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RRPA87041280 (17 .P)

行政院國家科學委員會專題研究計畫成果報告

送微縮小組

在不可度量化空間上的布朗伯格問題

計畫編號: NSC 87 - 2115 -M - 032 - 004
執行期間: 86 年 8 月 1 日 至 87 年 7 月 31 日

計畫主持人: 曾琇瑱

處理方式: 兩年後可對外提供參考
(必要時,本會得展延發表時限)

執行單位: 淡江大學數學系
中華民國 八十七 年 十 月 六 日

The Blumberg Problem in Nonmetrizable Spaces †‡

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Abstract. In this work, we investigate the correlation between the Blumberg property and Baire spaces. We modify Blumberg's idea which are descriptive and metric and replace the metrizable condition to the nonmetrizable condition. We show that real-valued functions defined on nonmetrizable spaces have some properties given by Blumberg. We also show such spaces have some property similar to Blumberg property.

† 1991 *Mathematics Subject Classification.* 26A21, 54C50

‡ Key Words. real-valued function, nonmetrizable space, weak Blumberg property

1. Introduction

We know that continuity plays an important role in Mathematics. In 1922, Blumberg [1] showed that for each real-valued function f defined on a separable complete and metrizable space X, there exists a dense subset D of X such that $f|_D$ is continuous. Hence, we say that this property is the *Blumberg property.* A space which has Blumberg property is called a *Blumberg space.* In 1990-1991, Baldwin[2] show that a real-valued function acting on a categorically dense subset has the same result as Blumberg's. He strengthened Blumberg's theorem. In our work, we weakened Blumberg's theorem. In 1960, Bradford and Goffman [3] showed that every Blumberg space is a Baire space and the converse is true if the Baire space is metrizable. Clearly, the structure of metrizable spaces having the Blumberg space is well developed in the literature. We notice that discussing the continuity of a function defined on a nonmetrizable space is more difficult. In 1974, White [7] proved that X is a Baire space if and only if for every function $f : X \to Y$ where Y is a countable topological space, then there is a dense set D such that $f|_D$ is continuous. However, it is not always true if Y is not countable. In 1977, Weiss [6] constructed a compact Hausdorff non-Blumberg space. We know that every compact Hausdorff space is a Baire space. Therefore, there exists a non-Blumberg Baire space. In this work, we consider spaces without metrizable property and show that most properties which Blumberg [1] did hold for these spaces. We modify Blumberg's idea by using densely approach to analyze the structures of real-valued functions defined on a nonmetrizable space and give a new property of any real-valued function defined on a nonmetrizable space. Also we extend the whole structure to functions with image in $\mathbf{R}^n$. We modify Blumberg's idea by using other topological properties to replace the metrizable condition. Then, it is possible for real-valued functions defined on nonmetrizable spaces to have the properties given by Blumberg in 1922. Also, we extend the whole structure to functions with image in $\mathbf{R}^n$.

2. Weak Blumberg Property

Definition 2.1. *Let $\Re$ be a binary relation between open sets and elements, let U be an open subset of a topological space X and let x be an element of X. $U\Re p$ means that the open set U has the relation $\Re$ to the element p. The relation $\Re$ is closed if for every subset A of X, the relationships $U\Re s$ for all $s \in A$ and $p \in \overline{A}$ imply $U\Re p$.*

Definition 2.2. *A partial neighborhood, denoted by $N_<$, of p is an open set of which p is an interior or a boundary element.*

Lemma 2.3. *[1] If $\Re$ is a closed relation, then the elements p for which*

(a) $N\Re p$ for every open neighborhood N of p, and

(b) a partial neighborhood $N_<$ of p exists such that $N_<\Re p$ is false,

constitute a nowhere dense set.

