行政院國家科學委員會專題研究計畫 成果報告

消去法之研究
研究成果報告(精簡版)

計 畫 類 別 ： 個別型
計 畫 編 號 ： NSC 97-2115-M-032-006-
執 行 期 間 ： 97 年 08 月 01 日至 98 年 08 月 31 日
執 行 單 位 ： 淡江大學數學系

計畫主持人： 吳孟年

計畫參與人員： 此計畫無其他參與人員

報 告 附 件 ： 國外研究心得報告

處 理 方 式 ： 本計畫可公開查詢

中 華 民 國 98 年 10 月 25 日

Research on Elimination Methods
消去法之研究
(NSC 97-2115-M-032-006)

Mengnien Wu

October 25, 2009

這裡有一個問題 是這樣的:

給定一組 資料點集 $D = \left\{ D_{i,j} \;\middle|\; {}^{i=0,\ldots,M}_{j=0,\ldots,N} \right\} \subset \mathbb{R}^3$, 並且 定義 $P = \left\{ P_{i,j} \;\middle|\; {}^{i=0,\ldots,M}_{j=0,\ldots,N} \right\}$, 其中 P 裡頭除了編號 $(0,0),(M,0),(0,N),(M,N)$ 的點與 D 裡同編號的點相同外 (這裡稱這四點爲「角點」), 其餘編號的 x,y 座標 與 D 裡同編號點的 x,y 座標相同, 但 z 座標皆設爲不定元。

如果 我們要求 以點集 P 爲控制點 定義的 Bézier 曲面

$$B(u,v) = \sum_{i=0}^{M} \sum_{j=0}^{N} P_{i,j} \binom{M}{i} u^i (1-u)^{M-i} \binom{N}{j} v^j (1-v)^{N-j}$$

通過 D, 要如何訂出 非角點的 z 座標呢?

以 $M=2, N=2$, 並且 D 排在 2×2 的網格上方。由於 Bézier 曲面只會通過四個角點, 我們必須調整其他五個控制點 來通過資料點。這裡的作法是 先把 u,v 的參數式化爲 x,y,z 的隱式, 然後把資料點帶入, 求得控制點的 z 座標。

我們發現, 因爲 x,y 座標在格線上, 故 $x = x(v)$, $y = y(u)$, 並且 x,y,z 的 u,v 參數式 係數完全互不相關。爲了簡化起見, 可以假設一般爲

$$\begin{aligned}
x(v) &= av^2 + bv + c \\
y(u) &= du^2 + eu + f \\
z(u,v) &= u^2(gv^2 + hv + i) + u(jv^2 + kv + l) + (mv^2 + nv + o)
\end{aligned}$$

應用「吳方法」一連串的僞除法來降低階數, 達成消去 參數 u,v 的目的:

$$\begin{array}{lll}
r_1 := \operatorname{psrem}(z,x;v), & 即\ r_1 = m_1 z - q_1 x, & \deg r_1 = (2,1), \deg m_1 = (0,0), \deg q_1 = (2,0) \\
r_2 := \operatorname{psrem}(r_1,y;u), & 即\ r_2 = m_2 r_1 - q_2 y, & \deg r_2 = (1,1), \deg m_2 = (0,0), \deg q_2 = (0,1) \\
r_3 := \operatorname{psrem}(x,r_2;v), & 即\ r_3 = m_3 x - q_3 r_2, & \deg r_3 = (2,0), \deg m_3 = (2,0), \deg q_3 = (1,1) \\
r_4 := \operatorname{psrem}(r_3,y;u), & 即\ r_4 = m_4 r_3 - q_4 y, & \deg r_4 = (1,0), \deg m_4 = (0,0), \deg q_4 = (0,0) \\
r_5 := \operatorname{psrem}(y,r_4;u), & 即\ r_5 = m_5 y - q_5 r_4, & \deg r_5 = (0,0), \deg m_5 = (0,0), \deg q_5 = (1,0) \\
\quad \vdots & &
\end{array}$$

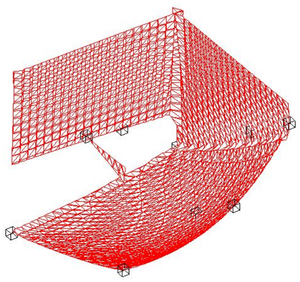
由於式子非常繁瑣, 就不將過程全部寫出。不過可以看到 僞餘式 的 u,v 次數逐漸下降。

