# 行政院國家科學委員會補助專題研究計畫成果報告

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※ 貝氏序列區間估計之研究 ※

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執行單位：淡江大學數學系

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# 行政院國家科學委員會專題研究計畫成果報告

## 貝氏序列區間估計之研究

## Bayes Sequential Interval Estimation

計畫編號： NSC89-2118-M-032-010
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一、中文摘要

對於常態分配的平均數之貝氏序列區間估計問題的探討, 本文提 出一個與事前機率分配無關且具有確定停止法則的區間估計程序。針對一個由許多事前機率分配所組成的集合來說, 此一區間估計程序 可証明出其具有 Bickel 和 Yahav 於1967及1968年所提出的漸近 點最優和漸近較佳的性質。

關鍵詞: 漸近較佳性, 漸近點最優, 貝氏區間估計, 停止法則

Abstract

The problem of Bayes sequential interval estimation of the mean of a normal distribution with known variance is considered. An interval estimation procedure, which does not depend on the prior distribution, with deterministic stopping rule is proposed in this paper. It is shown that the proposed procedure is asymptotically pointwise optimal and asymptotically Bayes in the sense of Bickel and Yahav (1967, 1968) for a large class of prior distributions.

Keywords: Asymptotically Bayes, asymptotically pointwise optimal, Bayes sequential interval estimation, stopping rule.

二、研究報告內容

### 1. Introduction

Let $X_1, X_2, \cdots$ be a sequence of independent observations from a normal population $N(\theta,\ \sigma^2)$ with the unknown parameter $\theta \in \Theta$ and the known variance $\sigma^2 > 0$, where $\Theta$ is a known subinterval of $R^1$. Here we treat $\theta$ as a realization of a random variable, and assume that $\theta$ has a continuous, positive and bounded prior density $\phi(\theta)$ on $\Theta$ with respect to Lebesgue measure.

Suppose that we are interested in finding an interval estimate of $\theta$ subject to $\theta \in \Theta$. Having recorded $n$ observations $X_1, \cdots, X_n$, we assume that the loss incurred in the interval estimate of $\theta$ by $I \in \mathcal{I}$, where $\mathcal{I}$ denotes the class of all subintervals (including point sets and the empty set) of $\Theta$, is

$$L(\theta, I, n) = al(I) + b(1 - \delta_I(\theta)) + cn.$$

Here $a,\ b,\ c$ are finite positive constants, $l(I)$ is the length of $I$, and $\delta_I(\theta)$ denotes the indicator function of $I$.

The Bayes sequential interval estimation problem is to seek an optimal sequential interval estimation procedure which includes an optimal stopping rule and an optimal interval estimate. It follows from Arrow, Blackwell and Girshick (1949) that the optimal interval estimate for any given stopping rule is the fixed-sample optimal interval estimate based on $n$ observations when the given stopping rule equals to $n$. Thus the main problem is to find an optimal stopping rule.

Optimal stopping rules usually exist under mild regularity conditions (c.f. Theorems 4.4 and 4.5 of Chow, Robbins and Siegmund (1971)), but the exact determination of the optimal stopping rules appears to be a formidable task, in practice. Bickel and Yahav (1967, 1968) provided an attractive large sample approximation to the optimal stopping rules which they called asymptotically pointwise optimal (A.P.O.) rules in the general setting. They also gave the A.P.O. rules in many point estimation and hypothesis testing problems, and showed that the A.P.O. rules are asymptotically Bayes (or asymptotically optimal). For sequential interval estimation of the positive mean of a normal distribution with known variance under a folded normal prior, an A.P.O. rule is given by Blumenthal (1970) and the A.P.O. rule is asymptotically Bayes. Gleser and Kunte (1976) extended the results of Bickel and Yahav (1967, 1968) to cover the problem of Bayes sequential interval estimation.

The A.P.O. rules proposed in Blumenthal (1970) and Gleser and Kunte (1976) depend on the prior distributions. An interval estimation procedure with deterministic stopping rule is proposed in this paper. The proposed interval estimation procedure does not depend on the prior, and it is shown to be A.P.O. and asymptotically Bayes for a large class of prior distributions.

