行政院國家科學委員會專題研究計畫 成果報告

二階多點界邊值函數的正解存在性
研究成果報告(精簡版)

計 畫 類 別 ：個別型
計 畫 編 號 ：NSC 96-2115-M-032-005-
執 行 期 間 ：96 年 08 月 01 日至 97 年 07 月 31 日
執 行 單 位 ：淡江大學數學系

計 畫 主 持 人 ：錢傳仁

計畫參與人員： 此計畫無其他參與人員

處 理 方 式 ：本計畫可公開查詢

中 華 民 國 98 年 05 月 06 日

Positive solutions of nonlinear second-order m-point boundary value problem

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Abstract

Under suitable conditions on $f(t, u)$, the nonlinear second-order m-point boundary value problem

$$u''(t) + f(t, u(t)) = 0$$
$$u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i)$$

has at least one positive solution.

1. Introduction

The study of multipoint boundary value problems for linear second-order ordinary value problems was initialed by Ilin and Moiseer [1, 2]. Then Gupta [5] studied three-point boundary value problems for nonlinear ordinary differential equations. We refer the reader to [3–10], for some recent results of nonlinear multipoint boundary value problems.

In this paper, we investigate the existence of positive solutions to nonlinear second-order m-point boundary value problem:

$$u''(t) + f(t, u(t)) = 0, \quad 0 < t < 1 \tag{1.1}$$

$$u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i) \tag{1.2}$$

where $a_i \geq 0$ for $i = 1, \ldots, m-3$ and $a_{m-2} > 0$, ξ_i satisfy $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$ and $\sum_{i=1}^{m-2} a_i \xi_i < 1$. We also make the following assumptions.

(A_1) $f \in C([0, 1] \times [0, \infty); [0, \infty))$, $f \not\equiv 0$.

(A_2) $\exists H_1, H_2$ satisfy $f(s,u) \le M_1 H_1$ on $[0,1]\times[0,H_1]$ and $f(s,u) \ge M_2 H_2$ on $[\xi_{m-2},1]\times[\Gamma H_2,\infty)$, where $M_1 = 2(1-\sum_{i=1}^{m-2} a_i\xi_i)$ and $M_2 = \frac{2(1-\sum_{i=1}^{m-2} a_i\xi_i)}{(1-\xi_{m-2})^2\cdot\sum_{i=1}^{m-2} a_i\xi_i}$ or

(A_2') $\exists H_3, H_4 > 0$ such that $f(s,u) \ge M_2 H_3$ on $[\xi_{m-2},1]\times[0,\Gamma H_3]$ and $f(s,u) \le M_1 H_4$ on $[0,1]\times[H_4,\infty)$.

The proof of the main result in this article is based on an application of the following well-known Guo-Krasioselskii fixed-point theorem [11].

Theorem 1.1. *Let E be a Banach space and $\mathcal{K} \subset E$ be a cone in E. Assume Ω_1 and Ω_2 are open subset of E with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$, $T : \mathcal{K}\cap(\overline{\Omega}_2\backslash\Omega_1) \to \mathcal{K}$ be a completely continuous operator such that*

(i) $\|Tu\| \le \|u\|$, $\forall u \in \mathcal{K}\cap\partial\Omega_1$ *and* $\|Tu\| \ge \|u\|$, $\forall u \in \mathcal{K}\cap\partial\Omega_2$*; or*

(ii) $\|Tu\| \le \|u\|$, $\forall u \in \mathcal{K}\cap\partial\Omega_2$ *and* $\|Tu\| \ge \|u\|$, $\forall u \in \mathcal{K}\cap\partial\Omega_1$.

Then T has a fixed point in $\mathcal{K}\cap(\overline{\Omega}_2\backslash\Omega_1)$.

2. The Preliminary Lemmas

Lemma 2.1 (Gupta et al. [5]). *Let $a_i \ge 0$ for $i = 1,2,\ldots,m-2$, and $\sum_{i=1}^{m-2} a_i\xi_i \ne 1$; then for $y \in C[0,1]$, the boundary value problem:*

$$u''(t) + y(t) = 0, \quad 0 < t < 1, \tag{2.1}$$

$$u(0) = (0), \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i), \tag{2.2}$$

has a unique solution

$$\begin{aligned} u(t) &= \frac{t}{1-\sum_{i=1}^{m-2} a_i u(\xi_i)}\int_0^1 (1-s)y(s)ds - \int_0^t (t-s)y(s)ds \\ &\quad -\frac{t}{1-\sum_{i=1}^{m-2} a_i u(\xi_i)}\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s)y(s)ds. \end{aligned}$$

Let $C^+[0,1]$ be the set of nonnegative function in $C[0,1]$.