給實際一組數據來計算。資料點 $D = \left\{ \begin{array}{ccc} (1,1,1), & (1,3,5), & (1,4,0), \\ (2,1,3), & (2,3,2), & (2,4,4), \\ (4,1,7), & (4,3,8), & (4,4,3) \end{array} \right\}$, 控制點 $P = \left\{ \begin{array}{ccc} (1,1,1), & (1,3,a), & (1,4,0), \\ (2,1,b), & (2,3,c), & (2,4,d), \\ (4,1,7), & (4,3,e), & (4,4,3) \end{array} \right\}$, 要求 以 P 定義的 Bézier 曲面 通過 D。

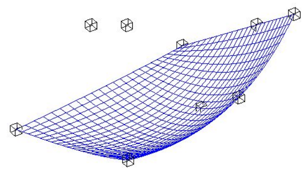
用以上方法求出的 x, y, z 隱式共有 55 項, 係數裡頭含著 a, b, c, d, e, 且 a, b, c, d, e 的次數都在 2 次以上:

$$\begin{aligned} &\ast + \ast x + \ast x^2 + \ast x^3 + \ast x^4 + \ast y + \ast xy + \ast x^2y + \ast x^3 + \ast x^4y \\ &+ \ast y^2 + \ast xy^2 + \ast x^2y^2 + \ast x^3y^2 + \ast x^4y^2 + \ast y^3 + \ast xy^3 + \ast x^2y^3 + \ast x^3y^3 + \ast x^4y^3 \\ &+ \ast y^4 + \ast xy^4 + \ast x^2y^4 + \ast x^3y^4 + \ast x^4y^4 + \ast z + \ast z\,x + \ast x^2z + \ast x^3z + \ast z\,y \\ &+ \ast z\,xy + \ast x^2z\,y + \ast x^3z\,y + \ast z\,y^2 + \ast xy^2z + \ast x^2y^2z + \ast x^3y^2z + \ast z\,y^3 + \ast z\,xy^3 + \ast z\,x^2y^3 \\ &+ \ast z\,x^3y^3 + \ast z^2 + \ast xz^2 + \ast x^2z^2 + \ast yz^2 + \ast xyz^2 + \ast x^2yz^2 + \ast z^2y^2 + \ast z^2xy^2 + \ast z^2x^2y^2 \\ &+ \ast z^3 + \ast z^3x + \ast z^3y + \ast z^3xy + \ast z^4 = 0 \end{aligned}$$

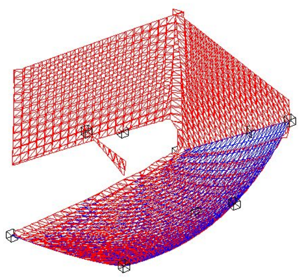
把資料點帶入, 解出控制點的 z 座標並不唯一, 而得到的一組解所構成的參數曲面, 也不過是由此隱式所定義的代數曲面 其中一片罷了。如 $(a, b, c, d, e) = (5, 0.525\ldots, -1.885\ldots, 2.232\ldots, 1.636\ldots)$,



x, y, z 隱式代數曲面



u, v 參數曲面



一起比較

二〇〇九年七月赴美研究心得報告

Mengnien Wu

Let $\square = \text{conv}\{(0,0),(0,1),(1,0),(1,1)\}$ for simplicity.
On $\square$, we define a polynomial $C(x,y)$ of multi-degree $\preccurlyeq (k,k)$

$$C(x,y) := \sum_{i=0}^{k}\sum_{j=0}^{k} c_{i,j}(x-0)^i(y-0)^j$$

For $k=3$, we made up a specific directional derivative on the boundary of $\square$(with the direction perpendicular to the boundary), that is

$$\Lambda(t) := \quad 3\left[(1-t)^2 m_0 + t^2 m_1\right] - 2\left[(1-t)^3 m_0 + t^3 m_1\right]$$

on one of four edges, where m_0 and m_1 are directional derivatives at both ends, and demand

$$\begin{array}{ll} C(i,j) & = f(i,j) \\ C_x(i,j) & = f_x(i,j) \quad \text{for } i,j=0,1, \\ C_x(i,j) & = f_x(i,j) \end{array}$$

we have obtained a unique formula for such C.

