (\tilde{X}_j - \tilde{X}_i)/\sqrt{z^*} \ge c_{\alpha,k,\nu} | H_0) = \alpha \qquad (10)$$

with k treatments and ν df. The critical value $c_{\alpha,k,\nu}$ is the solution to

$$Pr\{\max_{1 \le i < j \le k} (T_j - T_i) > c_{\alpha,k,\nu}\} = \alpha. \qquad (11)$$

which is the same as in Section 2.

The range (9) can be inverted to produce the following $(1-\alpha)$-level simultaneous one-sided confidence intervals for the ordered pairwise differences of the treatment means $\mu_j - \mu_i (1 \le i < j \le k)$:

$$\mu_j - \mu_i \ge \tilde{X}_j - \tilde{X}_i - \sqrt{z^*} c_{\alpha,k,\nu}, 1 \le i < j \le k$$

which are often of the most interest to an experimenter. The estimated power of the test (9) can be found in Table 2 for a specified alternative hypothesis with $\mu_k - \mu_1 = \delta$.

4 Discussion on Two-Stage and One-Stage

The two-stage test procedure can control both the level and the power of the test without having the influence of the unequal and unknown variances. It is an important design-oriented statistical method used in an experiment. However, in situations where the experiment is terminated earlier due to budget restriction, time limitation, or some other uncontrollable factors, the required sample size N_i in (2) in the two-stage procedure cannot be reached. Then the two-stage test procedure cannot work. One may have to, in this situation, use the available sample data based on the sample size $n_i (\ge n_0 + 1)$ on hand and recalculate weighted sample mean using the coefficients W_{ij} in (8) according to the one-stage sampling procedure described in Section 3. Thus, given the sample data, the estimated minimum power can be determined by, letting $z^* = \max_{1 \le j \le k}(S_j^2 / n_j)$,

$$Pr\{\max_{1 \le i < j \le k} (\tilde{X}_j - \tilde{X}_i) > \sqrt{z^*} c_{\alpha,k,\nu} | H_a\}. \qquad (12)$$

The power so determined is a data-dependent power which could be larger than, equal to, or smaller than the originally specified one by the two-stage procedure. This is elaborated as follows: If the sample size $n_i > n_0 + 1$, $i = 1, \ldots, k$, were given by the one-stage procedure and

Case 1. If $S_i^2/n_i = S_j^2/n_j$ for all i, j, then the two-stage and one-stage procedures have the same power due to $S_i^2/n_i = z$, and $z = z^*$. The power can be calculated by (12) or approximated by Table 2.

Case 2. If $z^* = \max_{1 \le j \le k}(S_j^2/n_j) < z$, then the one-stage test procedure has a power larger than that of the two-stage test procedure. A smaller z^*-value results in a larger sample size n_i than the required one by the two-stage one.

Case 3. If $\min_{1 \le j \le k}(S_j^2/n_j) > z$, then the power of the one-stage test procedure is smaller than that of the two-stage test procedure.

Case 4. In all other situations, the one-stage test procedure could have power larger than, smaller than, or equal to the two-stage depending on the actual samples and the true population variances.

5 A Numerical Example

Bishop and Dudewicz (1978) studied the data of bacterial killing ability of four solvents (Tables 3 and 4) using the two-stage sampling procedure. There were four types of solvents which can affect the ability of the fungicide methyl-2-benzimidazole-carbamate to destroy the fungus Penicillium expansum. For the purpose of illustration, we assumed that an active ingredient of the ascending levels were included in solvents I, II, III and IV. The fungicide was diluted in exactly the same manner in four different types of solvents and sprayed on the fungus. The percentage of fungus destroyed was measured and recorded. In their first stage of experiment $n_i = 15$ observations were run with each solvent. Wen and Chen (1994) have shown a significant difference among the variances by using Bartlett's χ^2 test for equality of variance. Suppose the aim of the experiment is to test the hypothesis that the mean percentages of fungus destroyed are all equal, $H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4$ against a simple ordered alternative, $H_a : \mu_1 \le \mu_2 \le \mu_4 \le \mu_4$, where μ_i denotes the mean percentage of fungus destroyed by solvent i. If the experimenter decides the level of the test to be 5% for testing H_0 with a power of at least .85 for a difference of $\mu_{\max} - \mu_{\min} \ge \sqrt{2}(= \delta)$. From Tables 1 and 2, the critical value $c = 3.63$ and $\delta/\sqrt{z} = 4.97$ can be found corresponding to the power of .85, so $z = .081$. The intermediate statistics S_i^2, a_i, b_i, and N_i which are necessary for the calculation of the final weighted means are given at the bottom of Table 3. The final sample sizes (N_i) required for each solvent were 40, 73, 16 and 27. The remaining $N_i - 15$ observations taken at the second stage are given in Table 4. The final weighted sample means are found to be $\tilde{X}_1 = 95.274, \tilde{X}_2 = 94.761, \tilde{X}_3 = 97.498$, and $\tilde{X}_4 = 97.593$. We found the test statistic $T = 9.95$, which exceeds the critical point of 3.63, so H_0 is rejected in favor of H_a with a power of at least 85% to detect a difference of $\sqrt{2}$ between the largest and the smallest means.

