計畫名稱:網格型微分方程動態行爲及腦資訊的非線性分析。

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1 中英文摘要及關鍵詞

1.1 計畫中文摘要

關鍵詞：混合型網格微分方程組、腦動態系統、單調行進波解、有限差分法、相關維度、熵、神經網路、 延遲座標、嵌入定理、Lyapunov指數、同步。

在神經網路方面，在有限個神經元，考慮週期解的存在、唯一性及穩定性，及當有 外力(forcing term)時，系統會有何分歧現象；在一維無窮個神經元，我們研究在不同的 耦合係數、不同的時間遲滯、非線性項有兩類：monostable type (Fisher type or KPP type) 及bistable type (Nagumo type)之下，單調行進波解(Traveling Wavefront Solution，TWS)的存在性、唯一性及穩定性，同時並考慮數值計算，所需用到的finite difference method 及 continuation method，我們將結果推廣至二維LDE。 關於Nonlinear Analysis of Brain Dynamics 方面我們要利用Dynamical Systems 及 Neural Network 的理論來計算實驗所量得的人類大腦實驗數據並做modeling。Dynamical Systems 的方法包括了資料處理及度量複雜性裡相關的特徵值兩部分；其中資料處理包 括了delay coordinate 和embedding theory 等方法，複雜性度量裡相關的特徵值，包含了 correlation dimension 、Lyapunov exponents 、entropy 、synchronization 及phase synchronization 等。希望藉由所計算出來的特徵值(D2, Entropy, Lyapnunov Exponents)改 變，應用在臨床醫學方面，能在病人發病早期的過渡時期，及早發現，及早治療。Neural Network 的方面，記錄人體運動的整個時程，大腦狀態(D2, Entropy)的變化，用Neural Network 的工具進而瞭解運動時，大腦的各系統在時間與空間交互作用的特性。

1.2 計畫英文摘要

Keywords：mixed type lattice differential equations, brain dynamics, monotone traveling wave solution, finite difference method, correlation dimension, entropy, neural network, delay coordinate, embedding theorem, Lyapunov exponents, sensorymotor system。

In the mixed type lattice differential equations, we have two considerations. First, for a system of finite neurons with time delay, we will investigate the existence, uniqueness and stability of periodic solutions. In particular, is there any bifurcation occur in the system with forcing term? Secondly, for one dimensional infinite neurons the existence of traveling wavefront solutions will be investigated in different coupling coefficients and different time delay, τ . Furthermore, two kinds of nonlinearity in one dimensional LDE, monostable type (Fisher type or KPP type) and bistable type (Nagumo type), will be considered. So far the existence of traveling wavefront solutions was be proved for monostable type and it is still open for bistable type. Therefore, we will investigate the existence of TWS for the bistable case numerically first. We will generalize the results to 2D-LDE. About the nonlinear analysis of brain dynamics, we will apply the theory of dynamical system to calculate some characteristics of experimental data acquired from human brains of normal subjects and patients. And try to model the relation between sensorymotor system of human brain and motor units in the techniques of neural networks. In the theory of dynamical system, Takens had developed the theory of delay coordinates and embedding theorem to reconstruct the phase space of the system we considered. Therefore, some characteristics like correlation dimension, Lyapunov exponents, entropy, synchronization and phase synchronization can be calculated. These characteristics can help us to detect transient state of some diseases and doctors can take some therapies early. Therefore, we hope these results can have some clinical applications. Since we can calculate the characteristics in the time course of movement, i.e. we can detect the interactions of states of human brain and state of human behavior. From here, we will try to use the technique of neural network to model the relation of state of sensorymotor system and motor unit.

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2 報告內容

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4 計畫成果自評

We investigate numerically and theoretically the existence of traveling wave solution of some mixed type functional differential equation on lattices named Lattice Differential Equation, LDE. There are two preliminary main results of this projects.

The first result is written as a mater thesis entitled *"Numerical Traveling Wave Solutions of Some Functional-coupling Mixed-type Lattice Differential Equations."* by Mr. 黃清郎.

The purpose of this work is to numerically investigate the existence of traveling plane wave solutions of the following functional coupling lattice differential equations of the form

$$\begin{aligned}\frac{du_{i,j}(t)}{dt} &= f(u_{i,j}(t)) + d_1 g(u_{i-1,j}(t)) + d_2 g(u_{i,j-1}(t)) - \\ &\quad d_3 g(u_{i,j}(t)) + d_4 g(u_{i+1,j}(t)) + d_5 g(u_{i,j+1}(t)). \end{aligned} \tag{1}$$

Here $u_{i,j}(t)$ indicate the state functions located at site $(i,j) \in Z^2$; the reaction function f is bistable type; the coupling function g is sigmoidal type, and the coupling coefficients $d_1, \cdots, d_5$ are non-negative constants.

Our main concern is to find a traveling wave solution of (1) numerically. Let $\xi = r_1 i + r_2 j + ct$ be the moving coordinate where $r_1 = \cos(\theta)$, $r_2 = \sin(\theta)$ and $0 \leq \theta \leq \pi/2$. We assume the wave speed c is positive. Under this coordinate $u_{i,j}(t) := \varphi(r_1 i + r_2 j + ct) := \varphi(\xi)$, we yield the following nonlinear mixed-type(delayed and advanced) functional-coupling profile equation

$$\begin{aligned}c\dot{\varphi}(\xi) &= f(\varphi(\xi)) + d_1 g(\varphi(\xi - r_1)) + d_2 g(\varphi(\xi - r_2)) - \\ &\quad d_3 g(\varphi(\xi)) + d_4 g(\varphi(\xi + r_1)) + d_5 g(\varphi(\xi + r_2)). \end{aligned} \tag{2}$$

It is clear that there are three homogenous stationary solutions of (1) denoted by

$$u^0 := 0,\ u^- := a \text{ and } u^+ := 1.$$

Hence, we impose the traveling wave solutions of (2) with the following asymptotically boundary conditions

$$\lim_{\xi \to -\infty} \phi(\xi) = u^0 \text{ and } \lim_{\xi \to \infty} \phi(\xi) = u^+. \tag{3}$$

The strategy for solving (2) with (3) is via finite difference method on a truncated finite interval. But there is a nonlinear term, the reaction function f, we can not solve it directly. Instead, Newton's method is applied to solve a nonlinear algebraic system induced by finite difference scheme. Using Continuation method to overcome the difficulty to find a good initial guess of Newton's method.

5 附錄

Numerical Traveling Wave Solutions of Some Nonlinear Mixed-type Lattice Differential Equations

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Abstract

We present a finite difference method for computing traveling wave front solutions of a two-dimensional lattice differential equations. In particular, the nonlinear reaction function is bi-stable type and the diffusion term is with function-coupling. Under some suitable conditions on the characteristic equation, we prove the existence of the positive wave speed. It can help us to approximate the asymptotically behavior on the boundaries of profile equation. Newton's method are used to find the solution of nonlinear algebraic equations inducing by the finite difference method. To overcome the difficulty of finding a good initial solution of Newton's iteration, the continuation method are implemented.

Keywords: Coupling profile equation, Finite difference method, Newton's iteration, Continuation method.

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1 Introduction

The purpose of this paper is to numerically investigate the existence of traveling plane wave solutions of the following functional coupling lattice differential equations of the form

$$\begin{aligned} \frac{du_{i,j}(t)}{dt} &= f(u_{i,j}(t)) + d_1 g(u_{i-1,j}(t)) + d_2 g(u_{i,j-1}(t)) - \\ & \quad d_3 g(u_{i,j}(t)) + d_4 g(u_{i+1,j}(t)) + d_5 g(u_{i,j+1}(t)). \end{aligned} \tag{1.1}$$

Here $u_{i,j}(t)$ indicate the state functions located at site $(i,j) \in Z^2$; the reaction function f is bistable type; the coupling function g is sigmoidal type, and the coupling coefficients $d_1, \cdots, d_5$ are non-negative constants.

The equations (1.1) is so-called lattice differential equations which are infinite system of ordinary differential equations indexed by integer points on $\mathbb{R}^2$. If the coefficients $d_1 = d_2 = d_4 = d_5 = 1$ and $d_3 = 4$ and the coupling function $g = id$, then equations (1.1) can be viewed as a discretization of the reaction diffusion equation

$$\frac{\partial u}{\partial t}(x,t) = \triangle u(x,t) + f(u(x,t)).$$

However, such systems can be derived independently from the study of population dynamics of biology, dynamics of neural networks, electro-cortical activity of mammalian neocortex, and so on.

Our main concern is to find a traveling wave solution of (1.1) numerically. Let $\xi = r_1 i + r_2 j + ct$ be the moving coordinate where $r_1 = \cos(\theta)$, $r_2 = \sin(\theta)$, $0 \le \theta \le \pi/2$. We assume the wave speed c is positive. Under this coordinate $x_{i,j}(t) := \varphi(r_1 i + r_2 j + ct) := \varphi(\xi)$, we yield the following nonlinear mixed-type(delayed and advanced) functional-coupling profile equation

$$\begin{aligned} c\dot{\varphi}(\xi) &= f(\varphi(\xi)) + d_1 g(\varphi(\xi - r_1)) + d_2 g(\varphi(\xi - r_2)) - \\ & \quad d_3 g(\varphi(\xi)) + d_4 g(\varphi(\xi + r_1)) + d_5 g(\varphi(\xi + r_2)). \end{aligned} \tag{1.2}$$

The following assumptions will be needed throughout the paper.

(A1) The bistable nonlinearity reaction function $f \in C^1[0,1]$ with $f(0) = f(a) = f(1) = 0$, $a \in (0, 0.5)$, $f'(0) < 0$, $f'(a) > 0$, and $f'(1) < 0$.

(A2) The coupling output function g is a monotone increasing $C^1(\mathbb{R})$ function.

(A3) The coupling coefficients are nonnegative real number with $d_1 + d_2 - d_3 + d_4 + d_5 = 0$.

By the assumption (A3), it follows that there exists homogeneous stationary solutions of (1.2) denoted by

$$\varphi^- := 0,\ \varphi^0 := a, \text{ and } \varphi^+ := 1.$$

Thus, we are interested in finding a numerical traveling wave solution of (1.2) connecting the two stable equilibria φ^- and φ^+ with the asymptotically boundary conditions

$$\varphi(-\infty) = 0 \text{ and } \varphi(+\infty) = 1. \tag{1.3}$$

The algorithm we discuss here consists of a combination of Newton's method and parameter continuation method. It is based upon the idea proposed in [9, 20]. Our main contribution is to generalize it in two parts. First, the traveling wave solution are computed in a two-dimensional lattice instead of one-dimensional grid. Secondly, functional-coupling effect is consider in discrete diffusion terms.

Recently, the investigation in lattice differential equations analytically[5, 11, 17, 19] or numerically[4, 10] are getting considerable attentions not only from mathematical theory but also from applications in many scientific disciplines. Models involving lattices differential equations can be found in a wealthy of research, including biology [2, 12], chemical reaction theory [8, 13], imaging processing and pattern recognition [6], material science [1, 3], and neural network[6, 15, 16].

The remainder of this paper is organized as follows. In section 2, we will introduce some numerical methods to approximate traveling wave solutions of (1.2). In section 3, some numerical results will be presented for fixed coefficients d_1, d_2, d_3, d_4, and d_5 while the wave speed $c > 0$ is solved.

2 Scheme of Solution Approximation

To approximate traveling wave solutions of (1.2) with the asymptotically boundary conditions (1.3), we truncate the real line into a finite interval $[-I, I]$ for I large enough. In section 2.1, we will prove the existence of characteristic roots, λ_1 and λ_2, of linearized profile equation for $\xi < -I$ and $\xi > I$ respectively. In section 2.2, traveling wave solutions outside the finite interval $[-I, I]$ can be approximated by exponential functions with λ_1 and λ_2. In section 2.3, we take advantage of the finite difference method[14] and the linear interpolation[18] to approximate the coupling profile equation (1.2) and solve the nonlinear discretized system of (1.2) by Newton's iteration. In section 2.4, we introduce the continuation method[7] to obtain a sufficiently accurate initial guess for Newton's iteration. Then we describe the numerical scheme as follows.

