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廣義不變量理論

A Study of generalized invariants

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一、中文摘要

令ρ: G ----> GL((n, F) 為有限群在 F 上之表現，而 F[V] 為 V 上之對稱代數，當 F 之特徵數可以整除 G 之秩時，吾人將研究不變量環 F[V]^G，並特別研究 G 為擬反射群時之情形。

關鍵詞：不變量環、擬反射群

Abstract

Let ρ: G ----> GL((n, F) be a representation of G. We shall study the ring of invariants F[V]^G in the case when the characteristic of F divides the order of G. We shall specially consider the case when G is a pseudo reflection group.

Keywords: ring of invariants, pseudo-reflection group

二、緣由與目的

Given $\rho : G \to GL(n,\mathbb{F})$ a representation of a finite group G over the field $\mathbb{F}$, it induces an action on the algebra $\mathbb{F}[V]$ of polynomial functions on $V = \mathbb{F}^n$. If $\rho : G \to GL(n,\mathbb{F})$ is a faithful representation, we denote by $\mathbb{F}[V]^G = \{f \in \mathbb{F}[V] \mid \rho(g)f = f \ \forall\ f \in G\}$ the ring of invariants of G. Notice that $\mathbb{F}[V]$ can be regarded as a graded algebra over $\mathbb{F}$ with homogeneous component of degree d, $\mathbb{F}[V]_d$, the homogeneous polynomial of degree d. If G is a finite group, Hilbert proved in 1890 [H], the main theorem of invariant theory that the ring of invariants is finitely generated. Noether [N] later produced an explicit set of basic invariants. It is always an interesting question to as the number of invariants needed to generate the ring of invariants and their explicit form. As we know the ring $\mathbb{F}[V]^G$ is known to be Cohen-Macaulay if the order of G is relatively prime to the characteristic of $\mathbb{F}$ [HE], S1 or if dimension of V is 1, 2 or 3 over $\mathbb{F}$ [S1, S3]. As to find a set of generators, in the case of finite pseudoreflection groups, a systematic method exists for producing a set of generators which is in some sense minimal.

三、結果與討論

If $\rho : G \to GL(n,\mathbb{F})$ is faithful representation and $\rho(G) \subseteq GL(n,\mathbb{F})$ is generated by pseudoreflections then G is called a *pseusoreflection group.* Chevalley, Shephard and Todd proved that $\mathbb{F}[V]^G$ is a polynomial ring if and only if *G is generated by* pseudoreflections. This theorem holds over a field of arbitrary characteristic p as long as p does not divide the order of G. [Ch], [ST]. In fact, the "only if" part of the theorem holds for arbitrary p [Se]. It is also known that the "if" part is false if $p|G$. An example is given by the Weyl group $W(\mathbf{F}_4)$.

Suppose G is a finite reflection group and $\mathbb{F}[V]^G = \mathbb{F}[I_1, \ldots, I_n]$. Let $d_1, d_2, \ldots, d_n$ be the respective degrees of the homogeneous invariants $I_1, I_2, \ldots, I_n$, where $d_1 \leq d_2 \leq \cdots \leq d_n$. It can be shown that the $d_i's$ are independent of the particular basis $I_1, \ldots, I_n$. Coexeter and Coleman [Cx2], [Co] gave a method for computing the $d_i' s$ which is restricted to the case where the underlying field $\mathbb{F}$ is real. If such a group G acts on the n-dimensional Euclidean space $\mathbb{R}^n$, then its reflection hyperplane divides $\mathbb{R}^n$ into $|G|$ components, called the chambers of G. Each chamber is bounded by n reflection hyperplanes called its walls. The reflections in these walls generate G. The d_i's and the eigenvalues of the product of these generators are related [Cx1]. Solomon [So] also obtained a formula for the $d_i s$ which works for all fields of characteristic 0 but is not as effective as the methos of Coxeter and Colman for the real case.

Let $S \subset GL(n, F)$ be a set of psuedoreflections.If s is a pseudoreflection with hyperplane defined by L_s, then for each $f \in \mathbb{F}[V]$

$$sf - f = \Delta_s(f) L_s$$

in $\mathbb{F}[V]$ for a unique $\Delta_s(f) \in \mathbb{F}[V]$. One can easily seen that

$$\Delta_s(f) = 0 \Longleftrightarrow sf = f$$

. Define the ideal of generalized invariants $I(S)$ as

$$I(S) = \{f \in \mathbb{F}[V] |\; deg(f) > 0 \text{ and } \Delta_{s_1}$$

$$\ldots \Delta_{s_{deg(f)}}(f) = 0 \,\forall\, s_1, \ldots, s_{deg(f)} \in S\}$$

where $\Delta_{s_1} \ldots \Delta_{s_{deg(f)}}$ denotes the composition of the operators $\Delta_{s_1}, \ldots, \Delta_{s_{deg(f)}}$. If $s \in GL(V)$ is a pseudoreflection then L_s and Δ_s are well-defined up to a nonzero scalar. Hence $I(S)$ depends only on S and not on the choice of L_s for $s \in GL(V)$. Kac and Peterson [KP] have shown that $I(S)$ is generated by a regular sequence $h_1, \ldots, h_n \in \mathbb{F}[V]$.

