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Hermite-Hadamard 不等式之研究

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ABSTRACT

If $f : [a,b] \to R$ is convex, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2} \quad \cdots (1)$$

is known in the literature as Hermite-Hadamard inequalities. We established, in this project, several new extensions of the inequalities (1), also we discovered some refinements of the inequalities (1).

Key words: Convex, Hermite-Hadamard inequalities

摘要

若 $f:[a,b]\to R$ 為凹函數，則

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)\,dx\le\frac{f(a)+f(b)}{2}\quad\cdots\quad(1)$$

不等式(1)稱為 Hermite-Hadamard 不等式。本研究中我們建立一些不等式(1)之新的推廣，而且也找出一些比不等式(1)更細緻的不等式。

關鍵詞：凹函數，Hermite-Hadamard 不等式

Hermite-Hadamard 不等式之研究

On certain integral inequalities related to Hermite-Hadamard inequalities

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一、計劃緣由與目的：

若 $f:[a,b]\to R$ 為凸函數，則下列不等式成立：

$$f\left(\frac{a+b}{2}\right)\le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2} \quad \cdots (1)$$

稱為 Hermite-Hadamard 不等式，有關不等式(1)之種種推廣可參閱參考文獻 [3]、[4] 及它們裡面所引用的文獻，本計劃之目的旨在以基本的理論，做出不等式 (1) 更新更一般化之推廣。

二、研究成果：

我們建立了一些新的，有趣的不等式，其中一部份推廣了不等式 (1)，另有一部份找出了比不等式(1)更细緻的不等式，這些結果，我們已發表在 Journal of Mathematical Analysis and Applications 239，180-187 (1999) 中，(見附錄)。

參 考 文 獻

※參考文獻之中外文期刊、書籍按文中出現先後次序排列編號，須依次列出作者、期刊名、卷冊數、年月等，文中引用時，一律用括號及號碼附在文中。

1. J. Hadamard, Etude sur les proprietes des fonctions entieres et en particulier d'une fonction consideree par Riemann, *J. Math. Pures Appl.* **58** (1893), 171–215.
2. D. S. Mitrinovic and I. B. Lackovic, Hermite and convexity, *Aequationes Math.* **28** (1985), 229–232.
3. L. Fejer, Uber die Fourierreihen, II, *Math. Naturwiss. Anz. Ungar. Akad. Wiss.* **24** (1906), 369–390. (In Hungarian)
4. J. L. Brenner and H. Alzer, Integral inequalities for concave functions with applications to special functions, *Proc. Roy. Soc. Edinburgh Sect. A* **118** (1991), 173–192.
5. S. S. Dragomir, Two mappings in connection to Hadamard's inequalities, *J. Math. Anal. Appl.* **167** (1992), 49–56.
6. G.-S. Yang and M.-C. Hong, A note on Hadamard's inequality, *Tamkang J. Math.* **28**, No. 1 (1997), 33–37.

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NOTE

On Certain Integral Inequalities Related to Hermite–Hadamard Inequalities

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In this paper, we establish some new Hermite–Hadamard inequalities. © 1999 Academic Press

1. INTRODUCTION

The inequalities

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}, \tag{1}$$

which hold for all convex mappings f: $[a,b] \to R$, are known in the literature as Hadamard inequalities [1]. We note that J. Hadamard was not the first to discover them. As is pointed out by Mitrinovic and Lackovic [2] the inequalities (1) are due to Hermite, who obtained them in 1883, 10 years before J. Hadamad. In [3], Fejer proved that if g: $[a,b] \to R$ is nonnegative integrable and symmetric to $x = \frac{a+b}{2}$, and if f is convex on $[a,b]$, then

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx. \tag{2}$$

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In [4], Brenner and Alzer asserted that if $g: [a, b] \to R^+$ is integrale and symmetric to $x = A = \frac{pa+qb}{p+q}$ with positive numbers p and q, then

$$f\left(\frac{pa+qb}{p+q}\right)\int_{A-y}^{A+y} g(t)\,dt \le \int_{A-y}^{A+y} f(t)g(t)\,dt$$

$$\le \frac{pf(a)+qf(b)}{p+q}\int_{A-y}^{A+y} g(t)\,dt, \qquad (3)$$

where $0 \le y \le \frac{b-a}{p+q}\min(p, q)$, and f is convex on $[a, b]$.