Lemma 2.4. *Let X be any topological space and $f : X \to \mathbf{R}$ be any real-valued function defined on X. We define as follows the relation $\Re_{r_1 r_2}$, where r_1 and r_2 are any two real numbers and $r_1 < r_2$: If p is an element of X and U is an open subset of X, then $U \Re_{r_1 r_2} p$ if and only if $p \in \overline{U}$ and an element q of U exists such that $r_1 \leq f(q) < r_2$. Then $\Re_{r_1 r_2}$ is closed.*

Let $\mathbf{Q}$ be the set of all rational numbers in $\mathbf{R}$. And let $\mathbf{N}$ be the set of all natural numbers.

Lemma 2.5. *If f is a real-valued function defined on a topological space X, then for every pair of rational numbers r_1,r_2 where $r_1 < r_2$, the elements p of X for which*

(a) $N \Re_{r_1 r_2} p$ for every open neighborhood N of p, and

(b) $N_< \Re_{r_1 r_2} p$ is false for some partial neighborhood $N_<$ of p,

constitute a nowhere dense set, say, $T_{r_1 r_2}$. Thus, $\cup_{r_1, r_2 \in \mathbf{Q}} T_{r_1 r_2}$ *is of the first category.*

Definition 2.6. *A function $f : X \to \mathbf{R}$ is densely approached at point p of X if and only if for each $\epsilon > 0$ there exists an open neighborhood N of p such that the elements q of N for which $|f(q) - f(p)| < \epsilon$ form a dense set in N.*

Definition 2.7. *A subset of a topological space is residual if its com-*

plement is of the first category.

Theorem 2.8. *For every real-valued function $f : X \to \mathbf{R}$ where X is a topological space, the elements x of X at which f is densely approached constitute a residual set.*

Proof. We have to show that the set of elements p of X at which f is not densely approached is of the first category. If f is not densely approached at p, then there exists a positive number ϵ such that for every open neighborhood N of p, $N \cap f^{-1}(f(p) - \epsilon,\ f(p) + \epsilon)$ is not dense in N. For each open neighborhood N of p, there exists an open set U_N such that $U_N \cap N$ is nonempty and $|f(q_N) - f(p)| \geq \epsilon$ for all q_N in $U_N \cap N$. Let $U = \cup_N(U_N \cap N)$, then U is a partial neighborhood of p and $|f(q) - f(p)| \geq \epsilon$ for all q in U. Let r_1 and r_2 be two rational numbers with $f(p) - \epsilon < r_1 < f(p) < r_2 < f(p) + \epsilon$. Then $U\Re_{r_1 r_2}p$ is false and $M\Re_{r_1 r_2}p$ for all neighborhood M of p. Therefore p is an element of $T_{r_1 r_2}$ (It was defined in Lemma 2.5). $\cup_{r_1 r_2 \in \mathbf{Q}} T_{r_1 r_2}$ is of the first category. Thus, the theorem holds.

Definition 2.9. *A function $f : X \to \mathbf{R}$ is exhaustibly approached at* point p *of X if and only if an open neighborhood N of p and a number $\epsilon > 0$ exist such that the set of elements q of N where $|f(q) - f(p)| < \epsilon$ is of the first category. If f is not exhaustibly approached at p then we say that f is inexhaustibly approached* at p.

Theorem 2.10. *For every real-valued function $f : X \to \mathbf{R}$ where X*

is separable, the set constituted by elements x of X at which f is exhaustibly approached is of the first category.

Proof. Let $S_{rn} = \left(r - \frac{1}{n},\ r + \frac{1}{n}\right)$ where $r \in \mathbf{Q}$, $n \in \mathbf{N}$. If f is exhaustibly approached at q, then there exist a positive number ϵ and an open neighborhood N_q of q, such that $f^{-1}\left(f\left(q\right) - \epsilon\ ,\ f\left(q\right) + \epsilon\right) \cap N_q$ is of the first category. Let's pick the set S_{rn} such that $f\left(q\right) \in S_{rn} \subset \left(f\left(q\right) - \epsilon,\ f\left(q\right) + \epsilon\right)$. Then $f^{-1}\left(S_{rn}\right) \cap N_q$ is of the first category. Let $E_{rn} = \{x \in f^{-1}(S_{rn}) \mid$ a neighborhood N_x of x exists such that $f^{-1}\left(S_{rn}\right) \cap N_x$ is of the first category$\}$. Then the countable union of E_{rn} where $r \in \mathbf{Q}$ and $n \in \mathbf{N}$ is the set of all elements at which f is exhaustibly approached. Now, it is sufficient to prove that every E_{rn} is of the first category.