2.1 Existence of Characteristic Roots

We first consider the condition $\xi < -I$. Since $\xi \to -\infty$ implies $\varphi(\xi) \to 0$ and $f(\varphi(\xi)) \to 0$, by the linearizaiton of (1.2) near $\varphi = 0$, we have

$$\begin{aligned} c\dot{\varphi}(\xi) &= f'(0)\varphi(\xi) + d_1 g'(0)\varphi(\xi - r_1) + d_2 g'(0)\varphi(\xi - r_2) - \\ &\quad d_3 g'(0)\varphi(\xi) + d_4 g'(0)\varphi(\xi + r_1) + d_5 g'(0)\varphi(\xi + r_2). \end{aligned} \tag{2.1}$$

Since $\varphi(\xi) \to 0$ as $\xi \to -\infty$, we substitute $\varphi(\xi) = e^{\lambda_1 \xi}$, $\lambda_1 > 0$ into the equation (2.1) and obtain the following characteristic equation

$$\begin{aligned} c\lambda_1 &= f'(0) + d_1 g'(0)e^{-r_1\lambda_1} + d_2 g'(0)e^{-r_2\lambda_1} - \\ &\quad d_3 g'(0) + d_4 g'(0)e^{r_1\lambda_1} + d_5 g'(0)e^{r_2\lambda_1}. \end{aligned} \tag{2.2}$$

To prove the existence of λ_1, we give the following lemma.

Lemma 2.1. *For given wave speed $c > 0$, if $d_4 g'(0) > 0$ or $d_5 g'(0) > 0$ then there exists $\lambda_1 > 0$ such that the characteristic equation* (2.2) *has a positive root.*

Proof. We suppose that

$$\begin{aligned} L(\lambda) &= f'(0) + d_1 g'(0)e^{-r_1\lambda} + d_2 g'(0)e^{-r_2\lambda} - \\ &\quad d_3 g'(0) + d_4 g'(0)e^{r_1\lambda} + d_5 g'(0)e^{r_2\lambda} \end{aligned}$$

is the real-value function then for $d_4 g'(0) > 0$ or $d_5 g'(0) > 0$, we have

$$L(0) = f'(0) < 0 \text{ and } \lim_{\lambda\to\infty} L(\lambda) = \infty \text{ exponentially.}$$

This implies $L(\lambda)$ and the equation $c\lambda$ hand over a point $\lambda > 0$. Then we can find a $\lambda_1 > 0$ such that the characteristic equation (2.2) has a positive root. This completes the proof. □

Similarly, if $\xi > I$ with $\varphi(\xi) \to 1$, $f(\varphi(\xi)) \to 0$ as $\xi \to +\infty$ then near $\varphi = 1$, we can obtain the following linearized profile equation

$$\begin{aligned} c\dot{\varphi}(\xi) &= -f'(1)(1 - \varphi(\xi)) - d_1 g'(1)(1 - \varphi(\xi - r_1)) \\ &\quad - d_2 g'(1)(1 - \varphi(\xi - r_2)) + d_3 g'(1)(1 - \varphi(\xi)) \\ &\quad - d_4 g'(1)(1 - \varphi(\xi + r_1)) - d_5 g'(1)(1 - \varphi(\xi + r_2)). \end{aligned} \tag{2.3}$$

Since $\varphi(\xi) \to 1$ as $\xi \to +\infty$, we substitute $\varphi(\xi) = 1 - e^{\lambda_2 \xi}$, $\lambda_2 < 0$ into the equation (2.3) and obtain the following characteristic equation

$$\begin{aligned} c\lambda_2 &= f'(1) + d_1 g'(1)e^{-r_1\lambda_2} + d_2 g'(1)e^{-r_2\lambda_2} - \\ &\quad d_3 g'(1) + d_4 g'(1)e^{r_1\lambda_2} + d_5 g'(1)e^{r_2\lambda_2}. \end{aligned} \tag{2.4}$$

By the same argument as lemma 2.1, we obtain the following lemma.

Lemma 2.2. *For given wave speed $c > 0$, if $d_1 g'(1) > 0$ or $d_2 g'(1) > 0$ then there exists $\lambda_2 < 0$ such that the characteristic equation* (2.4) *has a negative root.*

Proof. Let

$$\begin{aligned} L(\lambda) &= f'(1) + d_1 g'(1) e^{-r_1\lambda} + d_2 g'(1) e^{-r_2\lambda} - \\ &\quad d_3 g'(1) + d_4 g'(1) e^{r_1\lambda} + d_5 g'(1) e^{r_2\lambda} \end{aligned}$$

be the real-value function then for $d_1 g'(1) > 0$ or $d_2 g'(1) > 0$, we have

$$L(0) = f'(1) < 0 \text{ and } \lim_{\lambda \to -\infty} L(\lambda) = \infty \text{ exponentially.}$$

This implies $L(\lambda)$ and the equation $c\lambda$ hand over a point $\lambda < 0$. Thus, we can find a $\lambda_2 < 0$ such that the characteristic equation (2.4) has a negative root. □

2.2 Approximate Solutions outside the Finite Interval

We first consider the condition $\xi < -I$. Since $\varphi(\xi) \to 0$ and $f(\varphi(\xi)) \to 0$ as $\xi \to -\infty$, near $\varphi = 0$, we use the following Taylor expansions

$$f(\varphi) = a_1\varphi + a_2\varphi^2 + a_3\varphi^3 + O(\varphi^4),$$
$$g(\varphi) = b_0 + b_1\varphi + b_2\varphi^2 + b_3\varphi^3 + O(\varphi^4)$$

respectively. Then the coupling profile equation (1.2) becomes

$$\begin{aligned} c\dot{\varphi}(\xi) &= a_1\varphi(\xi) + a_2\varphi(\xi)^2 + a_3\varphi(\xi)^3 + \\ &\quad d_1[b_1\varphi(\xi - r_1) + b_2\varphi(\xi - r_1)^2 + b_3\varphi(\xi - r_1)^3] + \\ &\quad d_2[b_1\varphi(\xi - r_2) + b_2\varphi(\xi - r_2)^2 + b_3\varphi(\xi - r_2)^3] - \\ &\quad d_3[b_1\varphi(\xi) + b_2\varphi(\xi)^2 + b_3\varphi(\xi)^3] + \\ &\quad d_4[b_1\varphi(\xi + r_1) + b_2\varphi(\xi + r_1)^2 + b_3\varphi(\xi + r_1)^3] + \\ &\quad d_5[b_1\varphi(\xi + r_2) + b_2\varphi(\xi + r_2)^2 + b_3\varphi(\xi + r_2)^3]. \end{aligned} \tag{2.5}$$

Since $\varphi(-I) \to 0$ as $I \to +\infty$, we consider the formal expansion

$$\varphi(\xi) = \varepsilon u_1(\xi) + \varepsilon^2 u_2(\xi) + \varepsilon^3 u_3(\xi) + O(\varepsilon^4),\ \varepsilon = \varphi(-I) \tag{2.6}$$

which satisfies the following initial conditions

$$u_1(-I) = 1 \text{ and } u_2(-I) = u_3(-I) = 0.$$

By the formal expansion (2.6), we can rewrite (2.5) and obtain the following equations

$$c\dot{u}_1(\xi) - Su_1(\xi) = 0, \tag{2.7}$$

$$\begin{aligned} c\dot{u}_2(\xi) - Su_2(\xi) &= a_2u_1(\xi)^2 + b_2[d_1u_1(\xi - r_1)^2 + d_2u_1(\xi - r_2)^2 - \\ &\quad d_3u_1(\xi)^2 + d_4u_1(\xi + r_1)^2 + d_5u_1(\xi + r_2)^2], \end{aligned} \tag{2.8}$$

$$\begin{aligned} c\dot{u}_3(\xi) - Su_3(\xi) &= a_3u_1(\xi)^3 + b_3[d_1u_1(\xi - r_1)^3 + d_2u_1(\xi - r_2)^3 - \\ &\quad d_3u_1(\xi)^3 + d_4u_1(\xi + r_1)^3 + d_5u_1(\xi + r_2)^3] + \\ &\quad 2\{a_2u_1(\xi)u_2(\xi) + b_2[d_1u_1(\xi - r_1)u_2(\xi - r_1) + \\ &\quad d_2u_1(\xi - r_2)u_2(\xi - r_2) - d_3u_1(\xi)u_2(\xi) + \\ &\quad d_4u_1(\xi + r_1)u_2(\xi + r_1) + d_5u_1(\xi + r_2)u_2(\xi + r_2)]\}, \end{aligned} \tag{2.9}$$

where the operator S is defined by

$$\begin{aligned} Su(\xi) &= a_1u(\xi) + d_1b_1u(\xi - r_1) + d_2b_1u(\xi - r_2) - \\ &\quad d_3b_1u(\xi) + d_4b_1u(\xi + r_1) + d_5b_1u(\xi + r_2). \end{aligned}$$

Since the equation (2.7) is linear and homogeneous, a solution to (2.7) which satisfies $u_1(-I) = 1$ is

$$u_1(\xi) = e^{\lambda_1(\xi+I)},\ \lambda_1 > 0.$$

A particular solution of the nonhomogeneous equation (2.8) is $u_2^p(\xi) = k_1e^{2\lambda_1(\xi+I)}$ where

$$k_1 = \frac{a_2 + b_2\Gamma_1}{2c\lambda_1 - a_1 - b_1\Gamma_1} \quad \text{with}$$

$$\Gamma_1 = d_1e^{-2r_1\lambda_1} + d_2e^{-2r_2\lambda_1} - d_3 + d_4e^{2r_1\lambda_1} + d_5e^{2r_2\lambda_1}.$$

Since a solution of (2.8) satisfies $u_2(-I) = 0$, we have

$$u_2(\xi) = k_1(e^{2\lambda_1(\xi+I)} - e^{\lambda_1(\xi+I)}).$$

We suppose that $u_3^p(\xi) = k_2e^{2\lambda_1(\xi+I)} + k_3e^{3\lambda_1(\xi+I)}$ is a particular solution of the nonhomogeneous equation (2.9) where

$$k_2 = \frac{-2k_1(a_2 + b_2\Gamma_1)}{2c\lambda_1 - a_1 - b_1\Gamma_1} = -2k_1^2,$$

and

$$k_3 = \frac{(2a_2k_1 + a_3) + (2b_2k_1 + b_3)\Gamma_2}{3c\lambda_1 - a_1 - b_1\Gamma_2} \quad \text{with}$$

$$\Gamma_2 = d_1e^{-3r_1\lambda_1} + d_2e^{-3r_2\lambda_1} - d_3 + d_4e^{3r_1\lambda_1} + d_5e^{3r_2\lambda_1}.$$

Since a solution of (2.9) satisfies $u_3(-I) = 0$, we obtain

$$u_3(\xi) = k_2e^{2\lambda_1(\xi+I)} + k_3e^{3\lambda_1(\xi+I)} - (k_2 + k_3)e^{\lambda_1(\xi+I)}.$$

Thus, if $\xi < -I$ then with $\varepsilon = \varphi(-I)$,

$$\begin{aligned} \varphi(\xi) \;\approx\; & \varepsilon e^{\lambda_1(\xi+I)} + \varepsilon^2 k_1(e^{2\lambda_1(\xi+I)} - e^{\lambda_1(\xi+I)}) + \\ & \varepsilon^3[k_2e^{2\lambda_1(\xi+I)} + k_3e^{3\lambda_1(\xi+I)} - (k_2 + k_3)e^{\lambda_1(\xi+I)}]. \end{aligned} \quad (2.10)$$