$c_{0,0} = f(0,0),$

$c_{0,1} = f_y(0,0),$

$c_{0,2} = -2f_y(0,0)-3f(0,0)+3f(0,1)-f_y(0,1),$

$c_{0,3} = 2f(0,0)+f_y(0,0)-2f(0,1)+f_y(0,1),$

$c_{1,0} = f_x(0,0),$

$c_{1,1} = 0,$

$c_{1,2} = -3f_x(0,0)+3f_x(0,1),$

$c_{1,3} = 2f_x(0,0)-2f_x(0,1),$

$c_{2,0} = -2f_x(0,0)-3f(0,0)+3f(1,0)-f_x(1,0),$

$c_{2,1} = -3f_y(0,0)+3f_y(1,0),$

$c_{2,2} = 6f_x(0,0)+6f_y(0,0)-6f_x(0,1)+3f_y(0,1)-6f_y(1,0)+3f_x(1,0)-3f_y(1,1)-3f_x(1,1)-9f(0,1)-9f(1,0)+9f(0,0)+9f(1,1),$

$c_{2,3} = -4f_x(0,0)+6f(0,1)+4f_x(0,1)+2f_x(1,1)-6f(1,1)-3f_y(0,0)+3f_y(1,0)-6f(0,0)+3f_y(1,1)+6f(1,0)-2f_x(1,0)-3f_y(0,1),$

$c_{3,0} = 2f(0,0)+f_x(0,0)-2f(1,0)+f_x(1,0),$

$c_{3,1} = 2f_y(0,0)-2f_y(1,0),$

$c_{3,2} = -4f_y(0,0)+4f_y(0,1)+3f_x(0,1)+3f_x(1,1)-6f(1,1)+4f_y(1,0)-6f(0,0)-3f_x(0,0)+2f_y(1,1)+6f(1,0)-3f_x(1,0),$

$c_{3,3} = -4f(0,1)-2f_x(0,1)-2f_x(1,1)+4f(1,1)+2f_y(0,0)-2f_y(1,0)+4f(0,0)+2f_x(0,0)-2f_y(1,1)-4f(1,0)+2f_x(1,0)+2f_y(0,1),$

This C looks irrelavant to any second derivative, but indeed, second derivative $f_{xy} = 0$ at each grid point.

If we request C

$$\left\{\begin{array}{ll} C(i,j) & = f(i,j) \\ C_x(i,j) & = f_x(i,j) \\ C_x(i,j) & = f_x(i,j) \\ C_{xy}(i,j) & = f_{xy}(i,j) \end{array}\right\} \text{ for } i,j=0,1,$$

then such C is unique

$$\begin{aligned}
c_{0,0} &= f(0,0),\\
c_{0,1} &= f_y(0,0),\\
c_{0,2} &= -2f_y(0,0)-3f(0,0)+3f(0,1)-f_y(0,1),\\
c_{0,3} &= 2f(0,0)+f_y(0,0)-2f(0,1)+f_y(0,1),\\
c_{1,0} &= f_x(0,0),\\
c_{1,1} &= \underline{f_{xy}(0,0)},\\
c_{1,2} &= -3f_x(0,0)+3f_x(0,1)\underline{-2f_{xy}(0,0)-f_{xy}(0,1)},\\
c_{1,3} &= 2f_x(0,0)-2f_x(0,1)+\underline{f_{xy}(0,0)+f_{xy}(0,1)},\\
c_{2,0} &= -2f_x(0,0)-3f(0,0)+3f(1,0)-f_x(1,0),\\
c_{2,1} &= -3f_y(0,0)+3f_y(1,0)\underline{-2f_{xy}(0,0)-f_{xy}(1,0)},\\
c_{2,2} &= 6f_x(0,0)+6f_y(0,0)-6f_x(0,1)+3f_y(0,1)-6f_y(1,0)+3f_x(1,0)-3f_y(1,1)-3f_x(1,1)-9f(0,1)-9f(1,0)+9f(0,0)+9f(1,1)\\
&\quad \underline{+4f_{xy}(0,0)+2f_{xy}(0,1)+2f_{xy}(1,0)+f_{xy}(1,1)},\\
c_{2,3} &= -4f_x(0,0)+6f(0,1)+4f_x(0,1)+2f_x(1,1)-6f(1,1)-3f_y(0,0)+3f_y(1,0)-6f(0,0)+3f_y(1,1)+6f(1,0)-2f_x(1,0)-3f_y(0,1)\\
&\quad \underline{-2f_{xy}(0,0)-2f_{xy}(0,1)-f_{xy}(1,0)-f_{xy}(1,1)},\\
c_{3,0} &= 2f(0,0)+f_x(0,0)-2f(1,0)+f_x(1,0),\\
c_{3,1} &= 2f_y(0,0)-2f_y(1,0)+\underline{f_{xy}(0,0)+f_{xy}(1,0)}\\
c_{3,2} &= -4f_y(0,0)+4f_y(0,1)+3f_x(0,1)+3f_x(1,1)-6f(1,1)+4f_y(1,0)-6f(0,0)-3f_x(0,0)+2f_y(1,1)+6f(1,0)-3f_x(1,0),\\
&\quad \underline{-2f_{xy}(0,0)-f_{xy}(0,1)-2f_{xy}(1,0)-f_{xy}(1,1)},\\
c_{3,3} &= -4f(0,1)-2f_x(0,1)-2f_x(1,1)+4f(1,1)+2f_y(0,0)-2f_y(1,0)+4f(0,0)+2f_x(0,0)-2f_y(1,1)-4f(1,0)+2f_x(1,0)+2f_y(0,1)\\
&\quad \underline{+f_{xy}(0,0)+f_{xy}(0,1)+f_{xy}(1,0)+f_{xy}(1,1)},
\end{aligned}$$