Given E_{rn}. X is separable, i.e. X has a countable dense subset, say, $P = \{p_i \mid i \in \mathbf{N}\}$. For each x in E_{rn}, there exists an open neighborhood N_x of x such that $f^{-1}\left(S_{rn}\right) \cap N_x$ is of the first category. P is dense. So, for each N_x, there exists an element p_i of P such that p_i belongs to N_x. Let $\mathscr{C}_i = \{N_x \mid p_i \in N_x\}$. Take an element M_i from each nonempty family $\mathscr{C}_i$. Then we claim that $E_{rn} \setminus (\cup_i M_i)$ is nowhere dense. We prove it by contradiction. Suppose that the claim is not true. Then $\overline{E_{rn} \setminus (\cup_i M_i)}$ contains a nonempty open subset U. Pick an element y from the set $U \cap (E_{rn} \setminus (\cup_i M_i))$. Then $N_y \cap U$ is an open neighborhood of y. This implies that there exists an element p_k of P such that p_k is in $N_y \cap U$. $\mathscr{B}_k$ is nonempty and p_k belongs to $\cup_i M_i$. Thus, p_k belongs to $U \cap (\cup_i M_i)$. Since $\cup_i M_i$ is open, $X \setminus (\cup_i M_i)$ is closed. $U \subset \overline{E_{rn} \setminus (\cup_i M_i)} \subset \overline{X \setminus (\cup_i M_i)} = X \setminus (\cup_i M_i)$. Thus, $U \cap (\cup_i M_i)$ is empty. It contradicts to that p_k belongs to $U \cap (\cup_i M_i)$. Hence, $\cup_i \left(f^{-1}(S_{pn}) \cap M_i\right) \cup (E_{rn} \setminus (\cup_i M_i))$ is of the first category. Also,

$$\begin{aligned} E_{rn} &= E_{rn} \bigcap \left(\bigcup_i M_i\right) \bigcup \left(E_{rn} \setminus \bigcup_i M_i\right) \\ &\subset f^{-1}(S_{rn}) \bigcap \left(\bigcup_i M_i\right) \bigcup \left(E_{rn} \setminus \bigcup_i M_i\right) \\ &= \bigcup_i \left(f^{-1}(S_{rn}) \bigcap M_i\right) \bigcup \left(E_{rn} \setminus \bigcup_i M_i\right). \end{aligned}$$

We have that E_{rn} is of the first category. Hence, the theorem holds.

If M is a subset of X, we shall use, in connection with approach, the expression "via M" to indicate that p is restricted to range in M.

Definition 2.11. *A function f is inexhaustibly approached at p via M if for each open neighborhood N of p and each positive number ϵ, the set of elements q of $N \cap M$ where $|f(p) - f(q)| < \epsilon$ is of the second category; otherwise, f is exhaustibly approached at p via M.*

We know that if A is of the first category, then every subset of A is also of the first category. Thus, if f is exhaustibly approached at p, then f is exhaustibly approached at p via M. On the other hand, if f is inexhaustibly approached at p via M, then f is inexhaustibly approached at p.

Definition 2.12. *A function f is densely approached at p via M if and only if for each $\epsilon > 0$ there exists an open neighborhood N of p such that the elements q of $M \cap N$ for which $|f(p) - f(q)| < \epsilon$ form a dense subset of $M \cap N$.*

Definition 2.13. *If A is a subset of the topological space X and if x is a point of X, we say that x is a limit of A if and only if every open neighborhood of x intersects A in some element other than x itself.*

Lemma 2.14. *Let f be a real-valued function defined on a topological space X and p be an element of X. If M is a subset of X such that p is a limit of M, the following statements are equivalent.*

(1) $f : X \to \mathbf{R}$ is densely approached at p via M.