## 2. The interval estimation procedure

For convenience, let $\mathbf{X}_n = (X_1, X_2, \cdots, X_n)$, $\mathbf{x}_n = (x_1, x_2, \cdots, x_n)$, $\phi(\theta \mid \mathbf{X}_n)$ be the posterior density of $\theta$ given $\mathbf{X}_n$, and $G_n(\mathbf{x}_n)$ be the marginal distribution function of $\mathbf{X}_n$. For an interval estimate $I(\mathbf{X}_n) \in \mathcal{I}$, a simple application of Fubini's theorem yields the corresponding Bayes risk

$$\begin{aligned} R(\phi, I(\mathbf{X}_n)) &= E\ L(\theta, I(\mathbf{X}_n), n) \\ &= \int_{R^n} \Big(\rho(\phi, I(\mathbf{x}_n)) + cn\Big)\ dG_n(\mathbf{x}_n), \end{aligned}$$

where

$$\begin{aligned} \rho(\phi, I(\mathbf{x}_n)) &= a\ l(I(\mathbf{x}_n)) + b\Big(1 - \int_{I(\mathbf{x}_n)} \phi(\theta \mid \mathbf{x}_n)\ d\theta\Big) \\ &= b\Big(1 + \int_{I(\mathbf{x}_n)} (ab^{-1} - \phi(\theta \mid \mathbf{x}_n))\ d\theta\Big). \end{aligned}$$

Hence in the remainder of this paper we make the assumption as follows.

**Condition (A).** The closure in $\Theta$ of the set $\{\theta$: $\phi(\theta \mid \mathbf{x}_n) \geq ab^{-1}\}$ is a closed subinterval $I^*(\mathbf{x}_n) = [\alpha^*_{1n}(\mathbf{x}_n),\ \alpha^*_{2n}(\mathbf{x}_n)]$ of $\Theta$ for almost all $\mathbf{x}_n$ with respect to the probability measure corresponding to $G_n$.

One notes that $I^*(\mathbf{x}_n)$ may be empty or a one-point set. In general,

$$\begin{aligned} \alpha^*_{1n}(\mathbf{x}_n) &= \inf\{\theta : \phi(\theta \mid \mathbf{x}_n) \geq ab^{-1}\} \\ \alpha^*_{2n}(\mathbf{x}_n) &= \sup\{\theta : \phi(\theta \mid \mathbf{x}_n) \geq ab^{-1}\}. \end{aligned}$$

The Condition (A) holds if $\phi(\theta)$ is a normal or folded normal density function (see the example on page 688 in Gleser and Kunte (1976)), and the following example is to show that the Condition (A) does not restrictive.

**Example.** Let the prior density of $\theta$ be of the form

$$\phi(\theta) = \exp\{\sum_{i=1}^{k} C_i(\mathbf{w}) T_i(\theta) + D(\mathbf{w})\}, \quad \theta \in \Theta,$$

where $\mathbf{w}$ is a constant in $R^k$, and the functions $T_i(\theta)$ are continuous on $\Theta$. Then we get the posterior density

$$\phi(\theta|\mathbf{x}_n) = k(\mathbf{x}_n, \sigma, \mathbf{w}) \exp\{-\frac{n}{2\sigma^2}\theta^2 + \frac{n\bar{x}_n}{\sigma^2}\theta + \sum_{i=1}^{k} C_i(\mathbf{w})T_i(\theta)\}, \quad \theta \in \Theta,$$

for some positive function $k(\mathbf{x}_n, \sigma, \mathbf{w})$, where $\bar{x}_n = \frac{1}{n}\sum_{i=1}^{n} x_i$. The Condition (A) holds if for almost all $\mathbf{x}_n$, the function $F_n(\theta) = -\frac{n}{2\sigma^2}\theta^2 + \frac{n\bar{x}_n}{\sigma^2}\theta + \sum_{i=1}^{k} C_i(\mathbf{w})\mathbf{T}_i(\theta)$ is either unimodal on $\Theta$ or increasing or decreasing on $\Theta$ with finite right endpoint or left endpoint, respectively. Hence it is not difficult to obtain that the Condition (A) satisfies for the prior distributions of $\theta = a_1\eta + a_2$, where $a_1 \neq 0$, $a_2 \in R^1$ and $\eta$ has the gamma distribution with shape parameter $w_1 \geq 1$ and scale parameter $w_2 > 0$ or the beta distribution with parameters $w_1 \geq 1$ and $w_2 \geq 1$.

Because the Condition (A) is assumed to be held, the $I^*(\mathbf{X}_n)$ is an optimal interval estimate against $\phi$ based on the observations $\mathbf{X}_n$. Let

$$\begin{aligned} Y_n &= \rho(\phi, I^*(\mathbf{X}_n)) \\ &= a\Big(\alpha_{2n}^*(\mathbf{X}_n) - \alpha_{1n}^*(\mathbf{X}_n)\Big) + b\Big(1 - \int_{\alpha_{1n}^*(\mathbf{X}_n)}^{\alpha_{2n}^*(\mathbf{X}_n)} \phi(\theta \mid \mathbf{X}_n)\, d\theta\Big), \end{aligned}$$

which denotes the posterior Bayes risk of $I^*(\mathbf{X}_n)$.