Similarly, if $\xi > I$ with $\varphi(\xi) \to 1$, $f(\varphi(\xi)) \to 0$ as $\xi \to +\infty$ then near $\varphi = 1$, we consider the following Taylor expansions

$$f(\varphi) = A_1(1-\varphi) + A_2(1-\varphi)^2 + A_3(1-\varphi)^3 + O((1-\varphi)^4),$$
$$g(\varphi) = B_0 + B_1(1-\varphi) + B_2(1-\varphi)^2 + B_3(1-\varphi)^3 + O((1-\varphi)^4)$$

respectively. Then the coupling profile equation (1.2) becomes

$$\begin{aligned} c\dot\varphi(\xi) \;=\; & A_1(1-\varphi(\xi)) + A_2(1-\varphi(\xi))^2 + A_3(1-\varphi(\xi))^3 + \\ & d_1[B_1(1-\varphi(\xi-r_1)) + B_2(1-\varphi(\xi-r_1))^2 + B_3(1-\varphi(\xi-r_1))^3] + \\ & d_2[B_1(1-\varphi(\xi-r_2)) + B_2(1-\varphi(\xi-r_2))^2 + B_3(1-\varphi(\xi-r_2))^3] - \\ & d_3[B_1(1-\varphi(\xi)) + B_2(1-\varphi(\xi))^2 + B_3(1-\varphi(\xi))^3] + \\ & d_4[B_1(1-\varphi(\xi+r_1)) + B_2(1-\varphi(\xi+r_1))^2 + B_3(1-\varphi(\xi+r_1))^3] + \\ & d_5[B_1(1-\varphi(\xi+r_2)) + B_2(1-\varphi(\xi+r_2))^2 + B_3(1-\varphi(\xi+r_2))^3]. \end{aligned} \quad (2.11)$$

Since $\varphi(I) \to 1$ as $I \to +\infty$, we consider the formal expansion

$$\varphi(\xi) = 1 - \delta v_1(\xi) - \delta^2 v_2(\xi) - \delta^3 v_3(\xi) + O(\delta^4),\ \delta = 1 - \varphi(I) \quad (2.12)$$

which satisfy the following initial conditions

$$v_1(I) = 1 \text{ and } v_2(I) = v_3(I) = 0.$$

By the formal expansion (2.12), we can rewrite (2.11) and obtain the following equations

$$c\dot v_1(\xi) - Tv_1(\xi) = 0, \quad (2.13)$$

$$
\begin{aligned}
c\dot{v}_2(\xi) - Tv_2(\xi) &= -A_2 v_1(\xi)^2 - B_2[d_1 v_1(\xi - r_1)^2 + d_2 v_1(\xi - r_2)^2 - \\
&\quad d_3 v_1(\xi)^2 + d_4 v_1(\xi + r_1)^2 + d_5 v_1(\xi + r_2)^2], \qquad (2.14)
\end{aligned}
$$

$$
\begin{aligned}
c\dot{v}_3(\xi) - Tv_3(\xi) &= -A_3 v_1(\xi)^3 - B_3[d_1 v_1(\xi - r_1)^3 + d_2 v_1(\xi - r_2)^3 - \\
&\quad d_3 v_1(\xi)^3 + d_4 v_1(\xi + r_1)^3 + d_5 v_1(\xi + r_2)^3] - \\
&\quad 2\{A_2 v_1(\xi) v_2(\xi) + B_2[d_1 v_1(\xi - r_1) v_2(\xi - r_1) + \\
&\quad d_2 v_1(\xi - r_2) v_2(\xi - r_2) - d_3 v_1(\xi) v_2(\xi) + \qquad (2.15) \\
&\quad d_4 v_1(\xi + r_1) v_2(\xi + r_1) + d_5 v_1(\xi + r_2) v_2(\xi + r_2)]\},
\end{aligned}
$$

where the operator T is defined by

$$
\begin{aligned}
Tv(\xi) &= -A_1 v(\xi) - d_1 B_1 v(\xi - r_1) - d_2 B_1 v(\xi - r_2) + \\
&\quad d_3 B_1 v(\xi) - d_4 B_1 v(\xi + r_1) - d_5 B_1 v(\xi + r_2).
\end{aligned}
$$

Since the equation (2.13) is linear and homogeneous, a solution to (2.13) which satisfies $v_1(I) = 1$ is

$$v_1(\xi) = e^{\lambda_2(\xi - I)},\ \lambda_2 < 0.$$

A particular solution of the nonhomogeneous (2.14) is $v_2^p(\xi) = m_1 e^{2\lambda_2(\xi - I)}$ where

$$m_1 = \frac{-A_2 - B_2\eta_1}{2c\lambda_2 + A_1 + B_1\eta_1} \quad \text{with}$$

$$\eta_1 = d_1 e^{-2r_1\lambda_2} + d_2 e^{-2r_2\lambda_2} - d_3 + d_4 e^{2r_1\lambda_2} + d_5 e^{2r_2\lambda_2}.$$

Since a solution of (2.14) satisfies $v_2(I) = 0$, we have

$$v_2(\xi) = m_1(e^{2\lambda_2(\xi - I)} - e^{\lambda_2(\xi - I)}).$$

We suppose that $v_3^p(\xi) = m_2 e^{2\lambda_2(\xi - I)} + m_3 e^{3\lambda_2(\xi - I)}$ is a particular solution of the nonhomogeneous equation (2.15) where

$$m_2 = \frac{2m_1(A_2 + B_2\eta_1)}{2c\lambda_2 + A_1 + B_1\eta_1} = -2m_1^2,$$

and

$$m_3 = \frac{-(2A_2 m_1 + A_3) - (2B_2 m_1 + B_3)\eta_2}{3c\lambda_2 + A_1 + B_1\eta_2} \quad \text{with}$$

$$\eta_2 = d_1 e^{-3r_1\lambda_2} + d_2 e^{-3r_2\lambda_2} - d_3 + d_4 e^{3r_1\lambda_2} + d_5 e^{3r_2\lambda_2}.$$

Since a solution of (2.15) satisfies $v_3(I) = 0$, we obtain

$$v_3(\xi) = m_2 e^{2\lambda_2(\xi-I)} + m_3 e^{3\lambda_2(\xi-I)} - (m_2 + m_3) e^{\lambda_2(\xi-I)}.$$

Thus, if $\xi > I$ then with $\delta = 1 - \varphi(I)$,

$$\begin{aligned}\varphi(\xi) \approx\ & 1 - \delta e^{\lambda_2(\xi-I)} - \delta^2 m_1 (e^{2\lambda_2(\xi-I)} - e^{\lambda_2(\xi-I)}) - \\ & \delta^3 [m_2 e^{2\lambda_2(\xi-I)} + m_3 e^{3\lambda_2(\xi-I)} - (m_2 + m_3) e^{\lambda_2(\xi-I)}]. \quad (2.16)\end{aligned}$$

2.3 Finite Difference Method

To approximate traveling wave solutions inside $[-I, I]$, we first discretize the finite interval $[-I, I]$ and use a fixed, uniform step size $h = 2I/N$. Then we define the following notations

$$\xi_j = -I + jh,\ \varphi(\xi_j) := \varphi_j \text{ for } j = 0, 1, ..., N. \quad (2.17)$$

By the finite difference method for five-points formula, we can approximate the first derivative term $\dot{\varphi}(\xi)$ of (1.2) is then

$$\dot{\varphi}(\xi) = \frac{1}{12h}[\varphi(\xi - 2h) - 8\varphi(\xi - h) + 8\varphi(\xi + h) - \varphi(\xi + 2h)] + O(h^4).$$

Thus, by the notations of (2.17), we have

$$\dot{\varphi}(\xi) \approx \frac{1}{12h}(\varphi_{j-2} - 8\varphi_{j-1} + 8\varphi_{j+1} - \varphi_{j+2}). \quad (2.18)$$

As $j = 0$ and $j = 1$, terms φ_{-2} and φ_{-1} appear on the right-hand side of (2.18). Then we use the formula (2.10) for $\xi = -I - 2h$ and $\xi = -I - h$ to approximate these terms respectively. Similarly, if $j = N-1$ and $j = N$ then φ_{N+1} and φ_{N+2} can be approximated by the formula (2.16) for $\xi = I + h$ and $\xi = I + 2h$ respectively.

For the delayed and the advanced terms in (1.2), we take advantage of the cubic interpolation to approximate them. Let $M_1 = [\frac{r_1}{h}]$ and $M_2 = [\frac{r_2}{h}]$ with $s_1 = r_1 - M_1 h \geq 0$, $s_2 = r_2 - M_2 h \geq 0$ where "$[x]$" means the integer part of x. Then $\xi - r_1$ lies between $\xi - M_1 h - h$ and $\xi - M_1 h$. Similarly, $\xi - r_2$ lies between $\xi - M_2 h - h$ and $\xi - M_2 h$. By the cubic formula

$$\begin{aligned}\varphi(\xi - r_1) =\ & x_4 \varphi(\xi - M_1 h - 2h) + x_3 \varphi(\xi - M_1 h - h) + \\ & x_2 \varphi(\xi - M_1 h) + x_1 \varphi(\xi - M_1 h + h) + O(h^4)\end{aligned}$$

where the x_i's are the following constants

$$x_1 = \frac{-(2h - s_1)(h - s_1)s_1}{6h^3}$$
$$x_2 = \frac{(2h - s_1)(h - s_1)(h + s_1)}{2h^3}$$
$$x_3 = \frac{(2h - s_1)(h + s_1)s_1}{2h^3}$$
$$x_4 = \frac{-(h + s_1)(h - s_1)s_1}{6h^3}$$

and

$$\begin{aligned}\varphi(\xi - r_2) &= y_4\varphi(\xi - M_2h - 2h) + y_3\varphi(\xi - M_2h - h) + \\ &\quad y_2\varphi(\xi - M_2h) + y_1\varphi(\xi - M_2h + h) + O(h^4)\end{aligned}$$

where the y_i's are the following constants

$$y_1 = \frac{-(2h - s_2)(h - s_2)s_2}{6h^3}$$
$$y_2 = \frac{(2h - s_2)(h - s_2)(h + s_2)}{2h^3}$$
$$y_3 = \frac{(2h - s_2)(h + s_2)s_2}{2h^3}$$
$$y_4 = \frac{-(h + s_2)(h - s_2)s_2}{6h^3}$$

Thus, the delayed terms can be approximated by

$$\varphi(\xi - r_1) \approx x_4\varphi_{j-M_1-2} + x_3\varphi_{j-M_1-1} + x_2\varphi_{j-M_1} + x_1\varphi_{j-M_1+1} \tag{2.19}$$

and

$$\varphi(\xi - r_2) \approx y_4\varphi_{j-M_2-2} + y_3\varphi_{j-M_2-1} + y_2\varphi_{j-M_2} + y_1\varphi_{j-M_2+1}. \tag{2.20}$$

Similarly, we can approximate the advanced terms by

$$\varphi(\xi + r_1) \approx x_1\varphi_{j+M_1-1} + x_2\varphi_{j+M_1} + x_3\varphi_{j+M_1+1} + x_4\varphi_{j+M_1+2} \tag{2.21}$$

and

$$\varphi(\xi + r_2) \approx y_1\varphi_{j+M_2-1} + y_2\varphi_{j+M_2} + y_3\varphi_{j+M_2+1} + y_4\varphi_{j+M_2+2}. \tag{2.22}$$

By using (2.18) to (2.22), we can discretize the coupling profile equation (1.2) and obtain the following nonlinear discretized system

$$\begin{aligned}
&\frac{c}{12h}(\varphi_{j-2} - 8\varphi_{j-1} + 8\varphi_{j+1} - \varphi_{j+2}) - f(\varphi_j) + d_3 g(\varphi_j) \\
&-d_1 g(x_4\varphi_{j-M_1-2} + x_3\varphi_{j-M_1-1} + x_2\varphi_{j-M_1} + x_1\varphi_{j-M_1+1}) \\
&-d_2 g(y_4\varphi_{j-M_2-2} + y_3\varphi_{j-M_2-1} + y_2\varphi_{j-M_2} + y_1\varphi_{j-M_2+1}) \\
&-d_4 g(x_1\varphi_{j+M_1-1} + x_2\varphi_{j+M_1} + x_3\varphi_{j+M_1+1} + x_4\varphi_{j+M_1+2}) \\
&-d_5 g(y_1\varphi_{j+M_2-1} + y_2\varphi_{j+M_2} + y_3\varphi_{j+M_2+1} + y_4\varphi_{j+M_2+2}) = 0 \\
&\text{for } j = 0, 1, ..., N. \qquad (2.23)
\end{aligned}$$

In the system (2.23), the φ_j's for $j < 0$ are functions of φ_0 then we can use (2.10) with $\varepsilon = \varphi_0$ to approximate them. As the φ_j's for $j > N$ relate to φ_N, φ_j are evaluated by (2.16) with $\delta = 1 - \varphi_N$. However, there are $(N+1)$ equations and $(N+2)$ unknowns c, φ_0, φ_1, ..., φ_N in the discretized system (2.23). This causes the system (2.23) to have no the unique solution. Hence, we add in $\varphi_{N/2} - 0.5 = 0$ as the $(N+2)^{th}$ equation of (2.23) since the condition $\varphi(0) = 0.5$ is autonomous, time translates of solutions are also solutions.