Once the C, C_x, C_y, C_{xy} at grid points are assigned, apply the formula to each box independently, we will have $\mathscr{C}^2$ bi-cubic spline. Consequently, we found out that the previous C with made-up boundary derivative is a special case. This motivated us to check out the $k=5$ case.

For $k=5$, by making up the directional derivative on the boundary

$$\Lambda(t) := \quad 5\left[(1-t)^4 m_0 + t^4 m_1\right] - 4\left[(1-t)^5 m_0 + t^5 m_1\right]$$

and work similarly to $k=3$ case, we also have a unique C.

But if we request C

$$\left\{\begin{array}{ll} C(i,j) & = f(i,j) \\ C_x(i,j) & = f_x(i,j) \\ C_x(i,j) & = f_x(i,j) \\ C_{xy}(i,j) & = f_{xy}(i,j) \\ C_{xxx}(i,j) & = f_{xxx}(i,j) \\ C_{xxy}(i,j) & = f_{xxy}(i,j) \\ C_{xyy}(i,j) & = f_{xyy}(i,j) \\ C_{yyy}(i,j) & = f_{yyy}(i,j) \\ C_{xxyy}(i,j) & = f_{xxyy}(i,j) \end{array}\right\} \text{ for } i,j=0,1,$$

then such C is unique:

$c_{0,0} = f(0,0),$

$c_{0,1} = f_y(0,0),$

$c_{0,2} = \frac{5}{2}[-f(0,0)+f(0,1)]-\frac{7}{4}f_y(0,0)-\frac{3}{4}f_y(0,1)-\frac{1}{16}f_{yyy}(0,0)+\frac{1}{48}f_{yyy}(0,1),$

$c_{0,3} = \frac{1}{6}f_{yyy}(0,0),$

$c_{0,4} = \frac{5}{2}[f(0,0)-f(0,1)]+\frac{5}{4}[f_y(0,0)+f_y(0,1)]-\frac{7}{48}f_{yyy}(0,0)-\frac{1}{16}f_{yyy}(0,1),$

$c_{0,5} = -f(0,0)-\frac{1}{2}f_y(0,0)-\frac{1}{2}f_y(0,1)+f(0,1)+\frac{1}{24}f_{yyy}(0,0)+\frac{1}{24}f_{yyy}(0,1),$

$c_{1,0} = f_x(0,0),$

$c_{1,1} = f_{xy}(0,0),$

$c_{1,2} = \frac{1}{2}f_{xyy}(0,0),$

$c_{1,3} = -10f_x(0,0)+10f_x(0,1)-6f_{xy}(0,0)-4f_{xy}(0,1)-\frac{3}{2}f_{xyy}(0,0)+\frac{1}{2}f_{xyy}(0,1),$

$c_{1,4} = 8f_{xy}(0,0)-f_{xyy}(0,1)+15f_x(0,0)+7f_{xy}(0,1)-15f_x(0,1)+\frac{3}{2}f_{xyy}(0,0),$