Lemma 2.2 (Ma [9]). *Let $a_i \ge 0$, for $i = 1,2,\ldots,m-2$ and $\sum_{i=1}^{m-2} a_i\xi_i < 1$. Then for $y \in C^+[0,1]$, the unique solution $u(t)$ of (2.1), (2.2) is nonnegative and satisfies*

$$\min_{t\in[\xi_{m-2},1]} u(t) \ge \Gamma\|u\|$$

where

$$\Gamma = \min\left\{\frac{a_{m-2}(1-\xi_{m-2})}{1-a_{m-2}\xi_{m-2}},\ a_{m-2}\xi_{m-2},\ \xi_1\right\}.$$

3. Main results

In this section we show the existence of positive solution for the boundary value problem (1.1), (1.2).

Theorem 3.1. *Suppose (A_1), (A_2) hold, then (1.1), (1.2) has at least one positive solution.*

Proof. Let

$$\mathcal{K} = \left\{y : y \in C[0,1],\ y \geq 0,\ \min_{\xi_{m-2}\leq t\leq 1} y(t) \geq \Gamma\|y\|\right\}.$$

where Γ is described as in Lemma 2.2. It is clear that $\mathcal{K}$ is a cone. Define an integral operator $A : \mathcal{K} \to C^+[0,1]$.

$$\begin{aligned} A(y(t)) &= -\int_0^t (t-s)f(s,y(s))ds \\ &\quad -t\frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i}(\xi_i-s)f(s,y(s))ds}{1-\sum_{i=1}^{m-2} a_i\xi_i} + t\frac{\int_0^1(1-s)f(s,y(s))ds}{1-\sum_{i=1}^{m-2} a_i\xi_i} \end{aligned}$$

Now (1.2), (1.2) has a solution $y = y(t)$ if and only if y solves the operator equation $A(y(t)) = y(t)$. It is clear that $A\mathcal{K} \subseteq \mathcal{K}$. The details of the proof can be founded in [9]. It is routine to check that A is a completely continuous operator on $\mathcal{K}$. Let $\Omega_1 = \{y \in C[0,1] : \|y\| < H_1\}$ and choose $y \in \mathcal{K}$ such that $\|y\| = H_1$, then we see that

$$\begin{aligned} Ay(t) &\leq \frac{t\int_0^1(1-s)f(s,y(s))ds}{1-\sum_{i=1}^{m-2} a_i\xi_i} \\ &\leq \frac{t\int_0^1(1-s)M_1H_1ds}{1-\sum_{i=1}^{m-2} a_i\xi_i} \\ &= \frac{M_1H_1\cdot\frac{1}{2}}{1-\sum_{i=1}^{m-2} a_i\xi_i} \\ &= H_1. \end{aligned}$$

Thus, we have $\|Ay\| \leq \|y\|$ on $\partial\Omega_1 \cap \mathcal{K}$. Now, let $\Omega_2 = \{y \in C[0,1] : \|y\| < H_2\}$ and $y \in \partial\Omega_2 \cap \mathcal{K}$, that is, $\|y\| = H_2$, according to Lemma 2.1 there exists a unique solution of

