1. Introduction

Consider the estimation of the probability density function f of a continuous distribution. Histogram and frequency polygons (Scott 1985) are two widely used density estimators due to their simplicities. The former is produced by the grouping of data into non-overlapping intervals, known as bins, and the histogram value is the ratio of the bin counts to bin width. The latter is a smoother version of density estimator based on histogram produced by connecting the histogram's mid-bin values by straight lines. For computationally and statistically efficient estimation of the density, the histogram, or binned data, is further utilized to form smoother density estimators. See, for example, the average shifted histogram of Scott (1985, 1992) and edge frequency polygons (EFP) of Jones et al. (1998). See also Dong and Zheng (2001) for generalized edge frequency polygons. This paper considers three histogram-based density estimators which are all linear interpolants of the weighted average of histogram values. We now describe the ideas that give rise to the two estimators of interests and some messages about their properties.

The first estimator extends the idea of Jones et. al. (1998) by the repeatedly rejoining the mid-points of every two consecutive line segments of the frequency polygons previously produced. Let $\hat{f}_{r,BP1}$ denote the r-th frequency polygons obtained by rejoining the mid-points of line segments of the (r-1)-th one. The estimator $\hat{f}_{r,BP1}$ is the linear interpolants of the weighted average of the binned data. We found that the weights assigned to the binned data form the probability mass function (p.m.f) of the $Binomial(r-1,1/2)$ variable. Consequently, the optimal asymptotic integrated mean square error (AIMSE) of $\hat{f}_{r,BP1}$ converges to that of the regular Gaussian kernel density estimator based on original data, as $r \to \infty$. In addition, $\hat{f}_{r,BP1}$ improves the previous one $\hat{f}_{r-1,BP1}$, the way that EFP (or $\hat{f}_{2,BP1}$) improve FP (or $\hat{f}_{1,BP1}$) by having a smaller AIMSE.

The second estimator $\hat{f}_{BP2}$ is the linear interpolants of the binned kernel density estimates (BKDE) that are produced either at the bins' centers or at their edges. Lin, Wu and Yen (2006) use direct kernel smoothing of the original data to construct the kernel polygons. Based on the kernels properly chosen from the class of uniform and linear function, their kernel polygons are bona fide densities that can evade the normalization of kernel estimator. Instead of direct kernel smoothing of original data, we use the kernel function to smooth the BDKE so as to construct $\hat{f}_{BP2}$. In the class of uniform and linear kernels, we locate the kernel functions based on which the estimator $\hat{f}_{BP2}$ can be a bona fide density. When BKDE are produced at bin edges, then the choices of kernels for $\hat{f}_{BP2}$ are the same as those for the kernel polygons in Lin, Wu and Yen (2006). On the other hand, if BKDE are produced at bin center, then kernels identified in Lin, Wu and Yen (2006) require some modification to yields the proper kernels for $\hat{f}_{BP2}$.

The rest of this paper is organized as follows. Section 2 describes the formulation of $\hat{f}_{r,BP1}$ and proves the asymptotic equivalence of its AIMSE and that of the non-interpolated Gaussian kernel density estimates. For simplicity of presentation, we drop r from $\hat{f}_{r,BP1}$ in what follows. Section 3 considers the second estimator $\hat{f}_{BP2}$ and locates the kernel function that guarantees $\hat{f}_{BP2}$ be a probability density.

2. Binomial weights and the frequency polygons.

The BKDE use kernel function to assign weights to the binned data. The binning rule applied to the data can produce ASH and the binned values at the bin edge for constructing EFP, among others. We now use a quite different and interesting way to describe the binning rule as follows. Let $X_1, X_2, \cdots, X_n$ be a random sample from the distribution with density f. Given bin origin t_0 and bin width δ, the bin center for the *k*-th bin is denoted by $t_k = t_0 + k \cdot \delta$, $k \in N$, and the histogram $\hat{f}_1(x)$ for estimating $f(x)$ is defined as

$$\hat{f}_1(x) = \frac{1}{n\delta} \sum_{i=1}^{n} I_{[s_{k-1}, s_k)}(X_i) \quad \text{for} \quad x \in [s_{k-1}, s_k),$$

where $s_{k-1} = t_k - \delta/2$ and $s_{k+1} = t_k - \delta/2$ are bin edges of the k-th bin, I is the indicator function, namely $I_A(u)=1$, if $u \in A$ and 0 otherwise. For $r \geq 2$, $\hat{f}_{r+1}(t_k)=[\hat{f}_r(s_{k-1})+\hat{f}_r(s_k)]/2$ for odd r and $\hat{f}_{r+1}(s_k)=[\hat{f}_r(t_k)+\hat{f}_r(t_{k+1})]/2$ for even *r*. Here and through the paper, we use $f_r(u_k)$ to express $\hat{f}_r(t_k)$ and $\hat{f}_r(s_k)$, alternately for odd or even value of r. Then proper arrangement reveals that $\hat{f}_r(u_k)$ is the weighted average of the neighboring histogram values $n_k/(n\delta)$ at u_k, where n_k is the bin counts for the k-th bin. For illustration, the following table summarizes the weights associated with $\hat{f}_r(u_k),\ r=2,\ 3,\ 4$