$c_{1,5} = -6f_x(0,0)-3f_{xy}(0,0)-3f_{xy}(0,1)+6f_x(0,1)-\frac{1}{2}f_{xyy}(0,0)+\frac{1}{2}f_{xyy}(0,1),$

$c_{2,0} = -\frac{7}{4}f_x(0,0)+\frac{1}{48}f_{xxx}(1,0)-\frac{5}{2}f(0,0)-\frac{3}{4}f_x(1,0)+\frac{5}{2}f(1,0)-\frac{1}{16}f_{xxx}(0,0),$

$c_{2,1} = \frac{1}{2}f_{xxy}(0,0),$

$c_{2,2} = \frac{1}{4}f_{xxyy}(0,0),$

$c_{2,3} = 25[f(0,0)-f(1,0)+f(1,1)-f(0,1)]+\frac{35}{2}[f_x(0,0)-f_x(0,1)]+\frac{15}{2}[f_x(1,0)-f_x(1,1)]-3f_{xxy}(0,0)-2f_{xxy}(0,1)+\frac{5}{8}[f_{xxx}(0,0)-f_{xxx}(0,1)]$
$-\frac{5}{24}[f_{xxx}(1,0)-f_{xxx}(1,1)]-\frac{3}{4}f_{xxyy}(0,0)+\frac{1}{4}f_{xxyy}(0,1),$

$c_{2,4} = \frac{75}{2}[-f(0,0)+f(1,0)-f(1,1)+f(0,1)-\frac{105}{4}[f_x(0,0)-f_x(0,1)]-\frac{45}{4}[f_x(1,0)-f_x(1,1)]+4f_{xxy}(0,0)+\frac{7}{2}f_{xxy}(0,1)$
$+\frac{15}{16}[-f_{xxx}(0,0)+f_{xxx}(1,0)-f_{xxx}(1,1)+f_{xxx}(0,1)]-\frac{1}{2}f_{xxyy}(0,1)+\frac{3}{4}f_{xxyy}(0,0),$

$c_{2,5} = 15[f(0,0)-f(1,0)+f(1,1)-f(0,1)]+\frac{21}{2}[f_x(0,0)-f_x(0,1)]+\frac{9}{2}[f_x(1,0)-f_x(1,1)]-\frac{3}{2}[f_{xxy}(0,0)+f_{xxy}(0,1)]+\frac{3}{8}[f_{xxx}(0,0)-f_{xxx}(0,1)]$
$-\frac{1}{8}[f_{xxx}(1,0)-f_{xxx}(1,1)]-\frac{1}{4}[f_{xxyy}(0,0)-f_{xxyy}(0,1)],$

$c_{3,0} = \frac{1}{6}f_{xxx}(0,0),$

$c_{3,1} = -10f_y(0,0)+10f_y(1,0)-6f_{xy}(0,0)-4f_{xy}(1,0)-\frac{3}{2}f_{xxy}(0,0)+\frac{1}{2}f_{xxy}(1,0),$

$c_{3,2} = 25[f(0,0)-f(1,0)+f(1,1)-f(0,1)]+\frac{35}{2}[f_y(0,0)-f_y(1,0)]+\frac{15}{2}[f_y(0,1)-f_y(1,1)]-3f_{xyy}(0,0)-2f_{xyy}(1,0)+\frac{5}{8}[f_{yyy}(0,0)-f_{yyy}(1,0)],$
$+\frac{5}{24}[f_{yyy}(1,1)-f_{yyy}(0,1)]-\frac{3}{4}f_{xxyy}(0,0)+\frac{1}{4}f_{xxyy}(1,0),$

$c_{3,3} = 100[-f(0,0)+f(1,0)-f(1,1)+f(0,1)]+36f_{xy}(0,0)+24f_{xy}(1,0)+16f_{xy}(1,1)+24f_{xy}(0,1)+9f_{xyy}(0,0)+6f_{xyy}(1,0)-2f_{xyy}(1,1)$
$-3f_{xyy}(0,1)+9f_{xxy}(0,0)-3f_{xxy}(1,0)-2f_{xxy}(1,1)+6f_{xxy}(0,1)+\frac{5}{3}[-f_{xxx}(0,0)+f_{xxx}(0,1)-f_{yyy}(0,0)+f_{yyy}(1,0)]$
$+\frac{9}{4}f_{xxyy}(0,0)-\frac{3}{4}f_{xxyy}(1,0)+\frac{1}{4}f_{xxyy}(1,1)-\frac{3}{4}f_{xxyy}(0,1),$