There are many possible ways of solving the system (2.23). Here we design a nested iteration scheme in which c is fixed during the inner loop. The outer loop is the solution of an equation $h(c) = 0$ by the secant method where the function $h(c)$ is defined as follows.

Definition 2.3. *For fixed wave speed $c > 0$, we solve the nonlinear system of $(N+1)$ equations*

$$F = \begin{cases} (2.23) \text{ for } j = 0, 1, ..., (\frac{N}{2} - 1) \\ \varphi_{N/2} - 0.5 \\ (2.23) \text{ for } j = (\frac{N}{2} + 1), ..., N. \end{cases} \qquad (2.24)$$

Then we define $h(c)$ = left-hand side of (2.23) for $j = N/2$.

To solve the nonlinear system (2.24), we use Newton's iteration during the inner loop. Given the initial guess $\Phi^{(0)}$, a sequence of iteration $\Phi^{(k)} = \{(\varphi_0^{(k)}, \varphi_1^{(k)}, ..., \varphi_n^{(k)}), k = 0, 1, ...\}$ is generated. The sequence $\Phi^{(k)}$ converges to the solution if $\Phi^{(0)}$ is sufficiently close to the solution and the Jacobian matrix J which is approximated by centered-difference method is nonsingular at the solution. However, the weakness in Newton's iteration arises from the need to compute and invert the Jacobian matrix J at each step. Thus we perform a two-step operation to avoid explicit computation of J^{-1}. First, we solve the linear system $J(\Phi^{(k-1)})y = -F(\Phi^{(k-1)})$ for a vector y at each stage of iteration. Then the new approximation $\Phi^{(k)}$ is obtained by adding y to $\Phi^{(k-1)}$.

2.4 Continuation Method

How to get a good initial guess for the Newton's iteration? At first, we can solve the nonlinear function f_e with the exact solution, $\varphi(\xi) = (1+\tanh(\xi))/2$ is given. Let $k := \tanh(\xi) := 2\varphi(\xi) - 1$ then $\dot{\varphi}(\xi) = (1 - \tanh^2(\xi))/2 = (1-k^2)/2$ and the coupling profile equation (1.2) becomes

$$\begin{aligned} c(\frac{1-k^2}{2}) &= f_e(\varphi) + d_1 g(\frac{(1-P_1)\varphi}{1-P_1 k}) + d_2 g(\frac{(1-P_2)\varphi}{1-P_2 k}) - \\ & \quad d_3 g(\varphi) + d_4 g(\frac{(1+P_1)\varphi}{1+P_1 k}) + d_5 g(\frac{(1+P_2)\varphi}{1+P_2 k}). \end{aligned}$$

Hence, we obtain the exactly form of f is then

$$\begin{aligned} f_e(\varphi) &= c(\frac{1-k^2}{2}) - d_1 g(\frac{(1-P_1)\varphi}{1-P_1 k}) - d_2 g(\frac{(1-P_2)\varphi}{1-P_2 k}) + \\ & \quad d_3 g(\varphi) - d_4 g(\frac{(1+P_1)\varphi}{1+P_1 k}) - d_5 g(\frac{(1+P_2)\varphi}{1+P_2 k}) \end{aligned} \quad (2.25)$$

where $P_1 = \tanh(r_1)$ and $P_2 = \tanh(r_2)$. It's difficult for Newton's iteration to ensure the initial guess which is close to the solution. Thus, we take advantage of the continuation method to approach the solution we want. The way of the continuation method is as follows.

At first, we consider the nonlinear reaction function f of (1.2) which represents our "target function" and add the coupling profile equation (1.2) in the following one-parameter family of equations

$$\begin{aligned} c\dot{\varphi}(\xi) &= f_\alpha(\varphi(\xi)) + d_1 g(\varphi(\xi - r_1)) + d_2 g(\varphi(\xi - r_2)) - \\ & \quad d_3 g(\varphi(\xi)) + d_4 g(\varphi(\xi + r_1)) + d_5 g(\varphi(\xi + r_2)), \end{aligned} \quad (2.26)$$

with asymptotically boundary conditions, $\varphi(-\infty) = 0$ and $\varphi(+\infty) = 1$ where

$$f_\alpha(\varphi) = \alpha f(\varphi) + (1-\alpha) f_e(\varphi), \quad 0 \le \alpha \le 1,$$

and $f_e(\varphi)$ is given by (2.25).

Then we compute approximate solutions from $\alpha = 0$ to $\alpha = 1$. As $\alpha = 0$, we have $f_\alpha(\varphi) = f_e(\varphi)$ then the exactly solution $\varphi(\xi) = (1 + \tanh(\xi))/2$ is used as the initial guess to solve the correspond discrete system (2.23). Once a equation for $\alpha = 0$ is solved, we increase α adaptively and use the solution to previous equation as the initial approximation to obtain the solution to the next equation. As $\alpha = 1$, we just have solved the equation for $f_\alpha(\varphi) = f(\varphi)$ which is our "target function". Then we are done.

3 Numerical Results

In this section, we will present the numerical results with two different function-coupling in diffusion term, the identity and the hypertangent function, in Section 3.1 and 3.2. The algorithm are implemented in MATLAB©[1] program with the fixed uniformly step size $h = 0.05$ and the error accuracy of Newton's iteration $\varepsilon = 10^{-4}$.

3.1 The coupling function g is Identity

In this section, we consider the monotone increasing coupling output function $g(\varphi) = \varphi$. Given the classical coupling coefficients $d_1 = 1.0$, $d_2 = 1.0$, $d_3 = 4.0$, $d_4 = 1.0$, and $d_5 = 1.0$. It is interested for us to investigate the convergence of numerical traveling wave solutions as θ less vary from $\theta = 0$ to $\theta = \pi/2$. Thus, the conditions we test are as follows.

At first, we frame $a = 0.05$ and choose some particular angles to observe the convergence of numerical solutions for continuation method. Then we obtain the following results.

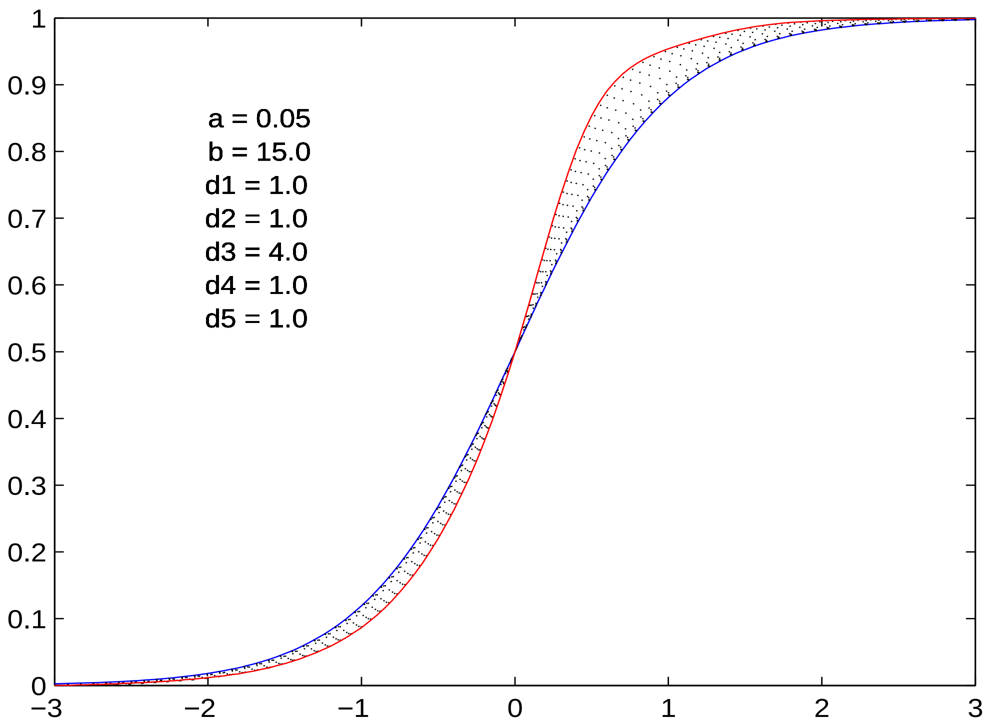


Figure 1. $\theta = 0$

[1]This is a high level general language software product of MathWork inc.

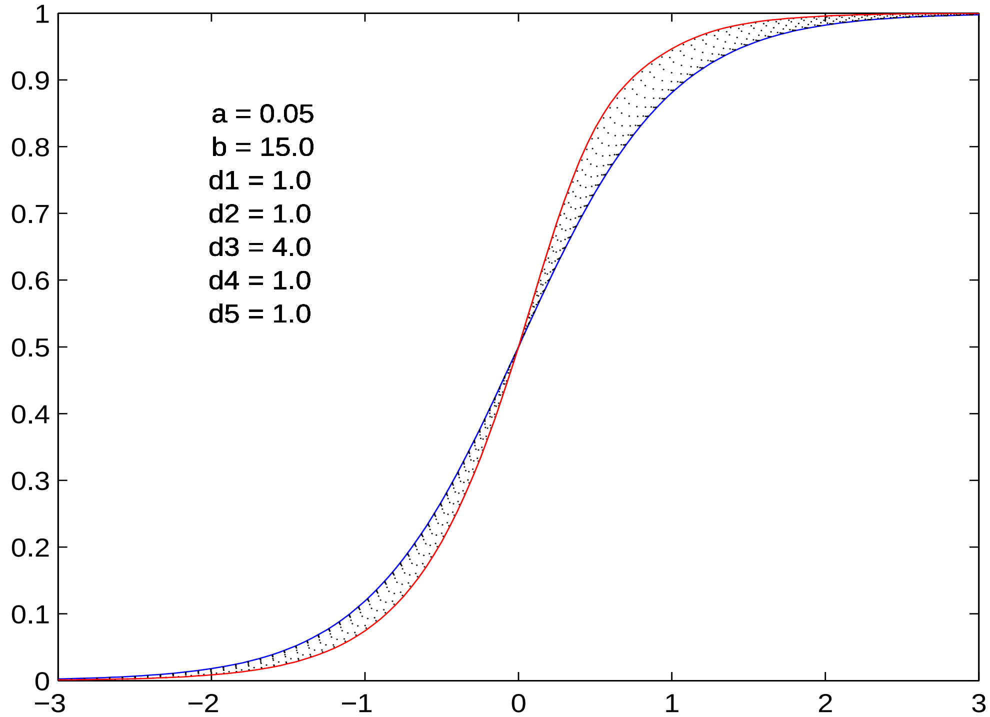


Figure 2. $\theta = \pi/4$

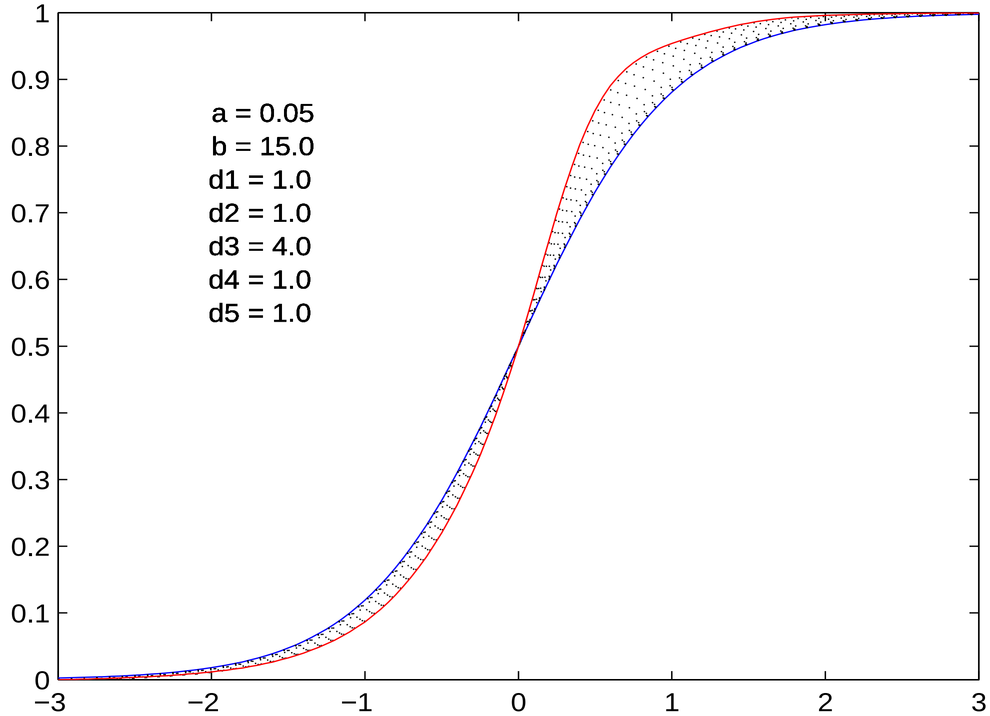


Figure 3. $\theta = \pi/2$

Table 1. $\theta = 0,\ \pi/4,\ \pi/2$

θ	Converges ?	α	c
0	Yes	1.00	2.289
$\pi/4$	Yes	1.00	2.368
$\pi/2$	Yes	1.00	2.289

As θ less increase from $\theta = 0$ to $\theta = \pi/2$, we can obtain the following $c - \theta$ polar coordinate figure.