$r=1,2$	$\frac{n_k}{n\delta}$	$\frac{n_{k+1}}{n\delta}$	$r=2,3$	$\frac{n_{k-1}}{n\delta}$	$\frac{n_k}{n\delta}$	$\frac{n_{k+1}}{n\delta}$	$r=3,4$	$\frac{n_{k-1}}{n\delta}$	$\frac{n_k}{n\delta}$	$\frac{n_{k+1}}{n\delta}$	$\frac{n_{k+2}}{n\delta}$
$\hat{f}_1(t_k)$	1	0	$\hat{f}_2(s_{k-1})$	1/2	1/2	0	$\hat{f}_3(t_k)$	1/4	2/4	1/4	0
$\hat{f}_1(t_{k+1})$	0	1	$\hat{f}_2(s_k)$	0	1/2	1/2	$\hat{f}_3(t_{k+1})$	0	1/4	2/4	1/4
$\hat{f}_2(s_k)$	1/2	1/2	$\hat{f}_3(t_k)$	1/4	2/4	1/4	$\hat{f}_4(s_k)$	1/8	3/8	3/8	1/8

Observe that the numerators of such weights' ratios for $\hat{f}_r(u_k)$ are exactly the coefficients of the r-th row of the Pascal triangle. Hence we can express $\hat{f}_r(t_k)$ and $\hat{f}_r(s_k)$ respectively by

$$\hat{f}_r(t_k)=\frac{1}{n\delta}\sum_{j=(1-r)/2}^{(r-1)/2} g\left(\frac{t_k-t_{k+j}}{\delta}+\frac{r-1}{2}\right)n_j$$

$$\hat{f}_r(s_k)=\frac{1}{n\delta}\sum_{j=1-r/2}^{r/2} g\left(\frac{s_k-t_{k+j}}{\delta}+\frac{r-1}{2}\right)n_j$$

where $g(i)=C_i^{r-1}(1/2)^{r-1},\ i=0,1,2,\cdots,r-1,$ and 0 otherwise, is the p.m.f. of Binomial(r-1,1/2) variable. Our binned kernel polygons (BKP) is defined as:

$$\hat{f}_{BKP(r)}(x)=\frac{u_{k+1}-x}{\delta}\ \hat{f}_r(u_k)+\frac{x-u_k}{\delta}\ \hat{f}_r(u_{k+1}),\quad x\in[u_k,u_{k+1})$$

One can rewrite BKP in form of linear interpolant of a kernel density estimates based a discretized kernel function $K_r^*(x)$ defined as

$$K_r^*(x)=\begin{cases}\sqrt{\frac{r-1}{4}}g(i+\frac{r-1}{2}), \frac{i-1/2}{\sqrt{(r-1)/4}}\le x<\frac{i+1/2}{\sqrt{(r-1)/4}} & \text{r is odd} \quad (2.1)\\ \sqrt{\frac{r-1}{4}}g(i+\frac{r}{2}), \frac{i}{\sqrt{(r-1)/4}}\le x<\frac{i+1}{\sqrt{(r-1)/4}} & \text{r is even.} \quad (2.2)\end{cases}$$

So we have the following alternative expression for $\hat{f}_r(u_k)$

$$\hat{f}_r(u_k)=\frac{1}{n}\sum_j\frac{1}{h}K_r^*(\frac{u_k-X_j}{h}), \quad (2.3)$$

where $h=\delta\sqrt{(r-1)/4}$

The benefit of this modification is that one can use the existing knowledge of kernel polygons (Lin, Wu and Yen) to understand $\hat{f}_{BKP(r)}(x)$.

Remark 2.1 As r is odd, $\hat{f}_r(t_k)$ is the binned data used for constructing the average shifted histogram (Scott 1985, 1992). So the polygon $\hat{f}_{BKP(r)}(x)$ is constructed from average-shift-histogram. As r is even, $\hat{f}_{BKP(r)}(x)$ is the generalize edged frequency polygon considered by Dong and Zheng (2001) and the Binomial p.m.f weight function.

Noted here that, as a direct result of AIMSE analysis of the kernel polygons (Lin, Wu and Yen 2006), the optimal asymptotic integrated mean square error (AIMSE) of $\hat{f}_{BKP(r)}(x)$ converges to that of the Gaussian kernel density estimator.

