行政院國家科學委員會補助專題研究計畫成果報告

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計畫類別：☒個別型計畫　　□整合型計畫

計畫編號：NSC 89－2115—M－032－007－

執行期間： 88 年 8 月 1 日至 89 年 7 月 31 日

計畫主持人：胡守仁

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執行單位：淡江大學數學系

中　華　民　國　89　年　10　月　31　日

行政院國家科學委員會專題研究計畫成果報告

不變量之計算

Computing of Invariants

計畫編號：NSC 89-2115-M-032-007-
執行期限：88 年 8 月 1 日至 87 年 7 月 31 日

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一、中文摘要

本計畫中，吾人將考慮群 $PSL_2(F_p)$ 之表現並探討其不變量環。特別是當 p=11 時，$PSL_2(F_{11})$為 一單群，其秩為 660，吾人將明白將其不變量環之生成元寫出。

關鍵詞：不變量環、Molien 級數

Abstract

In this study, we consider the representation of $PSL_2(F_p)$ over complex numbers and its ring of invariants. In particular, the group $PSL_2(F_{11})$ is a simple group of order 660. We shall consider a special representation and give explicit generator for its ring of invariants.

Keywords: ring of invariants, Molien series

二、緣由與目的

For a representation $\rho : G \to GL(n, \mathbb{F})$ of a finite group G over the field $\mathbb{F}$, we have an induced action on the algebra $\mathbb{F}[V]$ of polynomial functions on $V = \mathbb{F}^n$. If $\rho : G \to GL(n, \mathbb{F})$ is a faithful representation, we denote by $\mathbb{F}[V]^G = \{f \in \mathbb{F}[V] \mid \rho(g)f = f \; \forall f \in G\}$ the ring of invariants of G. If G is a finite group, Hilbert proved in 1890 [H], the main theorem of invariant theory that the ring of invariants is finitely generated. Noether [N] later produced an explicit set of basic invariants. Besides the finite generation problem, it is always interesting to determine the number of invariants needed to generate the ring of invariants as well as their explicit forms. It is also known that the ring $\mathbb{F}[V]^G$ is Cohen-Macaulay if the order of G is relatively prime to the characteristic of $\mathbb{F}$ (i. e. in the non-modular case) [HE], [S1] or if dimension of V is 1, 2 or 3 over $\mathbb{F}$ [S1, S3]. Also in the non-modular case, Chevalley-Shepard-Todd theorem tells us that the ring of invariants is a polynomial ring if and only if G is generated by pseudo-reflections [Ch], [ST]. In fact, a systematic method exists for producing a set of generators which is in some sense minimal.

Although theoretically the ring of invariants for finite groups is finitely generated and algorithms exist for finding the primary and secondary invariants, it is a task to find the generators for a given specific finite group. Denote by $SL_2(\mathbb{F}_p)$ the group of all 2×2 matrices of determinnt with entries in the field $\mathbb{F}_p$ where p is a prime. The center of $SL_2(\mathbb{F}_p)$ has order 2 and the quotient of $SL_2(\mathbb{F}_p)$ by its center is denoted by $PSL_2(\mathbb{F}_p)$. For example, the group $PSL_2(\mathbb{F}_p)$ is isomorphic to the symetric group S_3. It has a two dimesional irreducible complex representation. The group $PSL_2(\mathbb{F}_{11})$ is a simple group of order 660 and is the only simple group of that order. It has precisely two irreducible complex representations of degree 5 and they are equivalent under an outer automorphism. Therefore the rings of invariants for these two representations are the same [A1, A2]. We consider in this project, the ring of invariants for irreducible representations of $PSL_2(\mathbb{F}_p)$.

三、結果與討論

We shall first look at the dimension 5 representation of this group and study its ring of invariants.

Let ζ be a primitive 11-th root of unity. Let V be a vector space of dimension 5 over $\mathbb{F}_{11}$. Let g be a generator of the cyclic group $\mathbb{F}_{11}^*$ Consider the transformation defined by

$$A(y_k) = \zeta^{2k} y_k$$
$$B(y_k) = y_{|g \cdot k|}$$

$$C(y_k) = \frac{-1}{(-p)^{1/2}} \sum_{l=1}^{5} \left(\frac{kl}{p}\right) (\zeta^{2kl} - \zeta^{-2kl}) y_l$$

Here $|t|$ is t if $t = 1, \ldots 5$, $-t$ if $t = 6, \ldots 10$ and $\left(\frac{kl}{p}\right)$ denote the Legendre symbol. Note that $A^p = B^5 = C^2 = I_V$

Theorem. *There is one and only one irreducible representation ρ of $PSL_2(\mathbb{F}_{11})$ into $GL(V)$ such that*

$$\rho\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) = A$$
$$\rho\left(\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}\right) = B$$
$$\rho\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = C$$

Molien series is very useful in the study of ring of invariants. For a representation ρ of a group G on a complex vector space V of dimension n, the Molien series of the representation ρ is given by

$$\sum_{k=1}^{\infty} \dim Sym^k(V)^G \, t^k$$

Here $Sym^k(V)^G$ denote homogeneous polynomials of degree k.