If the experiment at the second stage is terminated earlier due to some uncontrollable factors, and we can only obtain smaller samples from each treatment, say, $N_1 = 34, N_2 = 64, N_3 = 16$, and $N_4 = 23$. We can apply the one-stage procedure with those observations available from each population. For this case we use the same number of observations used by Bishop and Dudewicz (1978), $n_0 = 15$ for initial estimation and the remaining for use in the final computation. The intermediate statistics S_i^2, U_i, V_i, and z^* are given at the bottom of Table 4. The final weighted sample means are $\tilde{X}_1 = 94.981, \tilde{X}_2 = 94.957, \tilde{X}_3 = 97.225$, and $\tilde{X}_4 = 97.311$. We found the test statistic $T = 7.71$, which exceeds the 5% critical value $c = 3.63$, so H_0 is rejected in favor of H_a subject to $\mu_{\max} - \mu_{\min} \ge \sqrt{2}$. Since the required sample sizes by the two-stage procedure were not met, we have obtained the estimated power being .80 according to $z^* = .0933$ by using Table 2 ($\delta/\sqrt{z^*} = 4.63$ corresponding to a power of .80). From this example we can clearly see that the one-stage procedure can both theoretically and practically provide a feasible solution to handle the difficulty of the two-stage procedure when an experiment is unexpectedly terminated earlier.

6 Acknowledgements

The author wishes to express his sincere thanks to his former major Professor, Dr. Hubert J. Chen at the University of Georgia, for his suggestions and careful examination into the statistical logic and mathematical derivations. The author also would like to thank an associate editor, the editor and referees for their valuable suggestions and comments which improved the applicability and readability of the manuscript.

BIBLIOGRAPHY

Bishop, T. A. (1979). Some results on simultaneous inference for analysis of variance with unequal variances. *Technometrics*, 21, 337-340.

Bishop, T. A. and Dudewicz, E. J. (1978). Exact analysis of variance with unequal variances: test procedures and tables. *Technometrics*, 20, 419-430.

Bishop, T. A. and Dudewicz, E. J. (1981). Heteroscedastic ANOVA. *Sankhya*, 43B, 40-57.

Chen, S. Y. and Chen, H. J. (1998). Single-stage analysis of variance under heteroscedasticity. *Communications in Statistics - Simulation and Computation*, 27(3), 641-666.

Chen, H.J. and Chen, S. Y. (1999). One-sided range test for testing against an ordered alternative under heteroscedasticity. Technical Report 99-5, Department of Statistics, University of Georgia.

Chen, H. J. and Lam, K. (1989). Single-stage interval estimation of the largest normal mean under heteroscedasticity. *Communications in Statistics - Theory and Methods*, 18(10), 3703-3718.

Dudewicz, E.J. and Dalal, S.R. (1975). Allocation of observations in ranking and selection with unequal variances. *Sankhya, 37B*, 28-78.

Hayter, A. J. (1990). A one-sided studentized range test for testing against a simple ordered alternative. *J. Ameri. Statist. Assoc.*, 85, 778-785.

Hayter, A. J. and Liu, W. (1996). Exact calculations for the one-sided studentized range test for testing against a simple ordered alternative. *Computational Statistics and Data Analysis*, 22, 17-25.

Wen, M. J. and Chen, H. J. (1994). Single-stage multiple comparison procedures under heteroscedasticity. *American Journal of Mathematical and Management Sciences*, 14, 1-48.