(2) For every partial neighborhood $N_<$ of p such that $N_< \cap M$ has p as a limit, the set

$$(N_< \cap M)' = \{(x, f(x)) | x \in N_< \cap M\}$$

has $p' = (p, f(p))$ as a limit.

Theorem 2.15. *For every real-valued function f defined on a separable Hausdorff space X, there exists a residual subset S of X such that if p is an element of S then the function f is inexhaustibly ,and therefore densely, approached at p via S.*

Proof. Let E_1 be the set of elements at which f is exhaustibly approached, and let $S_1 = X \setminus E_1$. By Theorem 2.10, we know that E_1 is of the first category. Then S_1 is residual and f is inexhaustibly approached at the elements of S_1. That is, if q is an element of S_1, then for each $\epsilon > 0$ and for each open neighborhood N of q, there exists a subset M of N such that M is of the second category and $|f(x) - f(q)| < \epsilon$ for all x in M. $M \cap S_1 = M \setminus E_1$ is of the second category since $M \cap E_1$ is of the first category and M is of the second category. Thus, f is inexhaustibly approached at the elements of S_1 via S_1.

Case1. Suppose that f is bounded. Then there exists a real number k such that $k > sup\{f(x)|x \in X\}$. Let's define a function $g : X \to \mathbf{R}$ by:

$$g(x) = \begin{cases} f(x) & \text{if } x \in S_1, \\ k & \text{if } x \in E_1. \end{cases}$$

Then, by Theorem 2.8, the elements of X at which g is densely approached constitute a residual set, say, S_g. For every element r in E_1 and for every s in S_1, $g(r) - g(s) \geq k - sup\{f(x)|x \in X\} > 0$. g is densely approached at the elements of $S_1 \cap S_g$ via S_1. f is densely approached at the elements of $S_1 \cap S_g$ via S_1. Furthermore, $S_g \cap S_1 = \left(S_g^c \cup S_1^c\right)^c = X \setminus \left(S_g^c \cup S_1^c\right)$ is a residual set since S_g^c and S_1^c are of the first category. It follows that the elements of S_1 at which f is densely approached via S_1 constitute a residual set, say, S. Let $E_2 = S_1 \setminus S$. Then $X = E_1 \cup E_2 \cup S$. It implies that E_2 is of the first category. For every element p of S, f is inexhaustibly approached at p via S since E_2 is of the first category and f is inexhaustibly approached at the elements of S_1 via S_1. Also, f is densely approached at p via S_1. For every partial neighborhood $N_<$ of p, if p is a limit of $N_< \cap S$ then p is a limit of $N_< \cap S_1$. By Lemma 2.14, it implies that $p' = (p, f(p))$ is a limit of $(N_< \cap S_1)' = \{(x, f(X)) | x \in N_< \cap S_1\}$. That is, every open neighborhood of p' intersects $(N_< \cap S_1)'$ in some element other than p' itself. For each $\epsilon > 0$ and for each open neighborhood N of p, there exists an element which is distinct from p, say, p^* such that p^* belongs to $N \cap (N_< \cap S_1)$ and $|f(p^*) - f(p)| < \epsilon/2$. It follows that f is inexhaustibly approached at p^* via S since p^* belongs to S_1 and $S_1 \setminus S = E_2$ is of the first category. X is Hausdorff so $\{p\}$ is closed. It follows that $(N \cap N_<) \setminus \{p\}$ is an open neighborhood of p^*. Thus, a subset M of $(N \cap N_< \cap S) \setminus \{p\}$ exists such that M is of the second category and for every elements q^* of M, $|f(q^*) - f(p^*)| < \epsilon/2$. By triangle inequality, we have that $|f(q^*) - f(p)| < \epsilon$ for all q^* in M. We can pick an element q from M such that q is distinct from p, $q \in N \cap N_< \cap S$ and $|f(q) - f(p)| < \epsilon$. It implies that p is a limit of $(N_< \cap S)'$. Hence, f is densely approached at p via S.