$$u''(t) + f(t, y(t)) = 0, \quad t \in (0,1) \tag{3.1}$$

$$u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i) \tag{3.2}$$

Thus, we have

$$\begin{aligned}
u(1) &= \sum_{i=1}^{m-2} a_i u(\xi_i) \\
&= \sum_{i=1}^{m-2} a_i \left[-\int_0^{\xi_i} (\xi_i - s) f(s, y(s)) ds - \xi_i \frac{\sum_{j=1}^{m-2} a_j \int_0^{\xi_j} (\xi_j - s) f(s, y(s)) ds}{1 - \sum_{j=1}^{m-2} a_j \xi_j} \right. \\
&\qquad \left. + \xi_i \frac{\int_0^1 (1-s) f(s, y(s)) ds}{1 - \sum_{j=1}^{m-2} a_j \xi_j} \right] \\
&= -\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s) f(s, y(s)) ds \\
&\quad - \left(\sum_{i=1}^{m-2} a_i \xi_i \right) \frac{\sum_{j=1}^{m-2} a_j \int_0^{\xi_j} (\xi_j - s) f(s, y(s)) ds}{1 - \sum_{j=1}^{m-2} a_j \xi_j} \\
&\quad + \left(\sum_{i=1}^{m-2} a_i \xi_i \right) \frac{\int_0^1 (1-s) f(s, y(s)) ds}{1 - \sum_{j=1}^{m-2} a_j \xi_j} \\
&= \frac{1}{1 - \sum_{j=1}^{m-2} a_j \xi_j} \left[-\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s) f(s, y(s)) ds \right. \\
&\qquad + \left(\sum_{j=1}^{m-2} a_j \xi_j \right) \left(\sum_{j=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s) f(s, y(s)) ds \right) \\
&\qquad - \left(\sum_{i=1}^{m-2} a_i \xi_i \right) \left(\sum_{j=1}^{m-2} a_i \int_0^{\xi_i} (\xi_i - s) f(s, y(s)) ds \right) \\
&\qquad \left. + \left(\sum_{j=1}^{m-2} a_j \xi_j \right) \int_0^1 (1-s) f(s, y(s)) ds \right] \\
&= \frac{1}{1 - \sum_{j=1}^{m-2} a_j \xi_j} \left[-\sum_{i=1}^{m-2} a_i \xi_i \int_0^{\xi_i} f(s, y(s)) ds + \sum_{i=1}^{m-2} a_i \int_0^{\xi_i} s f(s, y(s)) ds \right. \\
&\qquad \left. + \sum_{i=1}^{m-2} a_i \xi_i \int_0^1 f(s, y(s)) ds - \sum_{i=1}^{m-2} a_i \xi_i \int_0^1 s f(s, y(s)) ds \right]
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{1-\sum_{j=1}^{m-2} a_j\xi_j}\left[\sum_{i=1}^{m-2} a_i\xi_i \int_{\xi_i}^{1} f(s,y(s))ds - \sum_{i=1}^{m-2} a_i\xi_i \int_{\xi_i}^{1} sf(s,y(s))ds\right] \\
&\geq \frac{1}{1-\sum_{j=1}^{m-2} a_j\xi_j}\sum_{i=1}^{m-2} a_i\xi_i \int_{\xi_{m-2}}^{1} (1-s)f(s,y(s))ds \\
&\geq \frac{\sum_{i=1}^{m-2} a_i\xi_i \int_{\xi_{m-2}}^{1}(1-s)M_2H_2ds}{1-\sum_{j=1}^{m-2} a_j\xi_j} \\
&= \frac{\sum_{i=1}^{m-2} a_i\xi_i}{1-\sum_{i=1}^{m-2} a_i\xi_i}\left(\frac{1-\xi_{m-2}}{2}\right)^2 \cdot M_2H_2 \\
&= H_2\,.
\end{aligned}
$$

Hence $\|Ay\| \geq |Ay(1)| = |u(1)| \geq H_2 = \|y\|$. $\|Ay\| \geq \|y\|$ on $\partial\Omega_2 \cap \mathcal{K}$. Therefore, by (i) of the fixed-point theorem, it follows that A has a fixed point in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$. This completes the proof.

Theorem 3.2. *Suppose (A_1) and (A_2') hold, then (1.1), (1.2) has at least one positive solution.*

Proof. Let $\Omega_3 = \{y \in C[0,1] : \|y\| < H_3\}$. Thus, if $y \in K$ and $\|y\| = H_3$, we get

$$
\begin{aligned}
u(1) &\geq \frac{1}{1-\sum_{i=1}^{m-2} a_i\xi_i}\sum_{i=1}^{m-2} a_i\xi_i \int_{\xi_{m-2}}^{1}(1-s)f(s,y(s))ds \\
&\geq \frac{1}{1-\sum_{i=1}^{m-2} a_i\xi_i}\left(\sum_{i=1}^{m-2} a_i\xi_i\right) M_2H_3 \int_{\xi_{m-2}}^{1}(1-s)ds \\
&= \frac{1}{1-\sum_{i=1}^{m-2} a_i\xi_i}\left(\sum_{i=1}^{m-2} a_i\xi_i\right)\frac{(1-\xi_{m-2})^2}{2}M_2H_3 \\
&= H_3
\end{aligned}
$$

Hence, $\|Ay\| \geq |Ay(1)| = |u(1)| \geq H_3$, for all $y \in \partial\Omega_3 \cap \mathcal{K}$. Now let $\Omega_4 = \{y \in C[0,1] : \|y\| > H_4\}$. For $y \in K$ and $\|y\| = H_4$, we get

$$
\begin{aligned}
Ay(t) &= -\int_0^t (t-s)f(s,y(s))ds - t\frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i}(\xi_i - s)f(s,y(s))ds}{1-\sum_{i=1}^{m-2} a_i\xi_i} \\
&\quad + t\frac{\int_0^1(1-s)f(s,y(s))ds}{1-\sum_{i=1}^{m-2} a_i\xi_i} \\
&\leq \frac{\int_0^1(1-s)f(s,y(s))ds}{1-\sum_{i=1}^{m-2} a_i\xi_i}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{M_1 H_4 \int_0^1 (1-s)ds}{1-\sum_{i=1}^{m-2} a_i \xi_i} \\
&= M_1 H_4 \cdot \frac{1}{2(1-\sum_{i=1}^{m-2} a_i \xi_i)} \\
&= H_4
\end{aligned}
$$

Hence, $\|Ay\| \leq H_4 = \|y\|$ on $\partial\Omega_4 \cap K$.