In [5], Dragomir and in [6], Yang and Hong found convex monotonically real functions H and F defined on $[0, 1]$ by

$$H(t) = \frac{1}{b-a}\int_a^b f\left[tx + (1-t)\frac{a+b}{2}\right] dx, \qquad (4)$$

and

$$F(t) = \frac{1}{2(b-a)}\int_a^b \left[f\left(\left(\frac{1+t}{2}\right)a + \left(\frac{1-t}{2}\right)x\right)\right.$$

$$\left. + f\left(\left(\frac{1+t}{2}\right)b + \left(\frac{1-t}{2}\right)x\right)\right] dx, \qquad (5)$$

respectively, such that

$$f\left(\frac{a+b}{2}\right) = H(0) \le H(t) \le H(1) = \frac{1}{b-a}\int_a^b f(x)\,dx = F(0)$$

$$\le F(t) \le F(1) = \frac{f(a)+f(b)}{2}.$$

2. MAIN RESULTS

THEOREM 1. *Let $f: [a, b] \to R$ be a convex function, $0 < \alpha < 1$, $0 < \beta < 1$, $A = \alpha a + (1-\alpha)b$, $u_0 = (b-a)\min\{\frac{\alpha}{1-\beta}, \frac{1-\alpha}{\beta}\}$, and let h be defined by $h(t) = (1-\beta)f(A-\beta t) + \beta f(A + (1-\beta)t)$, $t \in [0, u_0]$.*

Then h is convex, increasing on $[0, u_0]$ and for all $t \in [0, u_0]$,

$$f[\alpha a + (1-\alpha)b] \le h(t) \le \alpha f(a) + (1-\alpha)f(b). \qquad (6)$$

Proof. We note that if f is convex and g is linear, then the composition $f \circ g$ is convex. Also we note that a positive constant multiple of a convex function and a sum of two convex functions are convex, hence h is convex on $[0, u_0]$. Next, if $t \in [0, u_0]$, it follows from the convexity of f that

$$\begin{aligned} h(t) &= (1-\beta)f(A-\beta t) + \beta f(A+(1-\beta)t) \\ &\geq f[(1-\beta)(A-\beta t) + \beta(A+(1-\beta)t)] \\ &= f(A) = f[\alpha a + (1-\alpha)b]. \end{aligned}$$

Also, we observe that $0 < \alpha \leq \frac{\alpha(b-a)+\beta t}{b-a} \leq 1$, $0 \leq \frac{(1-\alpha)(b-a)-\beta t}{b-a} \leq 1 - \alpha < 1$, $0 \leq \frac{\alpha(b-a)-(1-\beta)t}{b-a} \leq \alpha \leq 1$, and $0 < 1 - \alpha \leq \frac{(1-\alpha)(b-a)+(1-\beta)t}{b-a} \leq 1$, so that

$$\begin{aligned} h(t) &= (1-\beta)f[\alpha a + (1-\alpha)b - \beta t] \\ &\quad + \beta f[\alpha a + (1-\alpha)b + (1-\beta)t] \\ &= (1-\beta)f\left[\frac{\alpha(b-a)+\beta t}{b-a}a + \frac{(1-\alpha)(b-a)-\beta t}{b-a}b\right] \\ &\quad + \beta f\left[\frac{\alpha(b-a)-(1-\beta)t}{b-a}a \right.\\ &\qquad \left. + \frac{(1-\alpha)(b-a)+(1-\beta)t}{b-a}b\right] \\ &\leq (1-\beta)\left[\frac{\alpha(b-a)+\beta t}{b-a}f(a) + \frac{(1-\alpha)(b-a)-\beta t}{b-a}f(b)\right] \\ &\quad + \beta\left[\frac{\alpha(b-a)-(1-\beta)t}{b-a}f(a) \right.\\ &\qquad \left. + \frac{(1-\alpha)(b-a)+(1-\beta)t}{b-a}f(b)\right] \\ &= \alpha f(a) + (1-\alpha)f(b), \end{aligned}$$

hence (6) holds. Finally, for t_1, t_2 such that $0 < t_1 < t_2 \leq u_0$, since h is convex, it follows from (6) that

$$\frac{h(t_2)-h(t_1)}{t_2-t_1} \geq \frac{h(t_1)-h(0)}{t_1-0} = \frac{h(t_1)-f[\alpha a + (1-\alpha)b]}{t_1} \geq 0,$$

hence $h(t_2) \geq h(t_1)$. This shows that h is increasing on $[0, u_0]$, and the proof is completed.