It can be shown that the Molien series can be written as

$$\frac{1}{o(G)} \sum_{g \in G} \frac{1}{det(I - \rho(g)t)}$$

Hence it is a rational function. In fact it can be written as quotient of two polynomials with integer coefficients. Since the eigenvalues of $\rho(g)$ are roots of unity of order $o(g)$, where $o(g)$ is the order of $g \in G$, the Molien series can be written in the form

$$\frac{\sum_{i=0}^{M} u_i t^i}{(1 - t^{d_1} \cdot (1 - t^{d_2} \cdots (1 - t^{d_n})}$$

Usually one can find the $d_1, \ldots, d_n$ by inspecting. If $p_1, \ldots, p_n$ form a homopgeneous system of parameters, then the full ring of invariants is a free module of finite rank over the ring $\mathbb{C}[p_1, \ldots, p_n]$. If $q_1, \ldots, q_r$ is a basis for this free module consisting of homogeneous polynomials, the the Molien series can be written in the form

$$\frac{\sum_{i=1}^{r} deg(q_i) t^i}{\prod_{j=1}^{n} (1 - t^{deg(p_i)})}$$

If we can find an expression for the Molien series in the above form one then is able to see whether it corresponds to an actual structure of the ring of invariants. In [A]2, there is the following

Edge's Theorem. *The Molien series for an irreducible 5-dimensional complex reopresentation of $PSL_2(\mathbb{F}_{11})$ represents the rational function*

$$\frac{1 + t^7 + t^9 + t^{10} + t^{12} + 2t^{14} + t^{16}}{(1 - t^3)(1 - t^5)(1 - t^6)}$$
$$\frac{+ t^{18} + t^{19} + t^{21} + t^{28}}{(1 - t^8)(1 - t^{11})}$$

¿From this we know the degree of polynomials in a homogeneous system of parameters and the degree of basis of the ring of invariants over the algebra generated by the homogeneous system of parameters.

Theorem. *The ring of invariants R for an irreducible representation of degree 5 of $PSL_2(\mathbb{F}_{11})$ is generated by 10 invariants of degrees 3, 5, 6, 7, 8, 9, 10, 11, 12, 14 denoted by f_3, f_5, F_6, f_7, f_8, f_9, f_{11}, f_{12}, and f_{14}. The invariants f_3, f_5, f_6, f_8 and f_{11} form a homogeneous system of parameters for the ring of invariants. Let $A = \mathbb{C}[f_3, f_5, f_6, f_8, f_{11}]$, then R is a free module of rank 12 over A with basis 1,, f_7, f_9, f_{12}, f_{14}, f_7^2, f_7f_9, f_9^2, f_9f_{10}, f_7^3, $f_9^2f_{10}$. Generaters for the ideal of relations can be obtained by writing down the multiplications table for the basis.*

If we write $\mathbb{F}_{11}[V]$ as $\mathbb{F}_{11}[x_1, x_2, x_3, x_4, x_5]$ and let σ be the permutation (12345). For a polynomial $p \in \mathbb{F}_{11}[x_1, x_2, x_3, x_4, x_5]$, write $\sigma(p) = \sum_{i=0}^{4} \sigma(p)$. then

$$
\begin{aligned}
f_3 &= \sigma(x_1^2x_2) \\
&= x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^2x_5 + x_5^2x_1 \\
f_5 &= 3x_1x_2x_3x_4x_5 + \sigma(x_1^3x_3^2 - x_1^3x_4x_5) \\
f_6 &= \sigma(x_1^4x_3x_4 + x_1^3x_2x_5^2 - x_1^2x_2x_3^2x_4)
\end{aligned}
$$

The explicit form of other $f_i's$ can be found in [A2].

If we want to consider the invariants of $PSL_2(\mathbb{F}_p)$ for other p, we need to look more carefully into the representation given in [A1]. The calculation need more work and is still going on.

四、參考文獻

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