We now briefly established their asymptotic equivalence. Let $R(\eta)=\int\eta(x)^2dx$. Assume that f'' is absolutely continuous over the real line and that $R(f'')<\infty$ here and through the paper.The following quantities

$$\mu_2=\int t^2K(t)dt=1+\frac{1}{3(r-1)}, R(K^*)=\binom{2r-2}{r-1}(\frac{1}{4})^{r-1}\sqrt{\frac{r-1}{4}}\sim\frac{1}{\sqrt{4\pi}}, \quad R^*(K^*)=$$

$$\int K(t+1/(2r))K(t-1/(2r))dt=\binom{2r-2}{r-1}(\frac{1}{4})^{r-1}\sqrt{\frac{r-1}{4}}(\frac{r-1}{r}), \text{ and } C(r)=$$

$$\left[\left(1+\frac{1}{3(r-1)}\right)^2+\frac{1}{3r^2}\left(1+\frac{1}{3(r-1)}\right)+\frac{1}{30r^4}\right]^{1/5}\times\left[\binom{2r-2}{r-1}\left(\frac{1}{4}\right)^{r-1}\sqrt{\frac{r-1}{4}}\left(1-\frac{1}{3r}\right)\right]^{4/5}$$

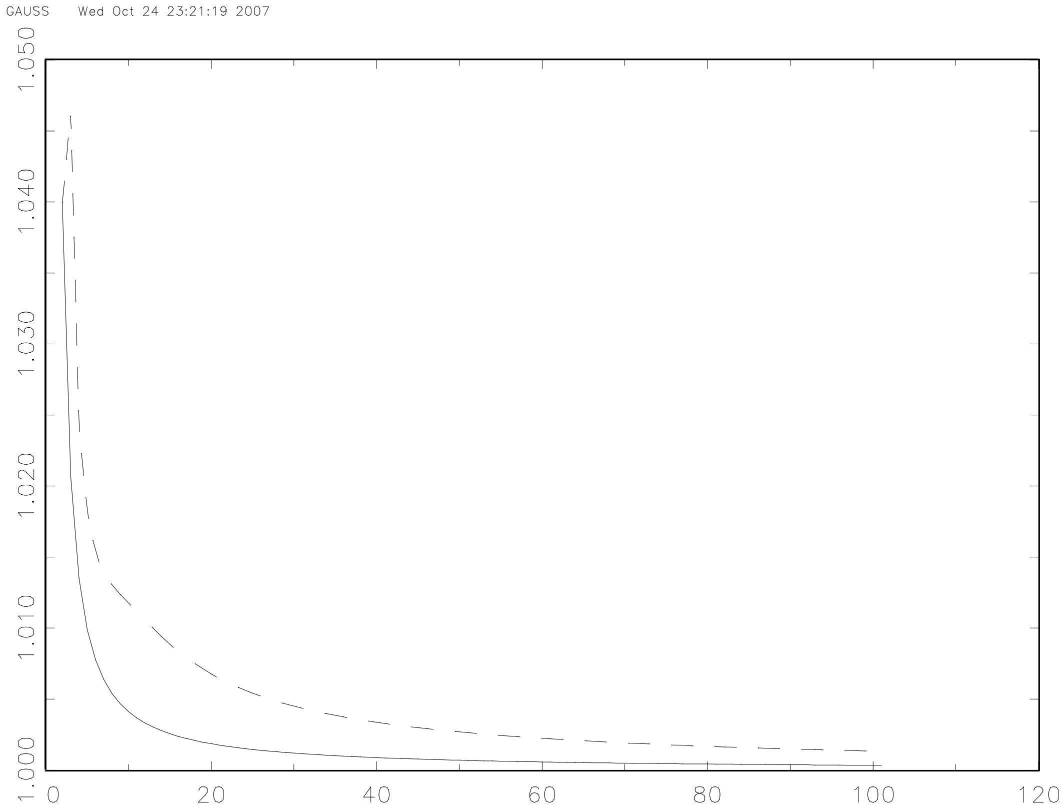
leads to the optimal AIMSE based on optimal bin width (Lin, Wu and Yen 2006)

$$\frac{5}{4}\left(\frac{R(f'')}{n^4}\right)^{1/5} C(r).$$

Since $\binom{2r-2}{r-1}(\frac{1}{4})^{r-1}\sqrt{\frac{r-1}{4}} \sim \frac{1}{\sqrt{4\pi}}$, where $\sim$ is used to indicate that the ratio of the two sides converges to unity as $r \to \infty$, we have

$$\frac{5}{4}\left(\frac{R(f'')}{n^4}\right)^{1/5} C(r) \sim \frac{5}{4}\left(\frac{R(f'')}{n^4}\right)^{1/5} \left(\frac{1}{\sqrt{4\pi}}\right)^{4/5},$$

where the latter is the optimal AIMSE of the Gaussian kernel density estimator. So, the optimal AMISE performance of the polygon type estimator $\hat{f}_{BKP(r)}(x)$ based on $\hat{f}_r$, which use the p.m.f. binomial (r-1, 1/2) to assign weight, can be approximated by that of the usual Gaussian kernel density estimator. Figure 2.1 exhibits the relative accuracy of two estimators, in terms of their AIMSE ratio. The solid exhibits the relative accuracies of $f_{BKP}(x)$ to the Gaussian kernel density estimator, and the dashed line does the case for the normalized kernel polygons using Gaussian kernel. Both AMISE ratios approach unity as *r* goes to infinity, however, the AMISE of $f_{BKP}(x)$ is always smaller for all values of *r.*



$$\mu_2 = \sqrt{(r-1)/4}\sum_i g(i+\frac{r}{2})\int_{i/\sqrt{(r-1)/4}}^{i+1/\sqrt{(r-1)/4}} x^2 dx$$

