行政院國家科學委員會專題研究計畫成果報告

計畫名稱：魔方陣的研究(Research on magic squares)

計畫編號: NSC 86-2115-M-032-003

執行期限: 86 年 8 月 1 日至 87 年 7 月 31 日

主持人: 高金美

E-mail: cmfu@mail.tku.edu.tw

執行機構及單位名稱: 淡江大學數學系

Some results on pandiagonal magic squares

一、中文摘要

一個 n 階魔方陣是指在一個方陣中填入 n^2 個相異正整數, 使其中每行每列及兩個主對角線上的數字和均相等的方陣. 一個 n 階泛魔方陣是指在一個方陣中填入 1 到 n^2 個相異正整數, 使其中每行每列及所有斷掉和未斷的對角線上的數字和均相等的方陣. 在這計劃中我們得到的結論是一個 4 階泛魔方陣其奇數次方之後仍然是一個泛魔方陣.

關鍵詞: 魔方陣, 泛魔方陣

Abstract

A magic square of order n is an n×n array such that each row, each column and two main diagonal have the same sum. A pandiagonal magic square of order n is an n×n array with integers 1, 2, …, n^2 such that each row, each column and each broken or unbroken diagonals have the same sum. In thithis plan, we got that any odd power of a papandiagonal magic square of order 4 is stistill a pandiagonal magic square.

K
m

Keywords: magic square, pandiagonal magic square

I. Introduction

A semi-magic square is an n×n array such that each row and each column have the same sum. A magic square of order n is an n×n array such that each row, each column and two main diagonal have the same sum. A pandiagonal magic square of order n is an $n \times n$ array with integers 1, 2, …, n^2 such that each row, each column and each broken or unbroken diagonals have the same sum. The set of magic squares of order n(resp., the set of semi-magic squares of order n, the set of pandiagonal magic squares of order n) we denote by Mag(n) (resp., Smag(n), Pmag(n)). Let J be the n-by-n matrix satisfying $J_{ij} = 1$ if $i + j = n+1$, otherwise $J_{ij} = 0$.

II. Main result

Proposition 1. Let s(M) be the sum of each column. Then the eigenvalues of any pandiagonal magic square are s(M), 0, λ, and $-\lambda$, where λ is any real number.

Proposition 2. The power of any odd number of pandiagonal magic squares of order 4 is a pandiagonal magic square.

Proof. Let M be a member of Pmag(4). Define $M^0 = M - tr(M)\cdot E/4$, where E is the 4-by-4 matrix with entries equal to 1 and tr(M) denotes the trace of M. Since E is a pandiagonal magic square of order 4, M^0 is a pandiagonal magic square of order 4. From the definition, we know that $tr(M^0) = 0$, since $tr(E) = 4$. Thus $s(M^0) = 0$. By Proposition 1, we obtain that the eigenvalues of M^0 are 0, 0, λ, and $-\lambda$, where λ is any real number. Hence the characteristic polynomial of M^0 is $p(x) = x^2(x^2 - \lambda^2)$.

From Caley-Hamilton theorem, we have $(M^0)^3 = \alpha M^0$, where α is a real number. Thus $(M^0)^3$ is a pandiagonal magic square of order 4. Consequently $(M^0)^5 = \alpha (M^0)^3 = \alpha^2 M^0$ is a pandiagonal magic square. Therefore $(M^0)^p$ is a pandiagonal magic square for all odd $p > 1$.

Now we have $M^0 = M - tr(M)\cdot E/4$, i.e., $M = M^0 + tr(M)\cdot E/4$.

$$M^p = (M^0 + tr(M)\cdot E/4)^p$$

$$= (M^0)^p + p^* (M^0)^{p-1}\, tr(M)\cdot E/4 + \ldots + p^* (M^0)\, (tr(M)\cdot E/4)^{p-1} + (tr(M)\cdot E/4)^p$$

Since $M^0\cdot E$ is equal to zero matrix, we have $M^p = (M^0)^p + (tr(M)\cdot E/4)^p$

$$= (M^0)^p + (tr(M)/4)^p \cdot E^p$$

$$= (M^0)^p + ((tr(M))^p/4)\cdot E$$

From the definition of E, we get

$E^p = 4^{p-1}\cdot E$. For any odd integer p, $(M^0)^p$ is a pandiagonal magic square. Therefore M^p is a pandiagonal magic square. Q.E.D.

For the discussion of pandiagonal magic squares of order 5, we can not get the same result. But we have the power of any odd number of pandiagonal magic squares of order 5 is a magic square. At the same time we generalize this result. For this part , you will see it in my student's master thesis.

III. Reference.

1. J.E. Ward III, Vector spacs of magic squares, Math. Mag.,53 (1980) 108-111.

2. A.V. D. Essen, Magic squares and Linear algebra, The teaching of Mathematics, January (1990) 60-62.