THEOREM 2. *Let f, α, β, A, and u_0 be defined as in Theorem* 1 *and let $g: [a,b] \to R$ be nonnegative and integrable and*

$$g(A - \beta u) = g(A + (1-\beta)u), \qquad u \in [0, u_0]. \tag{7}$$

Then

$$\begin{aligned} f[\alpha a + (1-\alpha)b] &\int_{A-\beta u}^{A+(1-\beta)u} g(t)\,dt \\ &\le \frac{1-\beta}{\beta}\int_{A-\beta u}^{A} f(t)g(t)\,dt + \frac{\beta}{1-\beta}\int_{A}^{A+(1-\beta)u} f(t)g(t)\,dt \\ &\le [\alpha f(a) + (1-\alpha)f(b)]\int_{A-\beta u}^{A+(1-\beta)u} g(t)\,dt. \end{aligned} \tag{8}$$

Proof. For every $u \in [0, u_0]$, we have the identity

$$\begin{aligned} \int_{A-\beta u}^{A+(1-\beta)u} g(t)\,dt &= \int_{A-\beta u}^{A} g(t)\,dt + \int_{A}^{A+(1-\beta)u} g(t)\,dt \\ &= \beta\int_0^u g(A-\beta t)\,dt + (1-\beta)\int_0^u g(A-\beta t)\,dt \\ &= \int_0^u g(A-\beta t)\,dt. \end{aligned} \tag{9}$$

Since g is nonnegative, multiplying (6) by $g(A-\beta t)$, integrating the resulting inequalities over $[0, u]$, and using (7) we have

$$\begin{aligned} f[\alpha a + (1-\alpha)b] &\int_0^u g(A-\beta t)\,dt \\ &\le (1-\beta)\int_0^u f(A-\beta t)g(A-\beta t)\,dt \\ &\quad + \beta\int_0^u f(A+(1-\beta)t)g(A+(1-\beta)t)\,dt \\ &= \frac{1-\beta}{\beta}\int_{A-\beta u}^{A} f(t)g(t)\,dt + \frac{\beta}{1-\beta}\int_{A}^{A+(1-\beta)u} f(t)g(t)\,dt \\ &\le [\alpha f(a) + (1-\alpha)f(b)]\int_0^u g(A-\beta t)\,dt, \end{aligned}$$

thus, the inequalities (8) follow by using the identity (9).

Remark 1. If we choose $\alpha = \frac{p}{p+q}$, $\beta = \frac{1}{2}$, and $u = 2y$ in Theorem 2, then the inequalities (8) reduce to the inequalities (3).

Remark 2. If we choose $\alpha = \beta = \frac{1}{2}$, and $u = u_0 = b - a$ in Theorem 2, then the inequalities (8) reduce to the inequalities (2).

Remark 3. If we choose $\alpha = \beta = \frac{1}{2}$, $u = u_0 = b - a$, and $g(x) \equiv 1$ in Theorem 2, then the inequalities (8) reduce to the inequalities (1).

THEOREM 3. *Let f, A, and u_0 be defined as in Theorem* 1, $0 < \alpha < 1$, $0 < \beta < 1$, $\alpha + \beta \leq 1$, *and let H be defined by*

$$H(t) = \frac{1-\beta}{\alpha(b-a)} \int_0^{\alpha(b-a)/(1-\beta)} [(1-\beta) f(A - \beta t x) + \beta f(A + (1-\beta) t x)]\, dx, \qquad 0 \leq t \leq 1. \quad (10)$$

Then, H is convex monotonically increasing on $[0, 1]$, *and*

$$\begin{aligned} f[\alpha a + (1-\alpha) b] = H(0) &\leq H(t) \\ &\leq H(1) = \frac{1-\beta}{\alpha(b-a)} \\ &\quad \times \int_0^{\alpha(b-a)/(1-\beta)} [(1-\beta) f(A - \beta x) + \beta f(A + (1-\beta) x)]\, dx \\ &\leq \alpha f(a) + (1-\alpha) f(b). \end{aligned} \quad (11)$$

Proof. That H is convex follows immediately from the convexity of f. Next, the condition $\alpha + \beta \leq 1$ implies that $u_0 = \frac{\alpha(b-a)}{1-\beta}$. It follows from Theorem 1 that $h(t) = (1-\beta) f(A - \beta t) + \beta f(A + (1-\beta) t)$ is increasing on $[0, u_0]$ and hence $H(t)$ is increasing on $[0, 1]$.