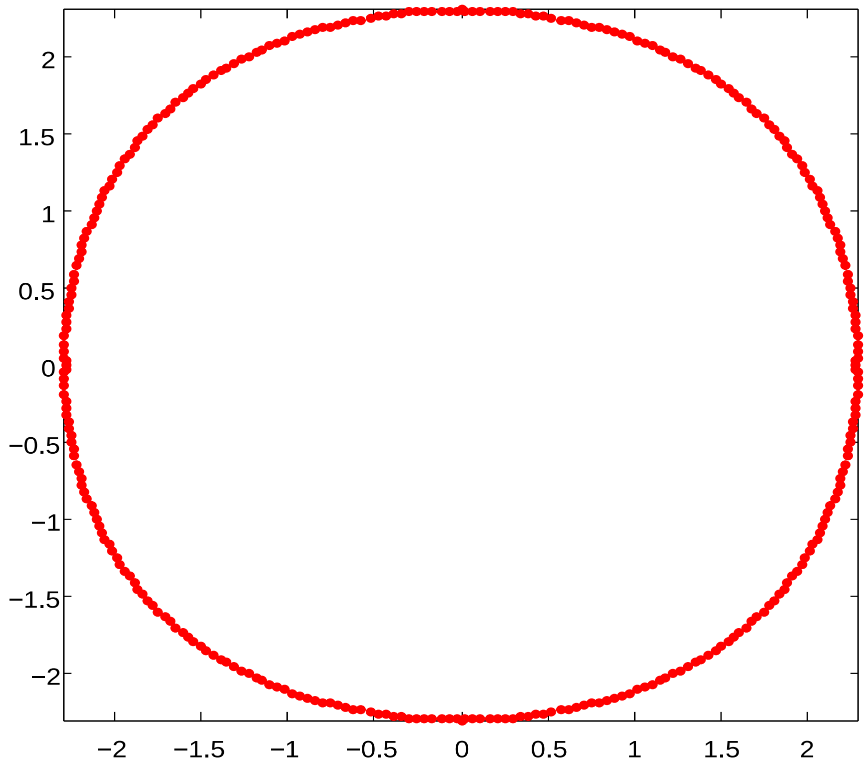


Figure 4. The polar gragh $c(\theta)$ for $a = 0.05$, $b = 15.0$

Consider $a = 0.1$ and choose the particular angles, $\theta = 0$, $\theta = \pi/4$, and $\theta = \pi/2$ to observe the convergence of numerical solutions for continuation method. Then we have the following results.

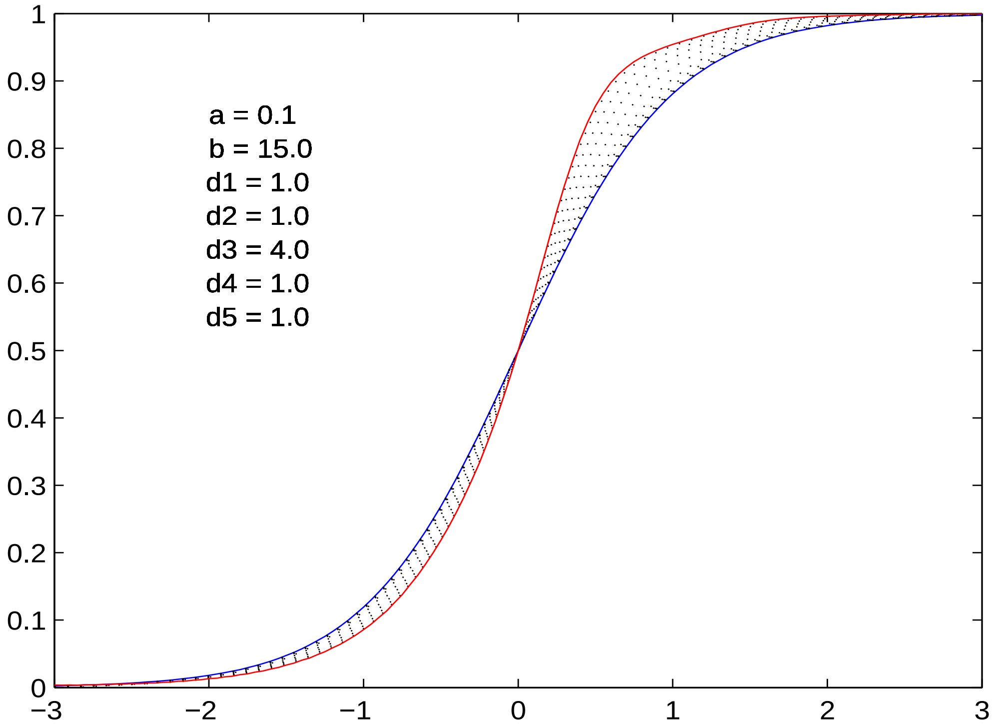


Figure 5. $\theta = 0$

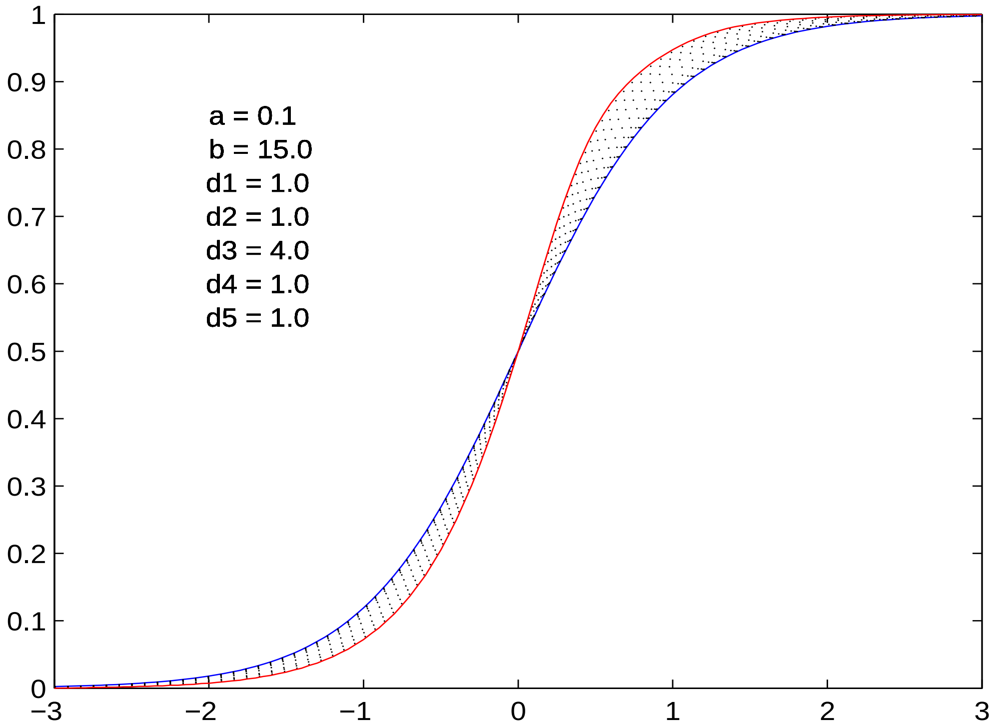


Figure 6. $\theta = \pi/4$

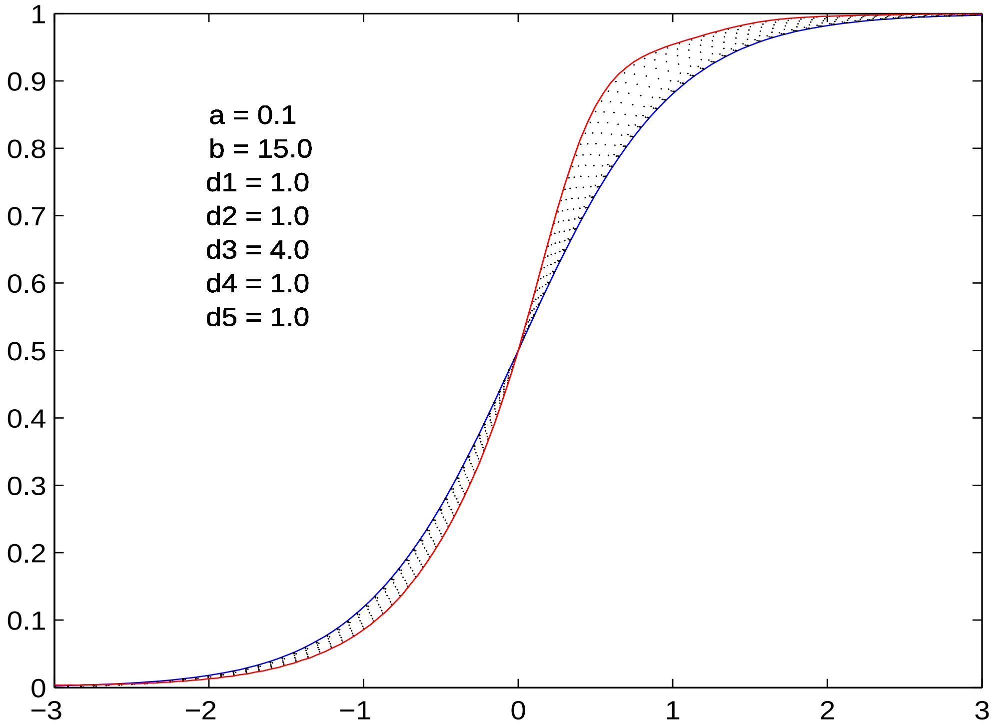


Figure 7. $\theta = \pi/2$

Table 2. $\theta = 0,\ \pi/4,\ \pi/2$

θ	Converges ?	α	c
0	Yes	1.00	1.982
$\pi/4$	Yes	1.00	2.071
$\pi/2$	Yes	1.00	1.982

As θ less increase from $\theta = 0$ to $\theta = \pi/2$, we can obtain the following $c - \theta$ polar coordinate figure.

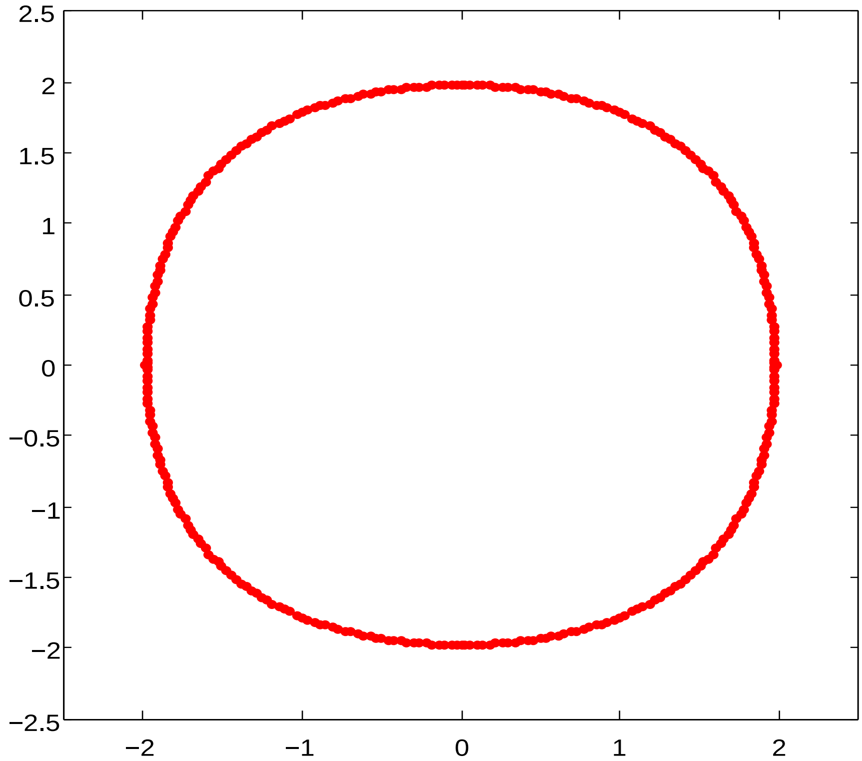


Figure 8. The polar gragh $c(\theta)$ for $a = 0.1$, $b = 15.0$

Similarly to previous analysis, we arrive at the following Figure 9 and Figure 10.