$c_{3,4} = 125[f(0,0)-f(1,0)+f(1,1)-f(0,1)]+\frac{25}{2}[-f_y(0,0)+f_y(1,0)+f_y(1,1)-f_y(0,1)]-48f_{xy}(0,0)-32f_{xy}(1,0)-28f_{xy}(1,1)-42f_{xy}(0,1)$
$-9f_{xyy}(0,0)-6f_{xyy}(1,0)+4f_{xyy}(1,1)+6f_{xyy}(0,1)-12f_{xxy}(0,0)+4f_{xxy}(1,0)+\frac{7}{2}f_{xxy}(1,1)-\frac{21}{2}f_{xxy}(0,1)+\frac{5}{2}f_{xxx}(0,0)-\frac{5}{2}f_{xxx}(0,1)$
$+\frac{35}{24}[f_{yyy}(0,0)-f_{yyy}(1,0)]-\frac{5}{8}f_{yyy}(1,1)+\frac{5}{8}f_{yyy}(0,1)-\frac{9}{4}f_{xxyy}(0,0)+\frac{3}{4}f_{xxyy}(1,0)-\frac{1}{2}f_{xxyy}(1,1)+\frac{3}{2}f_{xxyy}(0,1),$

$c_{3,5} = 50[-f(0,0)+f(1,0)-f(1,1)+f(0,1)]+5[f_y(0,0)-f_y(1,0)-f_y(1,1)+f_y(0,1)]+18f_{xy}(0,0)+12f_{xy}(1,0)+12f_{xy}(1,1)+18f_{xy}(0,1)$
$+3f_{xyy}(0,0)+2f_{xyy}(1,0)-2f_{xyy}(1,1)-3f_{xyy}(0,1)+\frac{9}{2}f_{xxy}(0,0)-\frac{3}{2}f_{xxy}(1,0)-\frac{3}{2}f_{xxy}(1,1)+\frac{9}{2}f_{xxy}(0,1)-f_{xxx}(0,0)+f_{xxx}(0,1)$
$+\frac{5}{12}[-f_{yyy}(0,0)+f_{yyy}(1,0)+f_{yyy}(1,1)-f_{yyy}(0,1)]+\frac{3}{4}f_{xxyy}(0,0)-\frac{1}{4}f_{xxyy}(1,0)+\frac{1}{4}f_{xxyy}(1,1)-\frac{3}{4}f_{xxyy}(0,1),$