Finally, the least inequality of (11) follows from (6), and the proof is completed.

Similarly, we have the following theorem:

THEOREM 4. *Let f, A, u_0, α, β be defined as in Theorem* 3. *If*

$$G(t) = \frac{1-\beta}{\alpha(b-a)} \times \int_0^{\alpha(b-a)/(1-\beta)} \left[(1-\beta) f\left(A - \beta\left(\frac{\alpha(b-a)}{1-\beta} - x(1-t) \right) \right) + \beta f\left(A + (1-\beta)\left(\frac{\alpha(b-a)}{1-\beta} - x(1-t) \right) \right) \right] dx,$$
$$0 \leq t \leq 1, \quad (12)$$

then G is convex and monotonically increasing on $[0,1]$, *and*

$$\frac{(1-\beta)^2}{\alpha\beta(b-a)}\int_{b-\alpha(b-a)/(1-\beta)}^{A} f(x)\,dx+\frac{\beta}{\alpha(b-a)}\int_A^b f(x)\,dx$$
$$=G(0)\le G(t)\le G(1)=(1-\beta)f\left(b-\frac{\alpha(b-a)}{1-\beta}\right)+\beta f(b)$$
$$\le \alpha f(a)+(1-\alpha)f(b),\qquad 0\le t\le 1. \tag{13}$$

Remark 4. The identity (4) is a special case of (10) taking $\alpha=\beta=\frac{1}{2}$.

Remark 5. The identity (5) is a special case of (12) taking $\alpha=\beta=\frac{1}{2}$.

THEOREM 5. *Let f,α,β,A,u_0 be defined as in Theorem 3 and let g be defined as in Theorem 2. Let P be a function defined on* $[0,1]$ *by*

$$P(t)=\int_0^u\{(1-\beta)f(A-\beta tx)g(A-\beta x)$$
$$+\beta f[A+(1-\beta)tx]g[A+(1-\beta)x]\}\,dx \tag{14}$$

for some $u\in[0,u_0]$. *Then P is convex and monotonically increasing on* $[0,1]$ *and*

$$f[\alpha a+(1-\alpha)b]\int_{A-\beta u}^{A+(1-\beta)u} g(x)\,dx$$
$$=P(0)\le P(t)\le P(1)=\frac{1-\beta}{\beta}\int_{A-\beta u}^{A} f(x)g(x)\,dx$$
$$+\frac{\beta}{1-\beta}\int_A^{A+(1-\beta)u} f(x)g(x)\,dx. \tag{15}$$

Proof. Since f is convex and g is nonnegative, we see that P is convex on $[0,1]$. Next, for each $x\in[0,u]$, where $u\in[0,u_0]$, it follows from Theorem 1 that $h(tx)=(1-\beta)f(A-\beta tx)+\beta f(A+(1-\beta)tx)$ is increasing for $t\in[0,1]$. Using the identity (7) we see that $P(t)$ is increasing on $[0,1]$. Therefore the inequalities (15) follow immediately.

THEOREM 6. *Let f,g,α,β,A,u_0 be defined as in Theorem 5 and let Q be defined on* $[0,1]$ *by*

$$Q(t)=\int_0^u[(1-\beta)f(A-\beta u+\beta x(1-t))g(A-\beta(u-x))$$
$$+\beta f(A+(1-\beta)u-(1-\beta)(1-t)x)$$
$$\times g(A+(1-\beta)(u-x))]\,dx \tag{16}$$

for some $u \in [0, u_0]$. Then Q is monotonically increasing and convex on $[0,1]$, and

$$\frac{1-\beta}{\beta}\int_{A-\beta u}^{A} f(x)g(x)\,dx + \frac{\beta}{1-\beta}\int_{A}^{A+(1-\beta)u} f(x)g(x)\,dx$$
$$= Q(0) \le Q(t) \le Q(1)$$
$$= \left[(1-\beta)f(A-\beta u) + \beta f(A+(1-\beta)u)\right]\int_{A-\beta u}^{A+(1-\beta)u} g(x)\,dx$$
$$\le \left[\alpha f(a) + (1-\alpha)f(b)\right]\int_{A-\beta u}^{A+(1-\beta)u} g(x)\,dx. \tag{17}$$