Table 1. The Average of 16 Critical Values $c_{\alpha,k,\nu}$ of
$$Q = \max_{1 \le i < j \le k} (t_i - t_j)$$
and Their Standard Errors in Parentheses.

k	ν	10%	5%	1%
3	4	3.32(.01)	4.19(.01)	6.45(.03)
3	6	2.98(.01)	3.67(.01)	5.21(.03)
3	9	2.80(.01)	3.38(.01)	4.66(.01)
3	14	2.68(.01)	3.21(.01)	4.29(.02)
3	19	2.62(.01)	3.13(.01)	4.15(.02)
3	24	2.60(.01)	3.10(.01)	4.07(.01)
3	29	2.58(.01)	3.06(.01)	4.01(.01)
3	59	2.53(.01)	2.99(.01)	3.89(.01)
3	∞	2.49(.00)	2.94(.01)	3.79(.01)
4	4	3.95(.01)	4.83(.01)	7.29(.05)
4	6	3.51(.01)	4.19(.01)	5.78(.02)
4	9	3.26(.01)	3.84(.01)	5.07(.02)
4	14	3.12(.01)	3.63(.01)	4.70(.02)
4	19	3.04(.01)	3.52(.01)	4.52(.02)
4	24	3.00(.01)	3.47(.01)	4.43(.01)
4	29	2.98(.01)	3.44(.01)	4.36(.01)
4	59	2.93(.01)	3.37(.01)	4.23(.01)
4	∞	2.88(.01)	3.31(.01)	4.13(.01)
6	4	4.85(.01)	5.84(.01)	8.56(.05)
6	6	4.21(.01)	4.93(.01)	6.59(.02)
6	9	3.87(.01)	4.45(.01)	5.67(.02)
6	14	3.65(.01)	4.14(.01)	5.19(.02)
6	19	3.55(.01)	4.02(.01)	4.99(.02)
6	24	3.52(.01)	3.96(.01)	4.86(.01)
6	29	3.48(.01)	3.91(.01)	4.79(.01)
6	59	3.41(.01)	3.81(.01)	4.62(.01)
6	∞	3.33(.01)	3.72(.01)	4.47(.01)
10	4	5.99(.01)	7.12(.02)	10.27(.06)
10	6	5.05(.01)	5.77(.01)	7.58(.03)
10	9	4.57(.01)	5.15(.01)	6.42(.02)
10	14	4.27(.01)	4.76(.01)	5.81(.02)
10	19	4.15(.01)	4.59(.01)	5.51(.01)
10	24	4.08(.01)	4.50(.01)	5.36(.01)
10	29	4.04(.01)	4.45(.01)	5.30(.01)
10	59	3.93(.01)	4.31(.01)	5.08(.01)
10	∞	3.83(.01)	4.19(.01)	4.90(.01)

Table 2. The Power-Related Ratio $\delta/\sqrt{z}$ of the One-Sided Range Test

ν	4			9			14			24		
α	10%	5%	1%	10%	5%	1%	10%	5%	1%	10%	5%	1%
power						$k=3$						
.80	4.47	5.42	7.83	3.89	4.52	5.90	3.69	4.30	5.47	3.58	4.17	5.21
.90	5.32	6.29	8.70	4.59	5.22	6.60	4.41	4.98	6.17	4.24	4.83	5.87
.95	6.00	7.07	9.48	5.19	5.83	7.20	4.98	5.53	6.74	4.82	5.36	6.47
						$k=4$						
.80	5.04	6.07	8.67	4.22	4.92	6.27	4.04	4.63	5.87	3.88	4.48	5.53
.90	5.85	6.84	9.48	4.88	5.63	7.00	4.75	5.32	6.50	4.57	5.15	6.23
.95	6.54	7.59	10.29	5.54	6.18	7.59	5.32	5.94	7.11	5.14	5.72	6.75
						$k=5$						
.80	5.48	6.59	9.40	4.53	5.21	6.58	4.30	4.90	6.81	4.13	4.70	5.79
.90	6.30	7.42	10.21	5.23	5.94	7.30	5.00	5.59	6.78	4.86	5.40	6.45
.95	7.04	8.20	10.98	5.87	6.56	7.90	5.58	6.19	7.36	5.37	5.95	7.04
						$k=6$						
.80	5.91	7.01	9.93	4.72	5.47	6.85	4.49	5.14	6.32	4.35	4.91	5.95
.90	6.72	7.88	10.76	5.48	6.21	7.56	5.22	5.84	7.02	5.02	5.57	6.61
.95	7.37	8.60	11.57	6.09	6.81	8.16	5.78	6.39	7.57	5.58	6.13	7.14
						$k=8$						
.80	6.50	7.70	10.86	5.09	5.81	7.24	4.78	5.44	6.65	4.64	5.19	6.24
.90	7.37	8.56	11.67	5.82	6.55	7.95	5.52	6.12	7.28	5.34	5.88	6.88
.95	8.08	9.34	12.54	6.45	7.15	8.57	6.08	6.73	7.88	5.90	6.42	7.42
						$k=10$						
.80	6.98	8.30	11.64	5.40	6.12	7.60	5.04	5.68	6.92	4.81	5.39	6.43
.90	7.80	9.10	12.50	6.14	6.85	8.28	5.76	6.40	7.63	5.55	6.07	7.09
.95	8.56	9.91	13.31	6.79	7.44	8.89	6.36	6.95	8.20	6.08	6.67	7.64
						$k=15$						
.80	7.89	9.29	12.86	5.90	6.63	8.14	5.49	6.10	7.30	5.19	5.78	6.84
.90	8.73	10.13	13.71	6.66	7.35	8.89	6.22	6.82	7.98	5.93	6.46	7.49
.95	9.42	10.95	14.54	7.30	8.05	9.42	6.80	7.38	8.59	6.48	7.03	8.01
						$k=20$						
.80	8.58	10.03	13.87	6.24	6.99	8.51	5.80	6.41	7.64	5.48	6.04	7.05
.90	9.46	10.90	14.73	7.00	7.69	9.23	6.56	7.12	8.34	6.20	6.73	7.73
.95	10.20	11.60	15.54	7.60	8.30	9.83	7.14	7.67	8.92	6.77	7.31	8.25