$$= \sqrt{(r-1)/4}\sum_i C_{i+\frac{r}{2}}^{r-1}(\frac{1}{2})^{r-1}\frac{1}{3}[(\frac{i+1}{\sqrt{(r-1)/4}})^3 - (\frac{i}{\sqrt{(r-1)/4}})^3]$$

$$= (4/3(r-1))\sum_i C_{i+\frac{r}{2}}^{r-1}(\frac{1}{2})^{r-1}(3i^2+3i+1)$$

$$= (4/3(r-1))\sum_i C_i^{r-1}(\frac{1}{2})^{r-1}(3(i-\frac{r}{2})^2+3(i-\frac{r}{2})+1)$$

$$= (4/3(r-1))\sum_i C_i^{r-1}(\frac{1}{2})^{r-1}(3i^2-3ir+3(\frac{r}{2})^2+3i-\frac{3r}{2}+1)$$

$$= (4/3(r-1))(\frac{3(r-1)}{4}+\frac{3(r-1)^2}{4}-\frac{3(r-1)^2}{2}+3\frac{(r-1)^2}{4}+\frac{1}{4})$$

$$= 1+\frac{1}{3(r-1)}$$

$$\| K \|^2 = ((r-1)/4)\sum_i (g(i+\frac{r}{2}))^2 \int_{i/\sqrt{(r-1)/4}}^{i+1/\sqrt{(r-1)/4}} 1dx$$

$$= \frac{\sqrt{(r-1)}}{2}\sum_i (C_{i+\frac{r}{2}}^{r-1})^2 (\frac{1}{2})^{2r-2}$$

$$= \frac{\sqrt{(r-1)}}{2}(\frac{1}{2})^{2r-2} C_{r-1}^{2r-2}$$

$$\| K \|^2 = \sum_{i=0}^{r-1} (\frac{\sqrt{(r-1)}}{2} g(i))^2 \int_{\frac{i-r/2}{\sqrt{(r-1)/4}}}^{\frac{i+1-r/2}{\sqrt{(r-1)/4}}} 1dx$$

$$= \frac{\sqrt{(r-1)}}{2}\sum_{i=0}^{r-1} (C_i^{r-1})^2 (\frac{1}{4})^{2r-2} = \frac{\sqrt{(r-1)}}{2} C_{r-1}^{2r-2} (\frac{1}{4})^{2r-2}$$

$$K_r^*(x) = \sqrt{\frac{r-1}{4}} g(i+\frac{r-1}{2}), \frac{i-1/2}{\sqrt{(r-1)/4}} \le x < \frac{i+1/2}{\sqrt{(r-1)/4}}, \ i \in \mathbb{Z}, \ \text{r is odd}$$

$$\mu_2 = \sqrt{(r-1)/4}\sum_i g(i+\frac{r-1}{2}) \int_{i-0.5/\sqrt{(r-1)/4}}^{i+0.5/\sqrt{(r-1)/4}} x^2 dx$$

$$= \sqrt{(r-1)/4}\sum_i C_{i+\frac{r-1}{2}}^{r-1} (\frac{1}{2})^{r-1} \frac{1}{3} [(\frac{i+0.5}{\sqrt{(r-1)/4}})^3 - (\frac{i-0.5}{\sqrt{(r-1)/4}})^3]$$

$$= (4/3(r-1))\sum_i C_{i+\frac{r-1}{2}}^{r-1} (\frac{1}{2})^{r-1} (3i^2 + 1/4)$$

$$= (4/3(r-1))\sum_i C_i^{r-1} (\frac{1}{2})^{r-1} (3(i-\frac{r-1}{2})^2 + 1/4)$$

$$= (4/3(r-1))\sum_i C_i^{r-1} (\frac{1}{2})^{r-1} (3i^2 - 3i(r-1) + \frac{3(r-1)^2}{4} + \frac{1}{4})$$

$$= (4/3(r-1))(\frac{3(r-1)}{4} + \frac{3(r-1)^2}{4} - \frac{3(r-1)^2}{2} + 3\frac{(r-1)^2}{4} + \frac{1}{4})$$

$$= 1 + \frac{1}{3(r-1)}$$

In the context of usual kernel density estimation, one has to use continuous kernel function compactly supported on [-1,1] to construct continuous kernel density estimates. However, such restriction on the choice of kernel function can be relaxed in the case of polygon type density estimators thanks to its interpolated nature.

3. Linear interpolant of binned kernel density estimates

Given any kernel function K, (2.3) can be rewrite to yield BKDE:

$$\hat{f}(u_k) = \frac{1}{nh}\sum_{j \in Z} K(\frac{u_k - t_{k+j}}{h}) n_{k+j} \quad .$$

where $h = m \cdot \delta, \ m > 0$. The second estimator $\hat{f}_{BP2}(x)$ is defined as:

$$\hat{f}_{BP2}(x) = \frac{u_{k+1} - x}{\delta} \hat{f}(u_k) + \frac{x - u_k}{\delta} \hat{f}(u_{k+1}) \quad , \quad x \in [u_k - \frac{\delta}{2}, u_k + \frac{\delta}{2}) .$$

Note that FP and EFP are two special cases of $\hat{f}_{BP2}(x)$ using uniform kernel that corresponds to $m = 1/2$ and 1, respectively.

