行政院國家科學委員會專題研究計畫成果報告

多項式方程式

Polynomial Equations

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一、 中文摘要

本計畫中吾人考慮實係數多項式方程組在複數及實數上解之個數，此一問題即使在項數不大的情形下，其實數解個數之上限仍為未知。吾人將利用牛頓多面體來處理此類問題。

關鍵詞：多項式方程式、牛頓多面體

Abstract

In this project, we shall study the number of solutions to a system of polynomial equations over complex and real numbers. This type of problems is open for real solutions even the number of terms involved in the polynomials is small. We shall use Newton polytopes to study the problem.

Keywords: Polynomial Equations, Newton Potytopes

二、 緣由與目的

In the project, we studied polynomial equations. Solving polynomial equations is one of the fundamental problems in mathematics. Galois theory tells us that for $n \geq 5$ there is no general formula of the solutions in terms of arithmetic operations and radicals. A natural question is how many of the roots lie in the field $\mathbb{R}$ of real numbers. Descartes' rule tells us that the number of positive roots of a polynomial $g(x)$ is bounded above by the number of its sign alternations, while Bezout's theorem says that if $f(x;y) = g(x;y) = 0$ are two polynomial equations in two unknowns and if they have only finitely many common complex zeros $(x;y) \in \mathbb{C}^2$, then the number of these zeros is at most $\deg(f)\cdot\deg(g)$.

Moreover, the study of real roots is considerable more difficulty than the study of complex roots. For example, consider the system of equations $f(x;y) = g(x;y) = 0$ where

$$f(x;y) = a_1x^{u_1}y^{v_1} + a_2x^{u_2}y^{v_2} + a_3x^{u_3}y^{v_3} + a_4x^{u_4}y^{v_4}$$

$$g(x;y) = b_1x^{r_1}y^{s_1} + b_2x^{r_2}y^{s_2} + b_3x^{r_3}y^{s_3} + b_4x^{r_4}y^{s_4}$$

$a_i; b_i$ are arbitrary real numbers and $u_i; v_i, r_i; s_i$ are arbitrary integers. What is the maximum number of isolated real roots in $(\mathbb{R}^*)^2$? This number certainly does not exist if we count the complex roots, because we can increase the number of complex roots by increasing the degree of the polynomials. However, such an unbound increase of roots over real numbers is impossible. A conjecture is that the number should be 36. It is easy to see that the following system has thirty-six roots:

$$F(x) = (x^2 - 1)(x^2 - 2)x^2 - 3)$$

$$g(x) = (y^2 - 1)(y^2 - 2)(y^2 - 3)$$

.

Writing R_+ for the set of positive real numbers, we have the following conjectures

Conjecture. (Kouchnirenko's conjecture) Consider any system of d polynomials in d unknowns, where the i-th equation has at most m_i terms. The number of isolated real roots in $(R_+)^d$ of such a system is at most $(m_1 - 1)(m_2 - 1)\cdots(m_d - 1)$.

The number is attained by equations in distinct variables, as the above example shows. If we wish to consider roots in $(R^*)^d$) instead of $(R_+)^d$) in the conjecture, then the asserted bound should be multiplied by 2^d. It is obvious that the conjecture is true for $d = 1$ by Descartes' rule of sign. Otherwise, the conjecture remaines open. Khovanskii found a bound on the number of real roots that does not depend on the degrees of the given equations [B-R, K1, K2]. That is the number of isolated roots in the positive orthant $(R_+)^d$ is $2^{\binom{n}{2}} \cdot (d+1)^n$. However, if we apply this bound to the example above, we have $d = 1$ and $n = 7$, so the bound should be 4,586,471,424 which is way above the conjectured 36.

Recent literature [L-W] suggests that Kouchinirenko's conjecture may be too optimistic.

三、 結果與討論

Definition. A Newton polytope of a system of equations

$$f_i(x) = \sum_{a \in A_i} c_{i,a} x^a$$

where $A_1, \ldots, A_n$ are fixed finite set in Z^n, $a = (a_1, \ldots, a_n) \in Z^n$ and $x = x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$. A_i is called the support of f_i. Its convex hull $Q_i = \mathrm{conv}(A_i)$

in R^n is called the Newton polytope of f_i.

Theorem (Bernstein Theorem). For almost all choices of coefficients $c_{i,a} \in C^*$, the number of common zeros in the torus $(C^*)^n$ equals the mixed volume $M(Q_1, \cdots, Q_n)$ of the Newton polytopes.

Example. (1) Let $f(x, y) = a_0 + a_1 x + a_2 x^n y^n$, $g(x, y) = b_0 + b_1 x + b_2 x^n y^n$. Then $M(f, g) = 2n$, while the Bezout bound equals $(2n)^2$ and their ratio tends to zero.

(2) The eigen value problem $Ax = \lambda x$ where A is a generic $n \times n$ matrix over $\mathbb{C}$. This can be regarded as a system of $(n+1)$ quadratic equations in $(n+1)$ variables: $\sum_j a_{ij} x_j - \lambda x_i = 0$; $\sum_i x_i^2 = 1$, with Newton polytopes $Q, \ldots, Q, Q'$. There are $2n$ distinct solutions to this system, so the mixed volume $M(Q, \ldots, Q, Q') = 2n$ while Bezout bound equals 2^{n+1}.

For real solutions, we have for the system

$$c_1 x^{a_1} y^{b_1} + c_2 x^{a_2} y^{b_2} = 0$$

$$c_3 x^{a_3} y^{b_3} + c_4 x^{a_4} y^{b_4} = 0$$

Propositiom ([S]). The above system has precisely one solution in $(\mathbb{R}_+)^2$ if and only if $c_1 c_2 < 0$ and $c_3 c_4 < 0$. In all other cases, it has no solution in $(\mathbb{R}_+)^2$.

If we consider the toric deformation

$$f_i(x) = \sum_{a \in A_i} c_{i,a} x^a t^{\omega_i(a)}$$

Then we have

Theorem ([P-S]). The asymptotic number of real roots of the above system is at most $\sum_C 2^{p(C)}$ where C is a mixed cell.

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