$$
\begin{aligned}
c_{4,0} =& \tfrac{5}{2}[f(0,0)-f(1,0)]+\tfrac{5}{4}[f_x(0,0)+f_x(1,0)]-\tfrac{7}{48}f_{xxx}(0,0)-\tfrac{1}{16}f_{xxx}(1,0),\\
c_{4,1} =& 15[f_y(0,0)-f_y(1,0)]+8f_{xy}(0,0)+7f_{xy}(1,0)+\tfrac{3}{2}f_{xxy}(0,0)-f_{xxy}(1,0),\\
c_{4,2} =& \tfrac{3}{4}f_{xxyy}(0,0)+\tfrac{75}{2}[-f(0,0)+f(1,0)-f(1,1)+f(0,1)]+\tfrac{105}{4}[-f_y(0,0)+f_y(1,0)]+\tfrac{45}{4}[f_y(1,1)-f_y(0,1)]+4f_{xyy}(0,0)+\tfrac{7}{2}f_{xyy}(1,0)\\
&+\tfrac{15}{16}[-f_{yyy}(0,0)+f_{yyy}(1,0)]+\tfrac{5}{16}[f_{yyy}(0,1)-f_{yyy}(1,1)]-\tfrac{1}{2}f_{xxyy}(1,0),\\
c_{4,3} =& 125[f(0,0)-f(1,0)+f(1,1)-f(0,1)]+\tfrac{25}{2}[-f_x(0,0)-f_x(1,0)+f_x(1,1)+f_x(0,1)]-48f_{xy}(0,0)-42f_{xy}(1,0)-28f_{xy}(1,1)-32f_{xy}(0,1)\\
&-12f_{xyy}(0,0)-\tfrac{21}{2}f_{xyy}(1,0)+\tfrac{7}{2}f_{xyy}(1,1)+4f_{xyy}(0,1)-9f_{xxy}(0,0)+6f_{xxy}(1,0)+4f_{xxy}(1,1)-6f_{xxy}(0,1)+\tfrac{35}{24}[f_{xxx}(0,0)-f_{xxx}(0,1)]\\
&+\tfrac{5}{8}[f_{xxx}(1,0)-f_{xxx}(1,1)]+\tfrac{5}{2}[f_{yyy}(0,0)-f_{yyy}(1,0)]-\tfrac{9}{4}f_{xxyy}(0,0)+\tfrac{3}{2}f_{xxyy}(1,0)-\tfrac{1}{2}f_{xxyy}(1,1)+\tfrac{3}{4}f_{xxyy}(0,1),\\
c_{4,4} =& 150[-f(0,0)+f(1,0)-f(1,1)+f(0,1)]+\tfrac{75}{4}[f_x(0,0)+f_x(1,0)-f_x(1,1)-f_x(0,1)+f_y(0,0)-f_y(1,0)-f_y(1,1)+f_y(0,1)]\\
&+64f_{xy}(0,0)+56f_{xy}(1,0)+49f_{xy}(1,1)+56f_{xy}(0,1)+12f_{xyy}(0,0)+\tfrac{21}{2}f_{xyy}(1,0)-7f_{xyy}(1,1)-8f_{xyy}(0,1)\\
&+12f_{xxy}(0,0)-8f_{xxy}(1,0)-7f_{xxy}(1,1)+\tfrac{21}{2}f_{xxy}(0,1)+\tfrac{35}{16}[-f_{xxx}(0,0)+f_{xxx}(0,1)]+\tfrac{15}{16}[-f_{xxx}(1,0)+f_{xxx}(1,1)]\\
&+\tfrac{35}{16}[-f_{yyy}(0,0)+f_{yyy}(1,0)]+\tfrac{15}{16}[f_{yyy}(1,1)-f_{yyy}(0,1)]+\tfrac{9}{4}f_{xxyy}(0,0)-\tfrac{3}{2}[f_{xxyy}(1,0)+f_{xxyy}(0,1)]+f_{xxyy}(1,1),\\
c_{4,5} =& 60[f(0,0)-f(1,0)+f(1,1)-f(0,1)]+\tfrac{15}{2}[-f_x(0,0)-f_x(1,0)+f_x(1,1)+f_x(0,1)-f_y(0,0)+f_y(1,0)+f_y(1,1)-f_y(0,1)]\\
&-24f_{xy}(0,0)-21f_{xy}(1,0)-21f_{xy}(1,1)-24f_{xy}(0,1)+4[-f_{xyy}(0,0)+f_{xyy}(0,1)]+\tfrac{7}{2}[f_{xyy}(1,1)-f_{xyy}(1,0)]-\tfrac{9}{2}[f_{xxy}(0,0)+f_{xxy}(0,1)]\\
&+3[f_{xxy}(1,0)+f_{xxy}(1,1)]+\tfrac{7}{8}[f_{xxx}(0,0)-f_{xxx}(0,1)]+\tfrac{3}{8}[f_{xxx}(1,0)-f_{xxx}(1,1)]+\tfrac{5}{8}[f_{yyy}(0,0)+f_{yyy}(0,1)-f_{yyy}(1,0)-f_{yyy}(1,1)]\\
&+\tfrac{3}{4}[-f_{xxyy}(0,0)+f_{xxyy}(0,1)]+\tfrac{1}{2}[f_{xxyy}(1,0)-f_{xxyy}(1,1)],\\