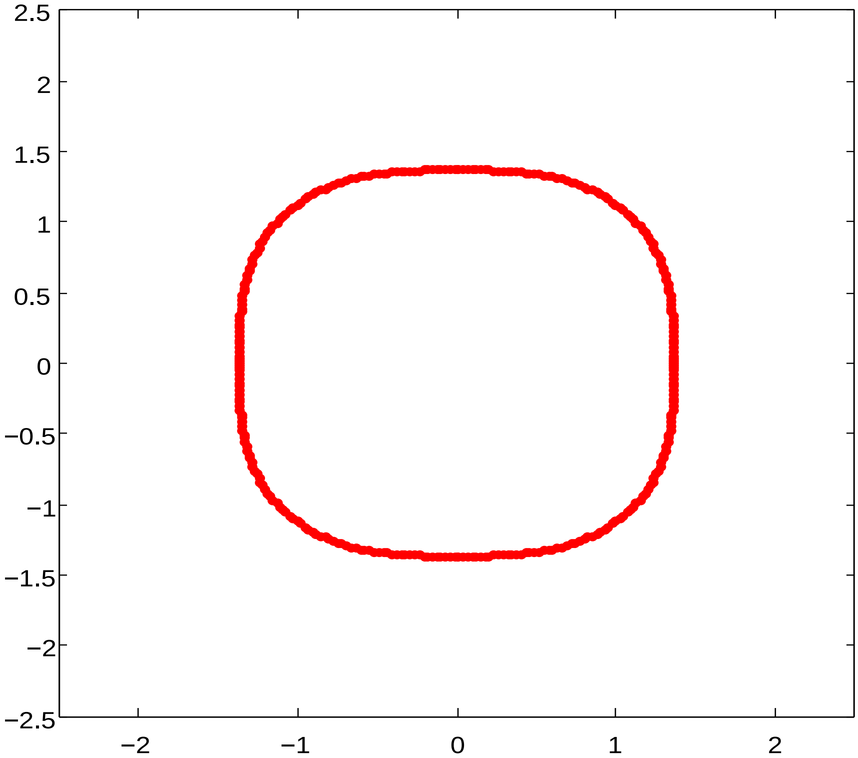
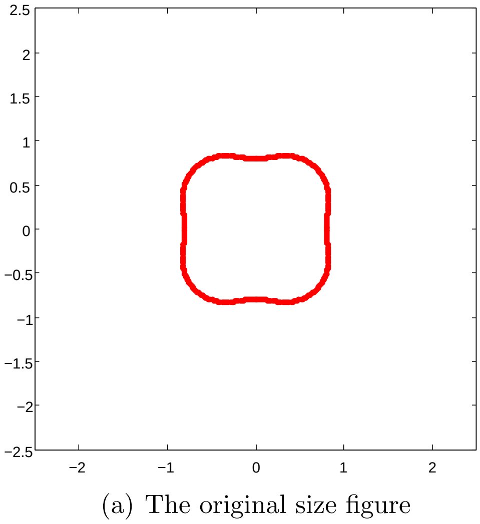
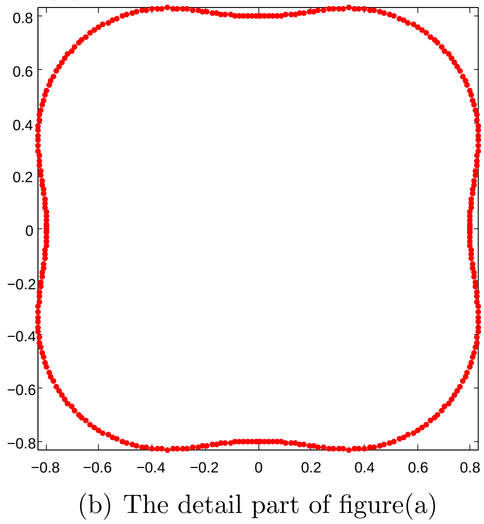


Figure 9. The polar gragh $c(\theta)$ for $a = 0.2$, $b = 15.0$

Figure 10. The polar gragh $c(\theta)$ for $a = 0.3$, $b = 15.0$

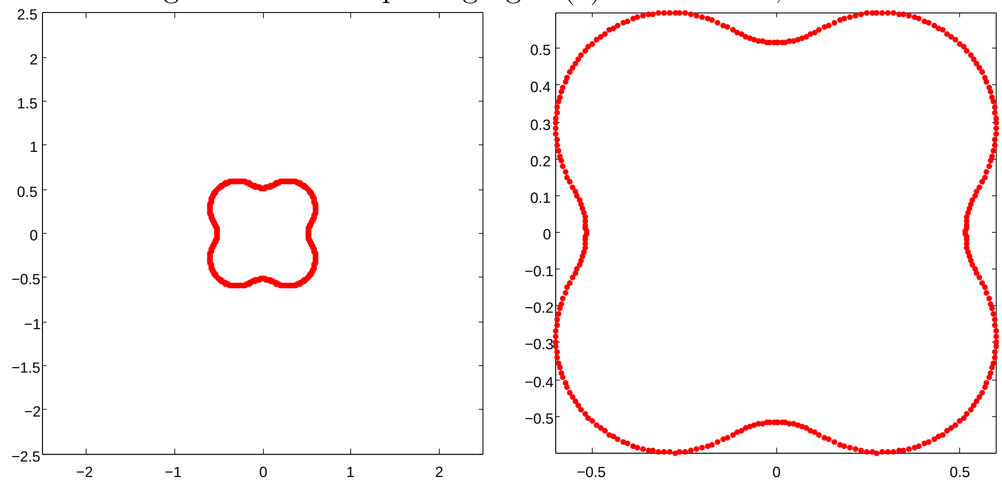


(a) The original size figure



(b) The detail part of figure(a)

Figure 11. The polar gragh $c(\theta)$ for $a = 0.35$, $b = 15.0$



(a) The original size figure (b) The detail part of figure(a)

3.2 The coupling function g is a Hyper-Tangent Map

Here we frame the coupling output function $g(\varphi) = (1 + \tanh(\varphi))/2$. Given the coupling coefficients $d_1 = 1.0$, $d_2 = 1.0$, $d_3 = 4.4$, $d_4 = 1.2$, and $d_5 = 1.2$. To investigate the convergence of numerical traveling wave solutions as θ less vary from $\theta = 0$ to $\theta = \pi/2$, we test the following conditions.

At first, we frame $a = 0.05$ and choose some particular angles to observe the convergence of numerical solutions for continuation method. Then the results we obtain are as follows.