One notes that the result in Theorem 5.1 of Gleser and Kunte (1976) also holds for the one-parameter exponential family, corresponding to Theorems 3.1 and 3.2 for sequential point estimation in Bickel and Yahav (1967). Hence we get that the A.P.O. rule with respect to $\{Y_n + cn;\ n \geq 1\}$ is

$$t_c = \inf\Big\{n \geq 3:\ \Big(1 - \frac{f(n)}{f(n+1)}\Big)Y_n \leq c\Big\},$$

where $f(n) = (\frac{n}{\log n})^{\frac{1}{2}}$ for $n \geq 3$, and $f(1)$and $f(2)$ are any positive constants. It means that the class of stopping rules $\{t_c; c > 0\}$ satisfies

$$P\Big\{\lim_{c\to 0} \frac{Y_{t_c} + ct_c}{\inf_s(Y_s + cs)} = 1\Big\} = 1,$$

where the infimum extends over all $\sigma(\mathbf{X}_n)$-stopping rules $s$.

In view of the condition $\sup_n f(n)E(Y_n) < \infty$ assured by Lemma 1 of Section 3, we know from Theorem 6.1 or Theorem 4.1 of Gleser and Kunte (1976) that the sequential interval estimation procedure $(I^*(\mathbf{X}_{t_c}),\ t_c)$ is asymptotically Bayes, that is

$$\lim_{c\to 0} \frac{E(Y_{t_c} + ct_c)}{\inf_s E(Y_s + cs)} = 1.$$

In fact, we see from (4.33) in the proof of Theorem 4.1 of Gleser and Kunte (1976) that

$$\lim_{c\to 0} f(\gamma_c)\, E(Y_{t_c} + ct_c) = \frac{3}{2}(2a\sigma)^{\frac{2}{3}},$$

where $\gamma_c = \inf\{n \geq 1 : nf(n) \geq \frac{1}{2c}\}$.

The sequential interval estimation procedure $(I^*(\mathbf{X}_{t_c}),\ t_c)$ depends on the prior density $\phi(\theta)$. Here we would like to find a procedure without using the prior information, but at the same time it can be used to replace the sequential interval estimation procedure $(I^*(\mathbf{X}_{t_c}),\ t_c)$.

Motivated by that $f(n)a(\alpha^*_{2n}(\mathbf{X}_n)-\alpha^*_{1n}(\mathbf{X}_n))$ and $f(n)Y_n$ converge to $2a\sigma$ with probability 1, we propose that the interval estimate based on $\mathbf{X}_n$ is $I^\circ(\mathbf{X}_n)=[\bar{X}_n-\frac{\sigma}{f(n)},\ \bar{X}_n+\frac{\sigma}{f(n)}]$ and the stopping rule is

$$u_c=\inf\Big\{n\geq 3:\ \Big(1-\frac{f(n)}{f(n+1)}\Big)\frac{2a\sigma}{f(n)}\leq c\Big\},$$

where $\bar{X}_n$ denotes the sample mean of $\mathbf{X}_n$. The statements involving a.s. or with probability 1 refer to the overall probability measure $P$ unless otherwise stated in this paper.

Note that the stopping rule $u_c$ is a constant depending on $c$, and thus the interval estimation procedure $(I^\circ(\mathbf{X}_{u_c}),\ u_c)$ is not sequential. Denote the posterior Bayes risk of $I^\circ(\mathbf{X}_n)$ by

$$\begin{aligned} Y_n^\circ &= \rho(\phi, I^\circ(\mathbf{X}_n)) \\ &= \frac{2a\sigma}{f(n)}+b\Big(1-\int_{\bar{X}_n-\frac{\sigma}{f(n)}}^{\bar{X}_n+\frac{\sigma}{f(n)}}\phi(\theta\mid\mathbf{X}_n)\,d\theta\Big). \end{aligned}$$

The class of stopping rules $\{u_c;\ c>0\}$ and the interval estimation procedure $(I^\circ(\mathbf{X}_{u_c}),\ u_c)$ are shown to be A.P.O. and asymptotically Bayes in the following Theorem 1 and Theorem 2, respectively.

**Theorem 1.** If $\phi(\theta)$ has continuous second derivatives in the interior of $\Theta$, then
(i) $\{u_c;\ c>0\}$ is an A.P.O. with respect to $\{Y_n+cn;\ n\geq 1\}$ and $\{Y_n^\circ+cn;\ n\geq 1\}$.
(ii) $\dfrac{Y^\circ_{u_c}+cu_c}{\inf_s(Y_s+cs)}\to 1$ with probability 1.

**Theorem 2.** We have $\lim_{c\to 0} f(\gamma_c)\,E(Y^\circ_{u_c}+cu_c)=\frac{3}{2}(2a\sigma)^{\frac{2}{3}}$.

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