c_{5,0} =& -f(0,0)+f(1,0)-\tfrac{1}{2}[f_x(0,0)+f_x(1,0)]+\tfrac{1}{24}[f_{xxx}(0,0)+f_{xxx}(1,0)],\\
c_{5,1} =& 6[-f_y(0,0)+f_y(1,0)]-3[f_{xy}(0,0)+f_{xy}(1,0)]+\tfrac{1}{2}[-f_{xxy}(0,0)+f_{xxy}(1,0)],\\
c_{5,2} =& 15[f(0,0)-f(1,0)+f(1,1)-f(0,1)]+\tfrac{21}{2}[f_y(0,0)-f_y(1,0)]+\tfrac{9}{2}[f_y(0,1)-f_y(1,1)]-\tfrac{3}{2}[f_{xyy}(0,0)+f_{xyy}(1,0)]+\tfrac{3}{8}[f_{yyy}(0,0)-f_{yyy}(1,0)]\\
&+\tfrac{1}{8}[f_{yyy}(1,1)-f_{yyy}(0,1)]-\tfrac{1}{4}[f_{xxyy}(0,0)-f_{xxyy}(1,0)],\\
c_{5,3} =& 50[-f(0,0)+f(1,0)-f(1,1)+f(0,1)]+5[f_x(0,0)-f_x(0,1)+f_x(1,0)-f_x(1,1)]+18[f_{xy}(0,0)+f_{xy}(1,0)]+12[f_{xy}(0,1)+f_{xy}(1,1)]\\
&+\tfrac{9}{2}[f_{xyy}(0,0)+f_{xyy}(1,0)]-\tfrac{3}{2}[f_{xyy}(1,1)+f_{xyy}(0,1)]+3[f_{xxy}(0,0)-f_{xxy}(1,0)]+2[f_{xxy}(0,1)-f_{xxy}(1,1)]\\
&+\tfrac{5}{12}[-f_{xxx}(0,0)-f_{xxx}(1,0)+f_{xxx}(1,1)+f_{xxx}(0,1)]-f_{yyy}(0,0)+f_{yyy}(1,0)+\tfrac{3}{4}[f_{xxyy}(0,0)-f_{xxyy}(1,0)+\tfrac{1}{4}[f_{xxyy}(1,1)-f_{xxyy}(0,1)],\\
c_{5,4} =& 60[f(0,0)-f(1,0)+f(1,1)-f(0,1)]+\tfrac{15}{2}[-f_x(0,0)-f_x(1,0)+f_x(1,1)+f_x(0,1)-f_y(0,0)+f_y(1,0)+f_y(1,1)-f_y(0,1)]\\
&-24[f_{xy}(0,0)+f_{xy}(1,0)]-21[f_{xy}(1,1)+f_{xy}(0,1)]-\tfrac{9}{2}[f_{xyy}(0,0)+f_{xyy}(1,0)]+3[f_{xyy}(1,1)+f_{xyy}(0,1)]-4[f_{xxy}(0,0)-f_{xxy}(1,0)]\\
&+\tfrac{7}{2}[f_{xxy}(1,1)-f_{xxy}(0,1)]+\tfrac{5}{8}[f_{xxx}(0,0)+f_{xxx}(1,0)-f_{xxx}(1,1)-f_{xxx}(0,1)]+\tfrac{7}{8}[f_{yyy}(0,0)-f_{yyy}(1,0)]-\tfrac{3}{8}[f_{yyy}(1,1)-f_{yyy}(0,1)]\\
&-\tfrac{3}{4}[f_{xxyy}(0,0)-f_{xxyy}(1,0)]+\tfrac{1}{2}[f_{xxyy}(0,1)-f_{xxyy}(1,1)],\\
c_{5,5} =& 24[-f(0,0)+f(1,0)-f(1,1)+f(0,1)]+3[f_x(0,0)+f_x(1,0)-f_x(1,1)-f_x(0,1)+f_y(0,0)-f_y(1,0)-f_y(1,1)+f_y(0,1)]\\
&+9[f_{xy}(0,0)+f_{xy}(1,0)+f_{xy}(1,1)+f_{xy}(0,1)]+\tfrac{3}{2}[f_{xyy}(0,0)+f_{xyy}(1,0)-f_{xyy}(1,1)-f_{xyy}(0,1)]\\
&+\tfrac{3}{2}[f_{xxy}(0,0)-f_{xxy}(1,0)-f_{xxy}(1,1)+f_{xxy}(0,1)]+\tfrac{1}{4}[-f_{xxx}(0,0)-f_{xxx}(1,0)+f_{xxx}(1,1)+f_{xxx}(0,1)]\\
&+\tfrac{1}{4}[-f_{yyy}(0,0)+f_{yyy}(1,0)+f_{yyy}(1,1)-f_{yyy}(0,1)+f_{xxyy}(0,0)-f_{xxyy}(1,0)+f_{xxyy}(1,1)-f_{xxyy}(0,1).
\end{aligned}
$$

Once the $C, C_x, C_y, C_{xy}, C_{xxx}, C_{xxy}, C_{xyy}, C_{yyy}, C_{xxyy}$ at grid points are assigned, apply the formula to each box independently, we will have $\mathscr{C}^4$ bi-quintic spline.