Case2. Suppose that f is unbounded. We define a function $\overline{f} : X \to \mathbf{R}$ by $\overline{f}(x) = f(x)/(1 + |f(x)|)$. Then $\overline{f}$ is bounded and the properties of densely approach, exhaustibly approach and inexhaustibly approach of f are preserved by $\overline{f}$. Thus, the proof is done.

Corollary 2.16. *For every real-valued function f defined on a separable*

Hausdorff Baire space X, there exists a dense subset S of X such that if p is an element of S and $N_<$ is a partial neighborhood of p then the function f is inexhaustibly approached at p via $S \cap N_<$.

Proof. By Theorem 2.15, there exists a residual set S of X, such that if p is an element of S, then f is densely approached at p via S. That is, if $M_<$ is a partial neighborhood of p such that p is a limit of $M_< \cap S$, then $p' = (p, f(p))$ is a limit of $(M_< \cap S)' = \{(x, f(x)) | x \in M_< \cap S\}$. Given a partial neighborhood $N_<$ of p, $N_< \setminus \{p\}$ is open since X is Hausdorff. The assumption that X is a Baire space implies that S is dense. So, p is a limit of $N_< \cap S$. It follows that p' is a limit of $(N_< \cap S)'$. So, for each open neighborhood N of p and for each $\epsilon > 0$, there exists an element p^* which is distinct from p such that p^* belongs to $N \cap N_< \cap S$ and $|f(p^*) - f(p)| < \epsilon/2$. p^* belongs to S, hence, by Theorem 2.15, f is inexhaustibly approached at p^* via S. Since $N_< \cap N$ is an open neighborhood of p, there exists a subset M of $N_< \cap N \cap S$ such that M is of the second category and $|f(p^*) - f(q)| < \epsilon/2$ for all q in M. It implies that $|f(q) - f(p)| < \epsilon$ for all q in M. Thus, f is inexhaustibly approached at p via $N_< \cap S$.

Definition 2.17. *We say that X is a weak Blumberg space or X has weak Blumberg property if and only if for each real-valued function f defined on X, there exists a dense subset D of X such that if p is an element of D then for each $\epsilon > 0$ there exists an open neighborhood*

N of p such that the elements q of $D \cap N$ for which $|f(p) - f(q)| < \epsilon$ constitute a dense subset of $D \cap N$.

Theorem 2.18. *Every separable Hausdorff Baire space has the weak Blumberg property.*

Proof. We know that every residual set of a Baire space is dense. Hence, by Theorem 2.15, the theorem holds.

Example 2.19. $[0,1]^{2^{\aleph_0}}$ *is a weak Blumberg space.*

Proof. $[0,1]^{2^{\aleph_0}}$ is a separable compact Hausdorff space. Every compact Hausdorff space is Baire. So, $[0,1]^{2^{\aleph_0}}$ is a separable Hausdorff Baire space. By Theorem 2.18, $[0,1]^{2^{\aleph_0}}$ is a weak Blumberg space.

$[0,1]^{2^{\aleph_0}}$ is nonmetrizable, so it is a nonmetrizable weak Blumberg space.

3. Multivalued Functions

We consider the functions with image in $\mathbf{R}^n$.

Definition 3.1. Let $F : X \to \mathbf{R}^n$ be given by

$$F(p) = (f_1(p), f_2(p), \ldots, f_n(p)), \forall p \in X$$

where $f_i : X \to \mathbf{R}$ are real-valued functions for all i. Then every $f_i : X \to \mathbf{R}$ is called a *coordinate function* of F.

Definition 3.2. A function $F : X \to \mathbf{R}^n$ is *densely approached* at p of X if and only if every coordinate function f_i of F is densely approached at p.

Theorem 3.3. *For every function $F : X \to \mathbf{R}^n$ where X is any topological space, the elements of X at which F is densely approached constitute a residual set.*

Proof. By Theorem 2.8, for each coordinate function f_i of F, there exists a residual set S_i of X such that f_i is densely approached at p_i for all p_i in S_i. Let $S = \cap S_i$. Then F is densely approached at p for all p in S. S^c is of the first category since $S^c = (\cap S_i)^c = \cup S_i^c$ is countable union of sets which are of the first category. Hence, the theorem holds.