Suppose that the kernel K is assumed to be a symmetric continuous function supported on [-1,1), (-1,1] or [-1,1]. The sufficient and necessary conditions to guarantee of $\hat{f}_{BP2}$ be a probability density function for odd and even value of m are respectively:

$$\frac{2}{m-1}\sum_j K\left(\frac{2j}{m-1}\right) = 1 \tag{3.1}$$

$$\frac{2}{m}\sum_j K\left(\frac{j+0.5}{m}\right) = 1 \tag{3.2}$$

Kernels in form of higher order polynomial often fail to satisfy (3.1) and (3.2) unless they are normalized. So, the goal of this section is to identify the classes of the uniform and linear kernel functions that satisfy condition (3.1) or (3.2). The results are now reported in Proposition 2.1 and 2.2.

Proposition 2.1. Suppose that a kernel function K can be expressed as a linear function of absolute-value variable. Then, kernel K that satisfies (3.1) for odd value of m are: are $K(y) = (1-|y|)I(-1 \le y \le 1)$ and $K(y) = (a + (1-2a)|y|)I_D(y)$, $0 \le a \le 1$, and $D = (-1,1]$ or [-1,1)

Remark 2.1. By taking the average of the linear kernels $K(y) = (a + (1-2a)|y|)$ $\times I_D$ for two case of *D*, the kernel function $K(y) = \frac{1}{2}(a + (1-2a)|y|)I(-1 \le y < 1)$ $+\frac{1}{2}(a + (1-2a)|y|)I(-1 < y \le 1)$, $0 \le a \le 1$, also satisfies the condition (3.1) for $m \in N$.

Proposition 2.2. Suppose that a kernel function K can be expressed as a linear function of absolute-value variable. Then, kernel K that satisfies (3.1) for even value of m are $K(x) = a + (1-2a)\,|\,x\,|\,I_{[-1,1]}$, $0 \le a \le 1$;

Let $r = m/2$ and $(m-1)/2$ respectively for even and odd value of *m*. Observe that $n_k = \sum_{i=1}^{n} I(t_k - \frac{1}{2} \le X_i < t_k + \frac{1}{2})$, we can rewrite $\hat{f}(u_k)$ as:

$$\hat{f}(u_k) = \frac{1}{nr\delta} \sum_{j=-\infty}^{\infty} K(\frac{u_k - t_{k+j}}{r\delta}) n_{k+j} = \frac{1}{nr\delta} \sum_{i=1}^{n} K^*(\frac{u_k - X_i}{r\delta})$$

, where

$$K_r^*(x) = \begin{cases} K_r^*(x) = \sum_j K(\frac{j}{r}) I(\frac{j-1/2}{r} \le x < \frac{j+1/2}{r}), & m \text{ is odd} \\ K_r^*(x) = \sum_j K(\frac{j+1/2}{r}) I(\frac{j}{r} \le x < \frac{j+1}{r}), & m \text{ is even} \end{cases}$$

The following notation is needed for describing the AIMSE of $\hat{f}_{BP2}(x)$

$$\alpha_1 = \left(\frac{1}{2} - \frac{a}{3} + \frac{7-8a}{12m^2}\right)^2 + \frac{1}{3m^2}\left(\frac{1}{2} - \frac{a}{3} + \frac{7-8a}{12m^2}\right) + \frac{1}{30m^4}$$

$$\beta_1 = \frac{2(a^2-a+1)}{3} - \frac{2(a-1)^2}{3m} + \frac{(2a-1)(a-1)}{3m^2}$$

$$\alpha_2 = \left(\frac{1}{2} - \frac{a}{3} + \frac{1-2a}{6m^2}\right)^2 + \frac{1}{3m^2}\left(\frac{1}{2} - \frac{a}{3} + \frac{1-2a}{6m^2}\right) + \frac{1}{30m^4}$$

$$\beta_2 = \frac{2(a^2-a+1)}{3} - \frac{(a-1)^2}{3m} + \frac{(2a-1)(4a-1)}{6m^2} + \frac{(1-2a)^2}{4m^3}$$

Straightforward from the result of Lin, Wu and Yen (2006), for the kernels in Cororrola (1) and proposition(2), the corresponding optimal AIMSE of $\hat{f}_{BP2}(x)$ is

$$\frac{5}{4}\left(\frac{R(f'')}{n^4}\right)^{1/5} C(a,m),$$

where $C(a,m) = \alpha_1^{1/5} \times \beta_1^{4/5}$ and $\alpha_2^{1/5} \times \beta_2^{4/5}$ respectively for odd and even value of *m*. Our approach to find the optimal AIMSE of $\hat{f}_{BP2}(x)$ based on such kernels is numerical. For optimal AIMSE performance for each value of *m*, it suffices to minimize $C(a,m)$ by finding its minimizer a_m^* through search.