If A is a graded connected commutative algebra over a field $\mathbb{F}$ and $I \subset A$ a proper ideal, we can associate a bigraded algebra defined by

$$gr_I(A)_{m,n-m} = (I^m / I^{m+1})_n$$

for $n, m \in \mathbb{N}$. We thus have a positively graded commutative algebra over $\mathbb{F}$ containing A/I as a subalgebra, and $\overline{gr}_I(A) = \mathbb{F} \otimes_{A/I} gr_I(A)$ is a graded connected commutative algebra over $\mathbb{F}$

Taking I to be $I(S)$ for pseudoreflections $S \subset GL(n, \mathbb{F})$, the ring of generalized invariants is $\overline{gr}_{I(S)}(\mathbb{F}[V])$. It can be shown that this is a polynomial algebra.

If $S \subset GL(n, \mathbb{F})$ is a collection of pseudoreflections generating the subgroup $G(S)$. The ideal $I(S)$ of generalized invariants of S contains the $G(S)$-invariant polynomials of degree > 0 and hence the ideal $(\overline{\mathbb{F}[V]^G})$ that they generate. If $G(S)$ is a finite group of order prime to the characteristic of the field $\mathbb{F}$, then $\mathbb{F}[V]^{G(S)} = \mathbb{F}[f_1, \ldots, f_n]$ and $I(S) = (f_1, \ldots, f_n)$. This is a generalization of Shephard and Todd [TS].

Suppose A is a graded connected algebra over a field $\mathbb{F}$. Let $\overline{A}$ be the augmentation ideal in A generated by the elements of positive degrees. If G is a group and $\alpha : G \rightarrow Aut(A)$ a faithful representation of G as a group of algebra automorphisms of A, denote by $A^G = \{a \in A \mid \alpha(g)a = a \;\forall g \in G\}$ the subalgebra of A left invariant by G. The algebra A^G is also graded and connected. The algebra of coinvariants $A_G := \mathbb{F} \otimes_{A^G} A$, where A is regarded as an A^G module via the inclusion and $\mathbb{F}$ via the augmentation. The algebra of coinvariants may also be viewed as A/I where I is the ideal generated by $\{a \in A |\; deg(a) > 0\}$. The ideal I is stable under the action of G on A so there is induced an action G on A_G. It is therefore possible to repeat the above constructions formating $(A_G)^G, A_{GG} := (A_G)_G$, etc. By iteration, we obtain a sequence of algebra epimorphisms

$$A \to A_G \to A_{GG} \to \ldots \to A_{G\ldots G} \to \ldots$$

Let J_m denote the kernel of the map $A \to A_{\underbrace{G\ldots G}_{m}}$. These ideals form an ascending chain

$$(0) = J_0 \subseteq J_1 \subseteq J_2 \subseteq \ldots \subseteq J_m \subseteq \ldots \subseteq A$$

in A. The ideal $J_\infty := \bigcup J_m \subset A$ is called the ideal of stable invariants. It can be characterized by the following conditions

(i) $J_\infty \subset A$ is stable under the action of G,

(ii) $(A/J_\infty)^G \cong \mathbb{F}$ (i. e., G acts fixed point free on $\overline{(A/J_\infty)}$

(iii) if $I \subset A$ is an ideal such that $(A/G)^G \cong \mathbb{F}$ then $J_\infty \subseteq I$.

The ring $\overline{gr}_{J_\infty(G)}$ is called the ring of stable invariants of G.

If $\rho : G \to GL(n, \mathbb{F})$ is a representation of a finite group then G acts on $\mathbb{F}[V]$ and there is the ideal of stable invariants $J_\infty(G) \subset \mathbb{F}[V]$. If G is a pseudoreflection group and $|G|$ is relatively prime to the characteristic of $\mathbb{F}$, Chevalley [Ch] has shown that $\mathbb{F}[V]_G$ is the regular representation of G. In particular, $(\mathbb{F}[V]_G)^G \cong \mathbb{F}$ and hence $J_\infty(G) = J_1(G) = (\overline{\mathbb{F}[V]^G})$ is the ideal of $\mathbb{F}[V]$ generated by the G-invariants of positive degree. If however, $|G| = 0 \in \mathbb{F}$ then $J_\infty(G)$ can be distinctly larger than $J_1(G)$.

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