Definition 3.4. *Let p be an element of a topological space X. We say that a function $F : X \to \mathbf{R}^n$ is exhaustibly approached at p if and only if there exists a coordinate function f_i of F such that f_i is exhaustibly approached at p. And if F is not exhaustibly approached at p, then we say that F is inexhaustibly approached at p. That is, F is inexhaustibly approached at p if and only if every coordinate function f_i of F is inexhaustibly approached at p.*

Theorem 3.5. *For every function $F : X \to \mathbf{R}^n$ where X is separable, the set constituted by elements x of X at which F is exhaustibly approached is of the first category.*

Proof. By Theorem 2.10, for each coordinate function f_i of F, there exists a subset E_i of X such that E_i is of the first category and is the set of elements at which f_i is exhaustibly approached. Let $E = \cup_{i=1}^{n} E_i$. Then E is the set of elements at which F is exhaustibly approached. Furthermore, E is of the first category since it is a finite union of sets which are of the first category.

Definition 3.6. *Let M be a subset of a topological space X. We say that $F : X \to \mathbf{R}^n$ is inexhaustibly approached at p via M if and only if every coordinate function f_i of F is inexhaustibly approached at p via M.*

Definition 3.7. *Let M be a subset of a topological space X. We say that $F : X \to \mathbf{R}^n$ is densely approached at p via M if and only if every coordinate function f_i of F is densely approached at p via M.*

Theorem 3.8. *For every function $F : X \to \mathbf{R}^n$ where X is a separable Hausdorff Baire space, there exists a residual subset S of X such that if p is an element of S and $N_<$ is a partial neighborhood of p then the function F is inexhaustibly approached at p via $S \cap N_<$ and therefore, densely approached at p via S.*

Proof. By Corollary 2.16, we have that for each coordinate function $f_i$of F, there exists a residual set S_i such that if p_i is an element of S_i and $N_{<i}$ is a partial neighborhood of p_i, then f_i is inexhaustibly approached at p_i via $S_i \cap N_{<i}$. Let $S = \cap S_i$. $S^c = \cup S_i^c$ is of the first category since

S_i^c are of the first category for all i. Then S is a residual set. Given an element p of S and a partial neighborhood $N_<$ of p, then for each i, f_i is inexhaustibly approached at p via $S_i \cap N_<$. $S_i \setminus S = S_i \cap S^c$ is of the first category. It follows that f_i is inexhaustibly approached at p via $S \cap N_<$ for all i. So F is inexhaustibly approached at p via $S \cap N_<$, for all p in S and for all partial neighborhood $N_<$ of p.

Let $M_<$ be a partial neighborhood of p such that p is a limit of $M_< \cap S$. Then f_i is inexhaustibly approached at p via $M_< \cap S$ for all i. Given i. For each positive number ϵ and for each open neighborhood N of p, there exists a subset B_i of $N \cap M_< \cap S$ such that $|f_i(b_i) - f_i(p)| < \epsilon$ for all b_i in B_i and B_i is of the second category. So, there exists an element q of $B_i \subset N \cap M_< \cap S$ which is distinct from p such that $|f_i(q) - f_i(p)| < \epsilon$. $p' = (p, f_i(p))$ is a limit of $(M_< \cap S)' = \{(x, f_i(x)) | x \in M_< \cap S\}$. Thus, f_i is densely approached at p via S for all i. It implies that F is densely approached at p via S for all $p \in S$.

Corollary 3.9. *If X is a separable Hausdorff Baire space then for each function $F : X \to \mathbf{R}^n$, there exists a dense subset D of X such that if p belongs to D then for each $\epsilon > 0$ there exists an open neighborhood N of p such that the elements q of $D \cap N$ for which $|f_i(p) - f_i(x)| < \epsilon$ constitute a dense subset of $D \cap N$ for all coordinate function f_i of F.*

Proof. By Theorem 3.8, the corollary holds since every residual set of a Baire space is dense and a finite intersection of open sets is open.

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