Bin number	Value of a	C(a,m)	Bin number	Value of a	C(a,m)
3	0.3796	0.36435138	15	0.7125	0.35080591

4	0.6641	0.35863201	16	0.7635	0.35119454
5	0.47955	0.35667522	17	0.724	0.35072941
6	0.7071	0.35405597	18	0.768	0.35114130
7	0.5858	0.35346028	19	0.733	0.35069697
8	0.72915	0.35246988	20	0.771	0.35111168
9	0.64268	0.35197414	21	0.74	0.35068858
10	0.742695	0.35179157	22	0.7735	0.35109603
11	0.6761	0.35129249	23	0.7455	0.35069354
12	0.752	0.35146236	24	0.775	0.35108888
13	0.698	0.35096585	25	0.75	0.35070592

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Appendix

(pf)

$k(x)= a+(1-2a)|x|$

$x=i/r$, $k=(r-1)/2$,

$$\mu_2 = \sum_{i=-k}^{k} K(\frac{i}{k})\int_{(i-0.5)/k}^{(i+0.5)/k} x^2$$

$$= \sum_{i=-k}^{k} K(\frac{i}{k})\frac{1}{3}[(\frac{i+0.5}{k})^3 - (\frac{i-0.5}{k})^3]$$

$$= \sum_{i=-k}^{k} K(\frac{i}{k})\frac{1}{3k^3}(3i^2 + \frac{1}{4})$$

$$= [\frac{1}{3k^3}\sum_{i=-k+1}^{k-1}(a+(1-2a)|\frac{i}{k}|)(3i^2+\frac{1}{4})] + \frac{1}{3k^3}(1-a)(3k^2+\frac{1}{4})$$

$$= [\frac{1}{3k^3}\sum_{i=-k+1}^{k-1}(3ai^2 + \frac{a}{4} + \frac{3-6a}{k}|i|^3 + \frac{1-2a}{4k}|i|)] + \frac{1-a}{k} + \frac{1-a}{12k^3}$$

$$= \frac{1}{3k^3}[(a(k-1)(2k-1)k + \frac{a}{4}(2k-1) + 2\frac{3-6a}{k}(\frac{(k-1)k}{2})^2 + 2(\frac{1-2a}{4k})\frac{(k-1)k}{2}] + \frac{1-a}{k} + \frac{1-a}{12k^3}$$

$$= \frac{1}{3k^3}[a(2k^3 - 3k^2 + k) + \frac{a}{2}k - \frac{a}{4} + \frac{3-6a}{2}(k^3 - 2k^2 + k) + \frac{1-2a}{4}k - \frac{1-2a}{4}] + \frac{1-a}{k} + \frac{1-a}{12k^3}$$

$$= (\frac{1}{3}\frac{4a+3-6a}{2}) + \frac{1}{3k}(-3a-3+6a+3-3a) + \frac{1}{3k^2}(a+\frac{a}{2}+\frac{3}{2}-3a+\frac{1}{4}-\frac{a}{2}) + \frac{1}{3k^3}(\frac{-a-1+2a+1-a}{4})$$

$$= (\frac{1}{2} - \frac{a}{3}) + \frac{1}{3k^2}(\frac{7}{4} - 2a)$$

$$\| K \|^2 = \sum_{i=-k}^{k}(K(\frac{i}{r}))^2 \int_{(i-0.5)/k}^{(i+0.5)/k} 1dx$$

$$= [\frac{1}{k}\sum_{i=-k+1}^{k-1}(a+(1-2a)|\frac{i}{k}|)^2] + \frac{2}{k}(\frac{(1-a)}{2})^2$$

$$= [\frac{1}{k}\sum_{i=-k+1}^{k-1}(a^2 + 2a(1-2a)|\frac{i}{k}| + (1-2a)^2|\frac{i}{k}|^2] + \frac{2}{k}(\frac{(1-a)}{2})^2$$

$$= \frac{1}{k}[(a^2(2k-1) + \frac{2a(1-2a)}{k}(k-1)k + 2(1-2a)^2(\frac{(k-1)(2k-1)k}{6k^2})] + \frac{1-2a+a^2}{2k}$$

$$= \frac{1}{k}[(2a^2 + 2a - 4a^2 + \frac{2}{3}(1-2a)^2)k + (-a^2 - 2a + 4a^2 - (1-2a)^2) + \frac{(1-2a)^2}{3k}] + \frac{1-2a+a^2}{2k}$$

$$= (\frac{2a^2 - 2a + 2}{3}) + \frac{1}{k}(3a^2 - 2a - 1 + 4a - 4a^2 + \frac{1}{2} - a + \frac{a^2}{2}) + \frac{(1-2a)^2}{3k^2}$$

$$= (\frac{2a^2 - 2a + 2}{3}) - \frac{1}{2k}(a^2 - 2a + 1) + \frac{(1-2a)^2}{3k^2}$$

$$K*K=\sum_{i=-r}^{r}(K(\frac{i}{r})K(\frac{i+1}{r}))\int_{(i-0.5)/r}^{(i+0.5)/r}1dx$$