Table 3. Bacterial killing ability example
(first 15 observations) and summary statistics

	Solvent I	Solvent II	Solvent III	Solvent IV
	93.63	93.58	97.18	96.44
	93.99	93.02	97.42	96.87
	94.61	93.86	97.65	97.24
	91.69	92.90	95.90	95.41
	93.00	91.43	96.35	95.29
	94.17	92.68	97.13	95.61
	92.62	91.57	96.06	95.28
	93.41	92.87	96.33	94.63
	94.67	92.65	96.71	95.58
	95.28	95.31	98.11	98.20
	95.13	95.33	98.38	98.29
	95.68	95.17	98.35	98.30
	97.52	98.59	98.05	98.65
	97.52	98.00	98.25	98.43
	97.37	98.79	98.12	98.41
		Summary statistics of the two-stage		
S_i^2	3.17085	5.88428	0.77969	2.10995
a_i	0.02023	0.01182	0.04937	0.03071
b_i	0.02786	0.01419	0.25948	0.04495
N_i	40	73	16	27
$\tilde{X}_i$	95.274	94.761	97.498	97.593
	$z = 0.081$		$T = 9.95$	

Table 4. Bacterial killing ability example

(second-stage observations)

Solvent I		Solvent II				Solvent III	Solvent IV
96.97	94.03	96.36	93.43	98.15	92.42	97.97	98.59
97.21	92.43	96.69	92.72	96.73	92.38		98.20
97.44	92.62	96.89	93.56	97.55	92.06		98.37
96.86	94.47	96.13	94.13	94.44	92.50		98.57
97.26	(94.14)	97.65	93.57	93.61	(92.54)		98.42
98.27	(93.09)	97.81	96.27	93.61	(92.52)		98.29
97.57	(98.47)	97.71	98.05	94.20	(91.80)		98.51
97.81	(98.06)	97.48	97.67	94.20	(93.00)		98.89
98.20	(98.35)	97.96	98.93	93.34	(91.69)		(98.66)
93.92	(97.09)	94.30	97.23	93.33	(95.42)		(97.39)
93.86		93.29	95.95	93.51	(92.29)		(97.41)
92.57		94.21	97.79	93.91	(92.57)		(97.52)
93.32		92.90	97.41	94.05	(94.96)		
92.15		93.02	96.94	93.76			
92.09		93.43	97.08	93.76			

Summary statistics of the one-stage

	Solvent I	Solvent II	Solvent III	Solvent IV
S_i^2	3.17085	5.88428	0.77969	2.10995
U_i	0.02941	0.01901	0.07793	0.04757
V_i	0.02941	0.01459	-0.16889	0.03581
N_i	34	64	16	23
$\tilde{X}_i$	94.981	94.957	97.225	97.311
$z^* = 0.0933$		$T = 7.71$		

† The data within parentheses are assumed not available in the calculation of the one-stage procedure.