4. Remarks

Remark 4.1. Recently, Ma [9] considered the second-order m-point boundary value problem.

$$u''(t) + a(t)g(u) = 0, \quad t \in (0,1) \tag{4.1}$$

$$u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i) \tag{4.2}$$

and gave the following result.

Theorem 4.1. *Assume*

(A_3) $g \in C([0,\infty),[0,\infty))$ *and both limits*

$$g_0 := \lim_{u\to 0^+} \frac{g(u)}{u}, \quad g_\infty := \lim_{u\to\infty} \frac{g(u)}{u}$$

exist.

(A_4) $a \in C([0,1],[0,\infty))$, *and there exists* $x_0 \in [\xi_{m-2}, 1]$ *such that* $a(x_0) > 0$, *and* $\sum_{i=1}^{m-2} a_i \xi_i < 1$.

Then problem (4.1), (4.2) has at least one positive solution in the case (i) $g_0 = 0$ *and* $g_\infty = \infty$ *or (ii)* $g_0 = \infty$ *and* $g_\infty = 0$.

Put $f(t,u) = a(t)g(u)$ in (1.1). Assumptions (A_1) and (A_2) will be fulfilled if the conditions (A_3), (A_4) together with case (i) are satisfied. Similarly, assumption (A_1) and (A_2') will be fulfilled if the conditions (A_3), (A_4) together with case (ii) are satisfied.

Remark 4.2. Y. Sun [10] considered the following boundary value problem

$$u''(t) + \lambda a(t)g(u) = 0, \quad t \in (0,1) \tag{4.3}$$

$$u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i) \tag{4.4}$$

where λ is a positive parameter, $a_i \geq 0$ for $i = 1, 2, \dots, m-3$ and $a_{m-2} > 0$, ξ_i satisfy $0 < \xi_1 < \xi_2 < \cdots, \xi_{m-2} < 1$ and $\sum_{i=1}^{m-2} a_i \xi_i < 1$. The author gave the following result.

Theorem 4.2. *Assume*

(c_1) $a \in C([0,\infty),[0,\infty))$ *and there exists* $x_0 \in [\xi_{m-2}, 1]$ *such that* $a(x_0) > 0$.

(c_2) $g \in C([0,1],[0,\infty))$ *and there exists nonnegative constants in the extended reals,* g_0, g_∞, *such that*

$$g_0 := \lim_{u \to 0^+} \frac{g(u)}{u}, \quad g_\infty := \lim_{u \to \infty} \frac{g(u)}{u}.$$

Let $A = \frac{1}{1-\sum_{i=1}^{m-2} a_i \xi_i} \int_0^1 (1-s)a(s)ds$, $B = \frac{(\sum_{i=1}^{m-2} a_i)\xi_1}{1-\sum_{i=1}^{m-2} a_i \xi_i} \int_{\xi_{m-2}}^1 (1-s)a(s)ds$. *The problem (4.3), (4.4) has at least one solution in the case*

(i) *If* $Ag_0 < \Gamma B g_\infty$, *then for each* $\lambda \in (\frac{1}{\Gamma B g_\infty}, \frac{1}{Ag_0})$, *the BVP (4.3), (4.4) has at least one positive solution.*

(ii) *If* $g_0 = 0$ *and* $g_\infty = \infty$, *i.e., g is super linear, then for any* $\lambda \in (0,\infty)$, *the BVP (4.3), (4.4) has at least one positive solution.*

(iii) *If* $g_\infty = \infty$, $0 < g_0 < \infty$, *then for each* $\lambda \in (0, \frac{1}{Ag_0})$, *the BVP (4.3), (4.4) has at least one positive solution.*

(iv) *If* $g_0 = 0$, $0 < g_\infty < \infty$, *then for each* $\lambda \in (\frac{1}{\Gamma B g_\infty}, \infty)$ *the BVP (4.3), (4.4) has at least one positive solution.*

Now if we put $g(u) = 1 + \sin^2 u$ and $a(t) \equiv 1$ in (4.3), it is easy to see that both g_0 and g_∞ do not exist, thus we fail to discuss such a problem by applying Theorem 4.2.

On the other hand, if we put $f(t,u) = \lambda(1 + \sin^2 u)$ in (1.1), then for all positive numbers a and b, $f(t,u) \leq 2\lambda$ on $[0,1] \times [0,a]$ and $f(t,u) \geq \lambda$ on $[0,1] \times (b,\infty)$. Thus, the existence of at least one positive solution is guaranteed by the Theorem 3.2.

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