$$=[\frac{1}{r}\sum_{i=-r+1}^{r-2}(a+(1-2a)|\frac{i}{r}|)(a+(1-2a)|\frac{i+1}{r}|)]+\frac{2}{r}(\frac{(1-a)}{2})(a+(1-2a)\frac{r-1}{r})$$

$$=[\frac{2}{r}\sum_{i=0}^{r-2}(a^2+a(1-2a)\frac{2i+1}{r}+(1-2a)^2\frac{i(i+1)}{r^2}]+\frac{(1-a)^2}{r}-\frac{(1-a)(1-2a)}{r^2}$$

$$=\frac{2}{r}[(a^2(r-1)+\frac{a(1-2a)}{r}((r-2)(r-1)+(r-1))+(1-2a)^2(\frac{(r-2)(2r-3)(r-1)}{6r^2}+\frac{(r-2)(r-1)}{2r^2})]+\frac{(1-a)^2}{r}-\frac{(1-a)(1-2a)}{r^2}$$

$$=\frac{2}{r}[(a^2(r-1)+\frac{a(1-2a)}{r}(r^2-2r+1)+(1-2a)^2\frac{1}{3r}(r^2-3r+2)]+\frac{(1-a)^2}{r}-\frac{(1-a)(1-2a)}{r^2}$$

$$=\frac{2}{r}[(a^2+a-2a^2+\frac{1-4a+4a^2}{3})r)+(-a^2-2a+4a^2-(1-2a)^2)+(\frac{a(1-2a)}{r}+\frac{2(1-2a)^2}{3r})]+\frac{(1-a)^2}{r}-\frac{(1-a)(1-2a)}{r^2}$$

$$=(\frac{2(a^2-a+1)}{3})+\frac{2}{r}(3a^2-2a-1+4a-4a^2+\frac{1}{2}-a+\frac{a^2}{2})+\frac{2(1-2a)(2-a)-3(1-a)(1-2a)}{3r^2}$$

$$=(\frac{2(a^2-a+1)}{3})-\frac{1}{r}(a^2-2a+1)+(\frac{1-a-2a^2}{3r^2})$$

(pf)

k(x)= a+(1-2a)|x| x=(2i+1)/r

$$\mu_2=\sum_{i=-r/2}^{r/2-1}K(\frac{2i+1}{r})\int_{2i/r}^{2(i+1)/r}x^2$$

$$=\sum_{i=-r/2}^{r/2-1}K(\frac{2i+1}{r})\frac{1}{3}[(\frac{2(i+1)}{r})^3-(\frac{2i}{r})^3]$$

$$=\sum_{i=-r/2}^{r/2-1}K(\frac{2i+1}{r})\frac{8}{3r^3}(3i^2+3i+1)$$

$$=\frac{8}{3r^3}\sum_{i=-r/2}^{r/2-1}(a+(1-2a)|\frac{2i+1}{r}|)(3i^2+3i+1)$$

$$
\begin{aligned}
&= \frac{16}{3r^3}\Big[\sum_{i=0}^{r/2-1}\Big(a+\frac{1-2a}{r}\Big)(3i^2+3i+1)+\frac{2-4a}{r}(3i^3+3i^2+i)\Big] \\
&= \frac{16}{3r^3}\Big[\Big(a+\frac{1-2a}{r}\Big)\Big(\frac{(r/2)(r/2-1)(r-1)}{2}+\Big(\frac{3(r/2-1)(r/2)}{2}\Big)+r/2\Big)+\frac{2-4a}{r}\Big[\frac{3(r/2-1)^2(r/2)^2}{4}+\frac{(r/2)(r/2-1)(r-1)}{2}+ (\\
&= \frac{16}{3r^3}\Big[\Big(a+\frac{1-2a}{r}\Big)\Big(\frac{r(r-2)(r-1)}{8}+\frac{3r(r-2)}{8}+\frac{4r}{8}\Big)+\frac{2-4a}{r}\Big[\frac{3(r-2)^2r^2}{64}+\frac{r(r-2)(r-1)}{8}+\frac{(r-2)r}{8}\Big] \\
&= \frac{16}{3r^3}\Big[\Big(a+\frac{1-2a}{r}\Big)\frac{r(r^2)}{8}+\frac{2-4a}{r}\Big(\frac{r(3r^3-12r^2+12r+8r^2-24r+16+8r-16}{64}\Big)\Big] \\
&= \frac{16}{3r^2}\Big[\Big(a+\frac{1-2a}{r}\Big)\frac{r^2}{8}+\frac{1-2a}{r}\Big(\frac{(3r^3-4r^2-4r)}{32}\Big)\Big] \\
&= \frac{2}{3r^2}\Big[\Big(a+\frac{1-2a}{r}\Big)r^2+(1-2a)\Big(\frac{3r^2-4r-4)}{4}\Big] \\
&= \frac{2}{3r^2}\Big[(ar^2+(1-2a)r+(1-2a)\Big(\frac{3r^2}{4}-r-1\Big)\Big] \\
&= \frac{2}{3r^2}\Big[\frac{3-2a}{4}r^2-(1-2a)\Big] \\
&= \Big(\frac{1}{2}-\frac{a}{3}\Big)-\Big(\frac{2-4a}{3r^2}\Big)
\end{aligned}
$$