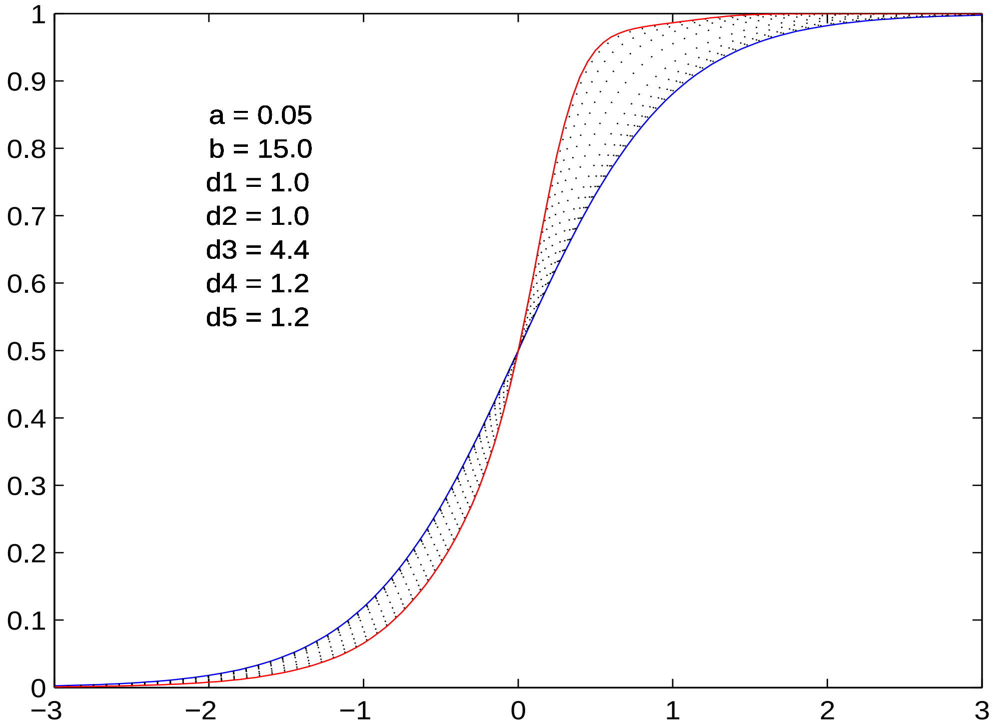


Figure 12. $\theta = 0$

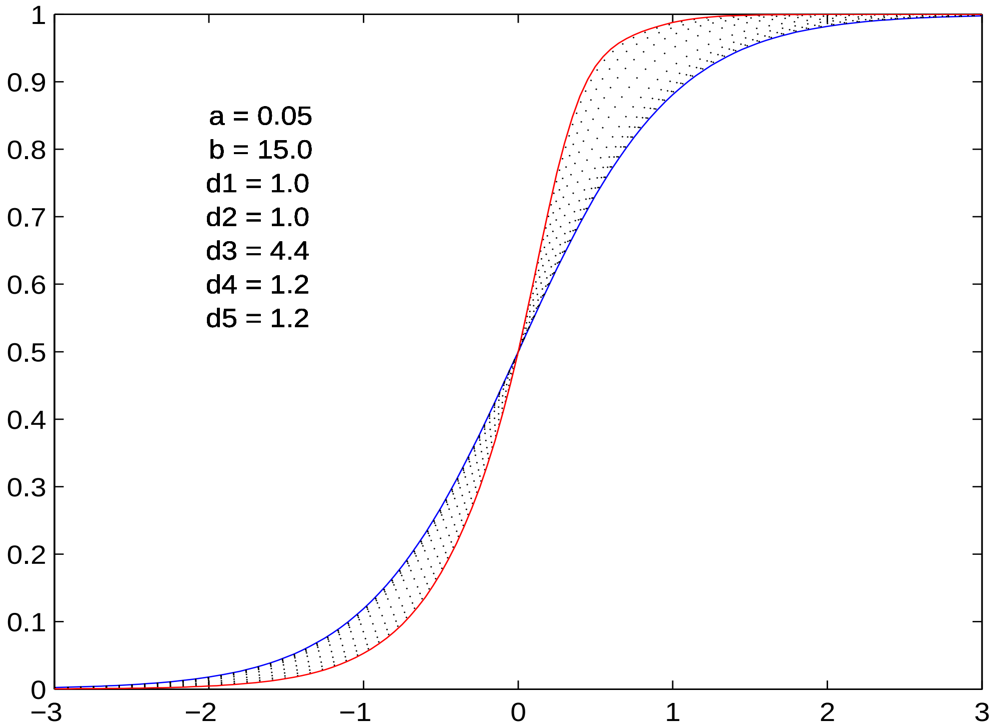


Figure 13. $\theta = \pi/4$

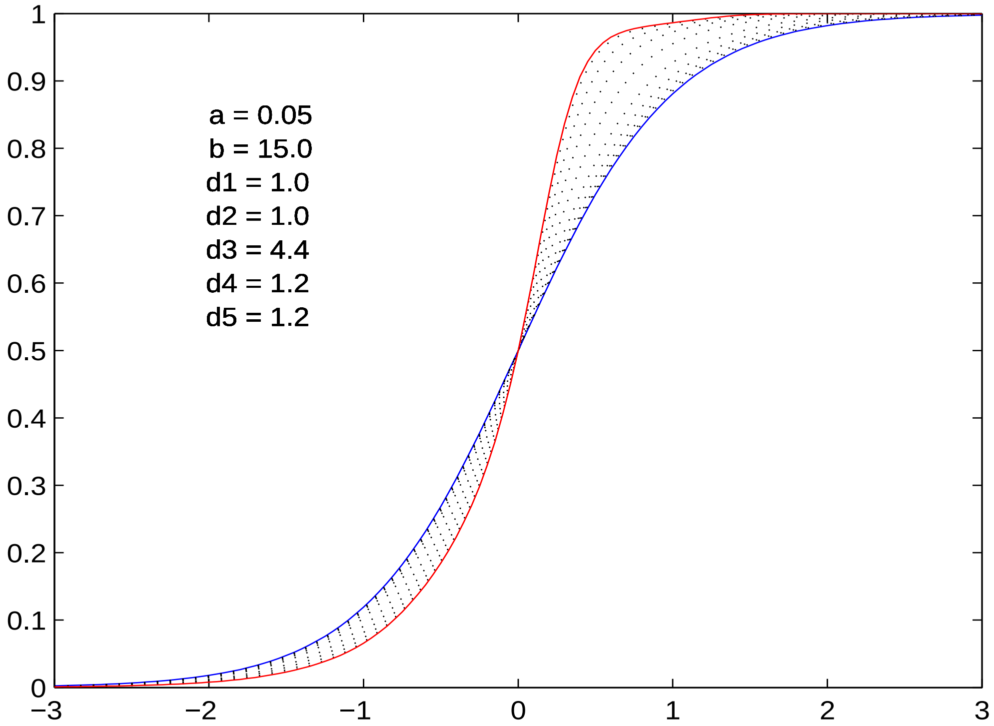


Figure 14. $\theta = \pi/2$

Table 3. $\theta = 0,\ \pi/4,\ \pi/2$

θ	Converges ?	α	c
0	Yes	1.00	1.580
$\pi/4$	Yes	1.00	1.680
$\pi/2$	Yes	1.00	1.580

As θ less increase from $\theta = 0$ to $\theta = \pi/2$, we can obtain the following $c - \theta$ polar coordinate figure.

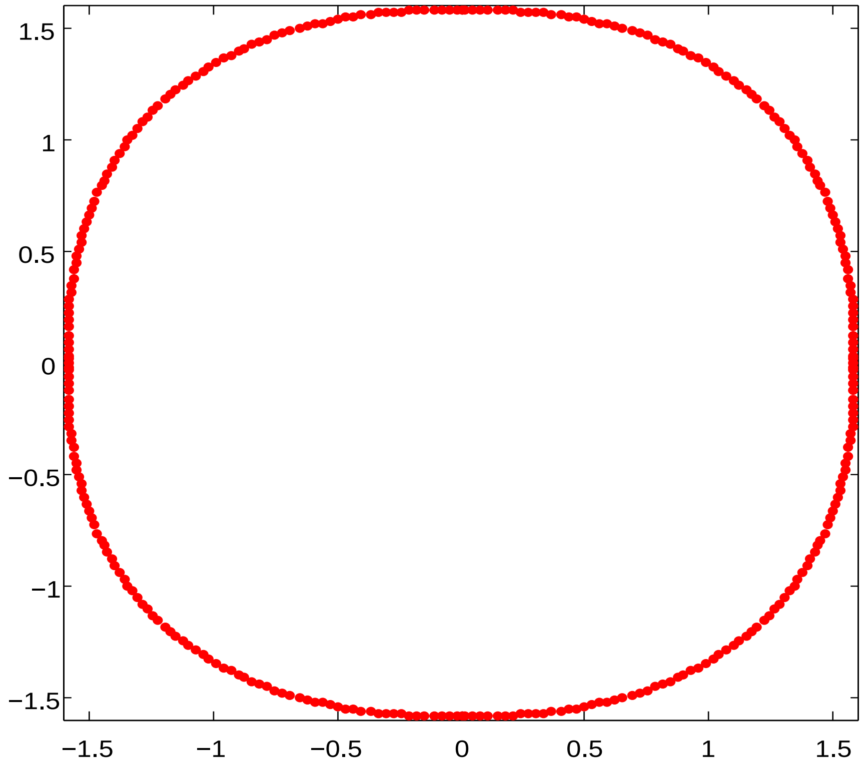


Figure 15. The polar gragh $c(\theta)$ for $a = 0.05$, $b = 15.0$

We set $a = 0.1$ and choose the particular angles, $\theta = 0$, $\theta = \pi/4$, and $\theta = \pi/2$ to observe the convergence of numerical solutions for continuation method. Then the results we obtain are as follows.

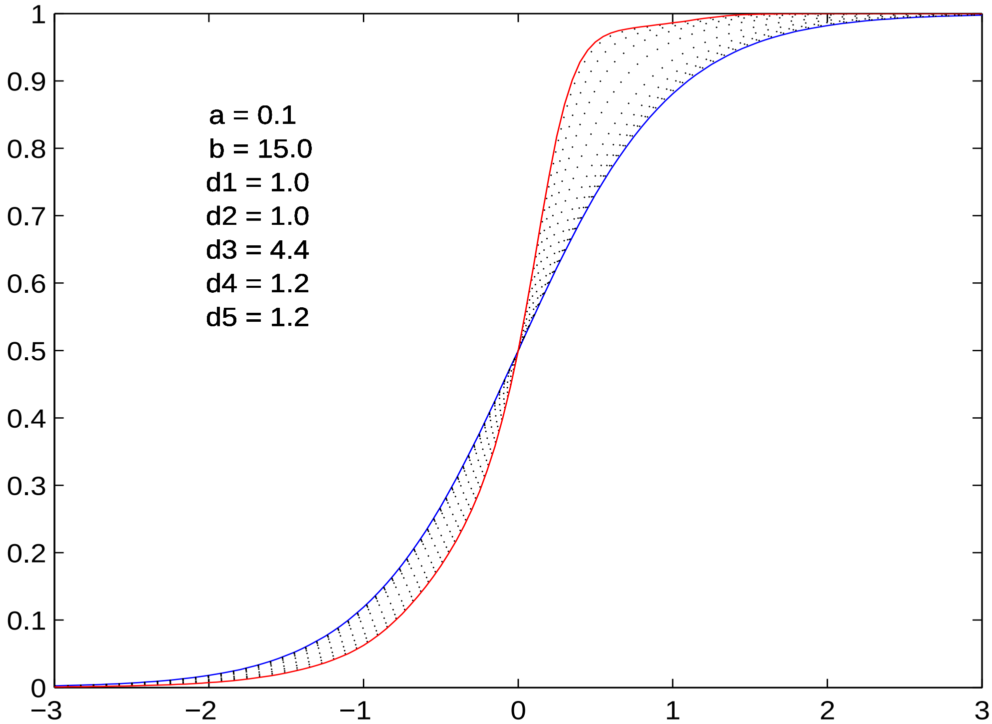


Figure 16. $\theta = 0$

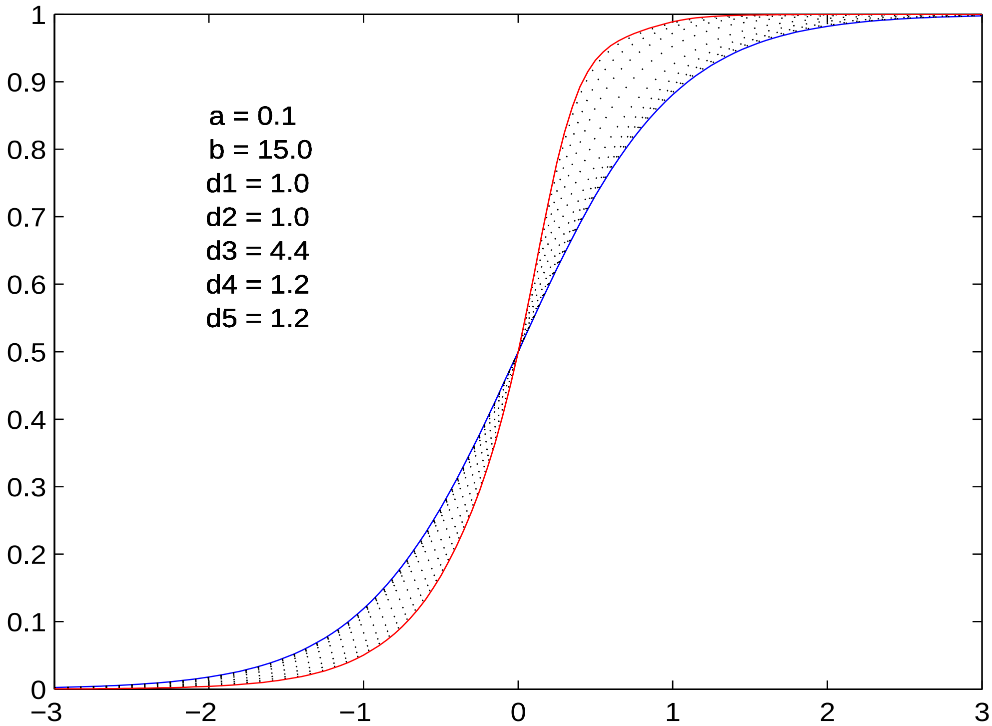


Figure 17. $\theta = \pi/4$

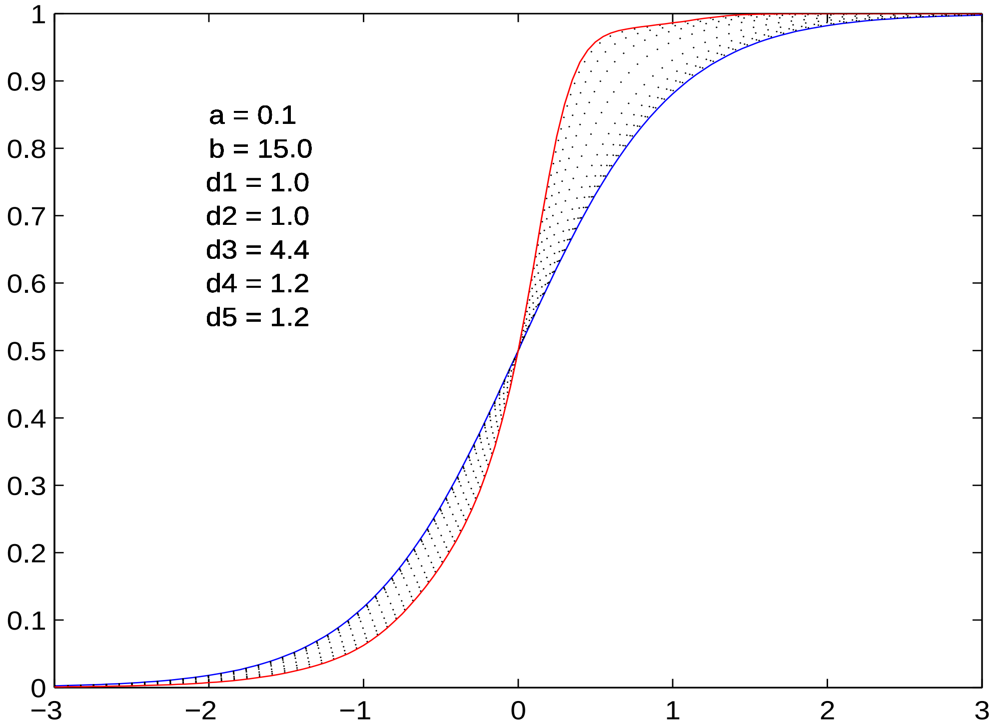


Figure 18. $\theta = \pi/2$

Table 4. $\theta = 0,\ \pi/4,\ \pi/2$

θ	Converges ?	α	c
0	Yes	1.00	1.288
$\pi/4$	Yes	1.00	1.423
$\pi/2$	Yes	1.00	1.288

As θ less increase from $\theta = 0$ to $\theta = \pi/2$, we can obtain the following $c - \theta$ polar coordinate figure.