Proof. That Q is convex follows immediately from the convexity of f. Next, for each $x \in [0,u]$, where $u \in [0,u_0]$, it follows from Theorem 1 that $h(t) = (1-\beta)f(A-\beta f) + \beta f(A+(1-\beta)t)$ and $k(t) = u-(1-t)x$ are increasing on $[0,u_0]$ and $[0,1]$, respectively. Hence $h(k(t)) = (1-\beta)f(A-\beta u+\beta x(1-t)) + \beta f(A+(1-\beta)u-(1-\beta)(1-t)x)$ is increasing on $[0,1]$. Since g is nonnegative and satisfies (7), it follows that $Q(t)$ is monotonically increasing on $[0,1]$. Finally, the last inequalities of (17) follow from (16) and (6).

Remark 6. Choose $\alpha = \beta = \frac{1}{2}$, $u = u_0 = b-a$ in Theorems 5 and 6. Then the inequalities (15) and (17) reduce to

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx = P(0) \le P(t) \le P(1) = \int_a^b f(x)g(x)\,dx$$
$$= Q(0) \le Q(t) \le Q(1)$$
$$= \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx,$$

where $P(t) = \int_a^b f[tx + (1-t)\frac{a+b}{2}]g(x)\,dx$ and

$$Q(t) = \frac{1}{2}\int_a^b\left[f\left(\frac{1+t}{2}a + \frac{1-t}{2}x\right)g\left(\frac{x+a}{2}\right)\right.$$
$$\left. + f\left(\frac{1+t}{2}b + \frac{1-t}{2}x\right)g\left(\frac{x+b}{2}\right)\right]dx,$$

which is a refinement of (2).

Remark 7. Choose $\alpha = \beta = \frac{1}{2}$, $u = u_0 = b-a$, and $g(x) \equiv 1$ in Theorems 5 and 6. Then $P(t) = (b-a)H(t)$, $Q(t) = (b-a)F(t)$, where $H(t)$

and $F(t)$ are defined in (4) and (5), respectively; hence (14) and (16) generalize (4) and (5), respectively.

Remark 8. Choose α, β such that $0 < \alpha < 1$, $0 < \beta < 1$, $\alpha + \beta \leq 1$, and choose $u = u_0 = \frac{\alpha(b-a)}{1-\beta}$, $A = \alpha a + (1-\alpha)b$, $g(x) \equiv 1$ in Theorems 5 and 6. Then $P(t) = \frac{\alpha(b-a)}{1-\beta}H(t)$ and $Q(t) = \frac{\alpha(b-a)}{1-\beta}G(t)$ where $H(t)$ and $G(t)$ are defined in Theorems 3 and 4, respectively; hence Theorem 5 generalizes Theorem 3 and Theorem 6 generalizes Theorem 4.

REFERENCES

1. J. Hadamard, Etude sur les proprietes des fonctions entieres et en particulier d'une fonction consideree par Riemann, *J. Math. Pures Appl.* **58** (1893), 171–215.
2. D. S. Mitrinovic and I. B. Lackovic, Hermite and convexity, *Aequationes Math.* **28** (1985), 229–232.
3. L. Fejer, Uber die Fourierreihen, II, *Math. Naturwiss. Anz. Ungar. Akad. Wiss.* **24** (1906), 369–390. (In Hungarian)
4. J. L. Brenner and H. Alzer, Integral inequalities for concave functions with applications to special functions, *Proc. Roy. Soc. Edinburgh Sect. A* **118** (1991), 173–192.
5. S. S. Dragomir, Two mappings in connection to Hadamard's inequalities, *J. Math. Anal. Appl.* **167** (1992), 49–56.
6. G.-S. Yang and M.-C. Hong, A note on Hadamard's inequality, *Tamkang J. Math.* **28**, No. 1 (1997), 33–37.