$$
\begin{aligned}
\|K\|^2 &= \sum_{i=-r/2}^{r/2-1}\Big(K\Big(\frac{2i+1}{r}\Big)\Big)^2\int_{2i/r}^{2(i+1)/r} 1\,dx \\
&= \frac{1}{r/2}\sum_{i=-r/2}^{r/2-1}\Big(a+(1-2a)\Big|\frac{2i+1}{r}\Big|\Big)^2 \\
&= \frac{1}{r/2}\Big[\sum_{i=-r/2}^{r/2-1}\Big(a^2+2a(1-2a)\Big|\frac{i+1/2}{r/2}\Big|+(1-2a)^2\Big|\frac{i+1/2}{r/2}\Big|^2\Big] \\
&= \frac{2}{r/2}\Big[\sum_{i=0}^{r/2-1}\Big(a^2+\frac{a(1-2a)}{r/2}+\frac{2a(1-2a)}{r/2}i+(1-2a)^2\frac{i^2+i+1/4}{(r/2)^2}\Big] \\
&= \frac{2}{r/2}\Big[\sum_{i=0}^{r/2-1}\Big(a^2+\frac{a(1-2a)}{r/2}+\frac{(1-2a)^2}{4(r/2)^2}\Big)+\Big(\frac{2a(1-2a)}{r/2}+\frac{(1-2a)^2}{(r/2)^2}\Big)i+\frac{(1-2a)^2}{(r/2)^2}i^2\Big] \\
&= \Big(2a^2+\frac{2a(1-2a)}{r/2}+\frac{(1-2a)^2}{2(r/2)^2}\Big)+\Big(\frac{2a(1-2a)}{r/2}+\frac{(1-2a)^2}{(r/2)^2}\Big)(r/2-1)+\frac{(1-2a)^2}{(r/2)^2}\Big(\frac{2(r/2)^2-3(r/2)+1}{3}\Big) \\
&= \Big(2a^2+2a(1-2a)+\frac{2(1-2a)^2}{3}\Big)+\big(2a(1-2a)-2a(1-2a)+(1-2a)^2-(1-2a)^2\big)\frac{1}{r/2} \\
&\quad +\frac{(1-2a)^2}{2(r/2)^2}-\frac{(1-2a)^2}{(r/2)^2}+\frac{(1-2a)^2}{3(r/2)^2} \\
&= \frac{2}{3}(a^2+a+1)-\frac{2}{3r^2}(1-2a)^2
\end{aligned}
$$

$$K*K=\sum_{i=-r/2+1}^{r/2-1}K(\frac{i+0.5}{r/2})K(\frac{i-0.5}{r/2})\int_{2i/r}^{2(i+1)/r}1dx$$

$$=[\frac{1}{r/2}\sum_{i=-r/2+1}^{r/2-1}(a+(1-2a)|\frac{i-0.5}{r/2}|)(a+(1-2a)|\frac{i+0.5}{r/2}|)]$$

$$=\frac{2}{r/2}[\sum_{i=1}^{r/2-1}(a^2+2a(1-2a)\frac{i}{r/2}+(1-2a)^2\frac{i^2-1/4}{(r/2)^2}]+\frac{1}{r/2}(a^2+\frac{a(1-2a)}{r/2}+\frac{(1-2a)^2}{4(r/2)^2})$$

$$=\frac{2}{r/2}[(a^2(r/2-1)+\frac{a(1-2a)}{r/2}(r/2)(r/2-1))+\frac{(1-2a)^2}{(r/2)}(\frac{(r/2-1)(2(r/2)-1)}{6})-(1-2a)^2\frac{(r/2-1)}{4(r/2)^2}]+(\frac{a^2}{r/2}+\frac{a(1-2a)}{(r/2)^2}+$$

$$=\frac{2}{r/2}[(a^2(r/2-1)+a(1-2a)(r/2-1)+\frac{(1-2a)^2}{12(r/2)}(4(r/2)^2-6(r/2)+2-3)]+\frac{a^2}{r/2}+\frac{a(1-2a)}{(r/2)^2}+\frac{3(1-2a)^2}{4(r/2)^3}$$

$$=\frac{2}{r/2}[(a^2+a-2a^2+\frac{1-4a+4a^2}{3})(r/2)+(-a^2-a+2a^2-\frac{(1-2a)^2}{2}+\frac{a^2}{2})+(\frac{a(1-2a)}{2(r/2)}-\frac{(1-2a)^2}{12(r/2)})+\frac{3(1-2a)^2}{8(r/2)^2}]$$

$$=(\frac{2(a^2-a+1)}{3})-\frac{(a-1)^2}{r/2}+\frac{(1-2a)(8a-1)}{6(r/2)^2}+\frac{3(1-2a)^2}{4(r/2)^3}$$