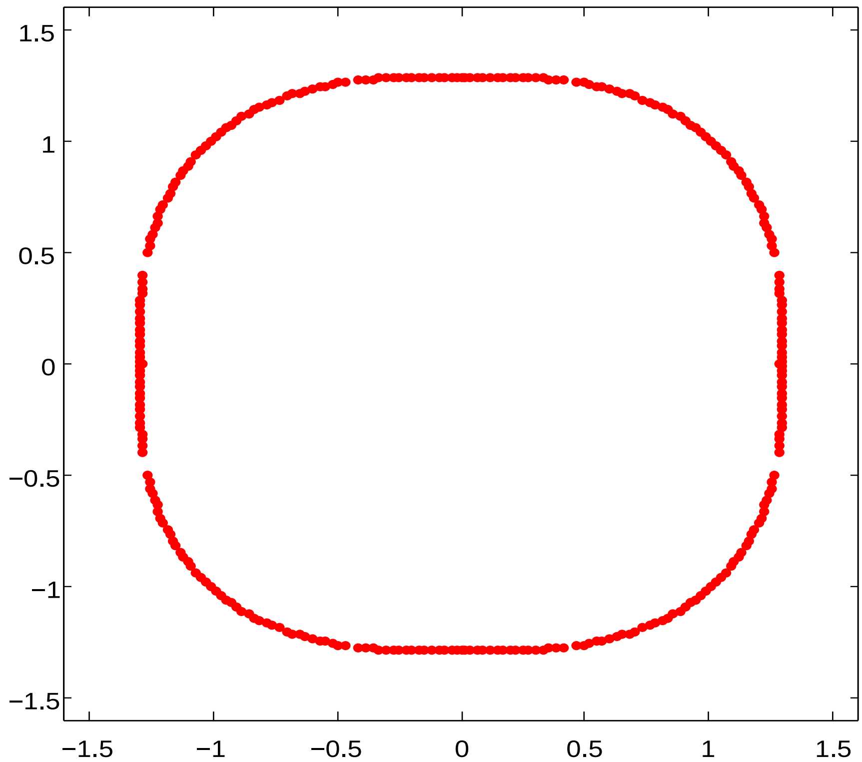


Figure 19. The polar gragh $c(\theta)$ for $a = 0.1$, $b = 15.0$

Table 5. Solution divergence for θ

θ	α	c	λ_1	λ_2
0.32	0.879	5.70	−0.060	NaN
0.34	0.832	5.33	−0.001	NaN
0.36	0.807	4.75	−0.002	NaN
1.24	0.876	6.06	−0.040	NaN

Appendixes

A The Algorithms

To indite our MATLAB program more conveniently, we use a pseudocode to describe the nested iteration scheme of continuation method containing the outer loop, secant method, for the wave speed c and Newton's iteration which is during the inner loop. Here we attempt to approach the following one-parameter family of equations

$$\begin{aligned} c\dot{\varphi}(\xi) &= f_\alpha(\varphi(\xi)) + d_1 g(\varphi(\xi - r_1)) + d_2 g(\varphi(\xi - r_2)) - \\ &\quad d_3 g(\varphi(\xi)) + d_4 g(\varphi(\xi + r_1)) + d_5 g(\varphi(\xi + r_2)), \end{aligned}$$

with $\varphi(-\infty) = 0$, $\varphi(+\infty) = 1$,

$$f_\alpha(\varphi) = \alpha f(\varphi) + (1-\alpha) f_e(\varphi), \quad 0 \le \alpha \le 1,$$

where $f_e(\varphi)$ is given by (2.25).

The nonlinear system

$$F = \begin{cases} \text{left-hand side of (2.23) for } j = 0, 1, ..., (\frac{N}{2} - 1) \\ \varphi_j - 0.5 \text{ for } j = N/2 \\ \text{left-hand side of (2.23) for } j = (\frac{N}{2} + 1), ..., N. \end{cases}$$

which is we are interesting to solve. For the continuation method, we use the following algorithms to solve the one-parameter family of equations and $F = 0$.

Algorithm 1: Continuation method.

Input: Initial guess $\Phi^{(0)}$; Error accuracy acy;
Maximum number of iterations N_1.

Output: Approximate solution Φ, wave speed c, or message of failure.

Set $\alpha = 0$ and $c_1 = 1.0$.

while $\alpha \leq 1$ **do**

- Set $c = c_1$.
- Use Newton's iteration to solve the system (2.24).
- Set $h_1 = h_1(c, \Phi)$.
- (Note: $h_i(c, \Phi)$ = left-hand side of (2.23) for $j = N/2$.)
- **if** $|h_1| < acy$ **then**
    - Set $c_1 = c$ and $\Phi^{(0)} = \Phi$.
- **else**
    - Set $c_2 = c_1 + 0.02$.
    - Set $c = c_2$.
    - Use Newton's iteration to solve the system (2.24).
    - Set $h_2 = h_2(c, \Phi)$.
    - **if** $|h_2| < acy$ **then**
        - Set $c_1 = c$ and $\Phi^{(0)} = \Phi$.
    - **else**
        - Set $i = 1$. (Note: Solve the wave speed c by secant method.)
        - **while** $(i \leq N_1)$ **do**
            - Set $c_3 = c_2 - \frac{h_2(c_2 - c_1)}{h_2 - h_1}$.
            - Set $c = c_3$.
            - Use Newton's iteration to solve the system (2.24).
            - Set $h_3 = h_3(c, \Phi)$.
            - **if** $|h_3| < acy$ **then**
                - Set $c_1 = c$ and $\Phi^{(0)} = \Phi$.
                - STOP.
            - **else**
                - Set $i = i + 1$.
                - **if** $i > N_1$ **then**
                    - STOP. (Note: Procedure failure.)
                - **end**
                - Set $c_1 = c_2$ and $h_1 = h_2$.
                - Set $c_2 = c_3$ and $h_2 = h_3$.
            - **end**
        - **end**
    - **end**
- **end**

end

Algorithm 2: Newton's iteration.

```
Input: Initial guess λ_1^(0), λ_2^(0), and Φ^(0);
       Error accuracy eps;
       Maximum number of iterations N_2.
Output: Approximate solution Φ; characteristic roots λ_1 and λ_2;
        or failed message.
if c < 0 then
  | STOP. (Note: Procedure failure.)
else
  | Estimate the value of λ_1 by 1-D Newton's method.
  | if λ_1 isn't convergent or λ_1 ≤ 0 then
  |   | STOP. (Note: Procedure failure.)
  | else
  |   | Estimate the value of λ_2 by 1-D Newton's method.
  |   | if λ_2 isn't convergent or λ_2 ≥ 0 then
  |   |   | STOP. (Note: Procedure failure.)
  |   | else
  |   |   | Start high dimensional Newton's iteration.
  |   |   | Set index = 1.
  |   |   | while (index ≤ N_2) do
  |   |   |   |   Calculate F(Φ^(0)) and the Jacobian matrix J(Φ^(0))
  |   |   |   |   to solve the (N + 1) × (N + 1) linear system
  |   |   |   |   J(Φ^(0))Y = −F(Φ^(0)).
  |   |   |   |   Set Φ^(0) = Φ^(0) + Y.
  |   |   |   | if ‖Y‖ < eps then
  |   |   |   |   | Set Φ = Φ^(0).
  |   |   |   |   | OUTPUT (Φ).
  |   |   |   |   | STOP.
  |   |   |   | else
  |   |   |   |   | Set index = index + 1.
  |   |   |   |   | if index > N_2 then
  |   |   |   |   |   | STOP. (Note: Procedure failure.)
  |   |   |   |   | end
  |   |   |   | end
  |   |   | end
  |   | end
  | end
end
```

B Data of Polar Figures

B−1 Identity Map to Coupling g

Table A. $a = 0.05$, $b = 15.0$

θ	converge ?	α	c	λ_1	λ_2
0.000	Yes	1.00	1.371	2.26	−3.81
0.020	Yes	1.00	1.391	2.28	−3.80
0.080	Yes	1.00	1.381	2.27	−3.81
0.140	Yes	1.00	1.388	2.29	−3.83
0.220	Yes	1.00	1.394	2.32	−3.86
0.260	Yes	1.00	1.396	2.33	−3.89
0.280	Yes	1.00	1.400	2.35	−3.90
0.320	Yes	1.00	1.403	2.37	−3.93
0.360	Yes	1.00	1.408	2.39	−3.97
0.400	Yes	1.00	1.412	2.42	−4.00
0.440	Yes	1.00	1.421	2.45	−4.04
0.480	Yes	1.00	1.427	2.48	−4.08
0.500	Yes	1.00	1.430	2.49	−4.10
0.540	Yes	1.00	1.436	2.52	−4.14
0.580	Yes	1.00	1.439	2.55	−4.19
0.640	Yes	1.00	1.443	2.58	−4.24
0.660	Yes	1.00	1.449	2.60	−4.25
0.700	Yes	1.00	1.451	2.61	−4.28
0.780	Yes	1.00	1.453	2.63	−4.30
0.840	Yes	1.00	1.452	2.62	−4.29
0.900	Yes	1.00	1.450	2.60	−4.26
0.960	Yes	1.00	1.444	2.57	−4.21
1.000	Yes	1.00	1.439	2.54	−4.18
1.040	Yes	1.00	1.432	2.51	−4.14
1.080	Yes	1.00	1.427	2.48	−4.09
1.160	Yes	1.00	1.424	2.43	−4.01
1.180	Yes	1.00	1.415	2.41	−3.99
1.220	Yes	1.00	1.407	2.39	−3.96
1.280	Yes	1.00	1.401	2.35	−3.91
1.360	Yes	1.00	1.394	2.31	−3.86
1.440	Yes	1.00	1.388	2.29	−3.82
1.520	Yes	1.00	1.382	2.27	−3.81
1.571	Yes	1.00	1.382	2.27	−3.80

B−2 Hyper-Tangent Map to Coupling g

Table E. $a = 0.05$, $b = 15.0$

θ	converge ?	α	c	λ_1	λ_2
0.000	Yes	1.00	1.371	2.26	−3.81
0.020	Yes	1.00	1.391	2.28	−3.80
0.080	Yes	1.00	1.381	2.27	−3.81
0.140	Yes	1.00	1.388	2.29	−3.83
0.220	Yes	1.00	1.394	2.32	−3.86
0.260	Yes	1.00	1.396	2.33	−3.89
0.280	Yes	1.00	1.400	2.35	−3.90
0.320	Yes	1.00	1.403	2.37	−3.93
0.360	Yes	1.00	1.408	2.39	−3.97
0.400	Yes	1.00	1.412	2.42	−4.00
0.440	Yes	1.00	1.421	2.45	−4.04
0.480	Yes	1.00	1.427	2.48	−4.08
0.500	Yes	1.00	1.430	2.49	−4.10
0.540	Yes	1.00	1.436	2.52	−4.14
0.580	Yes	1.00	1.439	2.55	−4.19
0.640	Yes	1.00	1.443	2.58	−4.24
0.660	Yes	1.00	1.449	2.60	−4.25
0.700	Yes	1.00	1.451	2.61	−4.28
0.780	Yes	1.00	1.453	2.63	−4.30
0.840	Yes	1.00	1.452	2.62	−4.29
0.900	Yes	1.00	1.450	2.60	−4.26
0.960	Yes	1.00	1.444	2.57	−4.21
1.000	Yes	1.00	1.439	2.54	−4.18
1.040	Yes	1.00	1.432	2.51	−4.14
1.080	Yes	1.00	1.427	2.48	−4.09
1.160	Yes	1.00	1.424	2.43	−4.01
1.180	Yes	1.00	1.415	2.41	−3.99
1.220	Yes	1.00	1.407	2.39	−3.96
1.280	Yes	1.00	1.401	2.35	−3.91
1.360	Yes	1.00	1.394	2.31	−3.86
1.440	Yes	1.00	1.388	2.29	−3.82
1.520	Yes	1.00	1.382	2.27	−3.81
1.571	Yes	1.00	1.382	2.27	−3.80

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