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不變量環之研究

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一、中文摘要

我們討論了不變量體的有理性問題。證明了如果 G 是一個不可交換的 p-群, 包含了一個指數爲 p的循環子群, 而 K爲包含了p^{n-2}次原根的體, 則 $K(G)$ 爲K的有理擴張。

關鍵字**:** 諾德問題、有理性問題,p-群作用

二、英文摘要

In this project, we studied rationality of some fields of invariants. We proved that if G is a non-abelian p-group of order p^n containing a cyclic subgroup of index p and K is any field containing a primitive p^{n-2}-th root of unity, then $K(G)$ is rational over K. In particular, if G is a non-abelian p-group of order p^3 and K is a field containing a p-th root of unity, then $K(G)$ is rational.

Key words: Noether's problem, rationality problem, p-group actions.

三、緣由與目的

Let K be any field and G be a finite group. Let G act on the rational function field $K(x_g : g \in G)$ by K-automorphisms such that $g \cdot x_h = x_{gh}$ for any $g,\ h \in G$. Denote by $K(G)$ the fixed field $K(x_g : g \in G)^G$. Noether's problem asks whether $K(G)$ is rational (=purely transcendental) over K. Noether's problem for abelian groups was studied by Swan, Voskresenskii, Endo, Miyataand Lenstra, etc. See the survey article [Sw] for more details. Hence we will restrict our attention to the non-ableian case in this article.

四、結果與討論

Our main result is the following.

Theorem 1. *Let G be a non-abelian p-group of order p^n such that G contains a cyclic subgroup of index p. Assume that K is any field such that either* (i) *char $K = p > 0$ or* (ii) *char $K \neq p$ and $[K(\zeta) : K] = 1$ or p where ζ is a primitive p^{n-1}-th root of unity. Then $K(G)$ is rational over K.*

As a corollary of a theorem by Chu and Kang Chu and Kang [CK, Theorem 1.6] and Theorem 1, we have

Theorem 2. *Let G be a non-ableian p-group of order p^3. Assume that K is any field such that either* (i) *char $K = p > 0$ or* (ii) *char $K \neq p$ and K contains a primitive p-th root of unity. Then $K(G)$ is rational over K.*

All the p-groups containing cyclic subgroups of index p are classified by the following theorem.

Theorem 3. ([Su, p.107]) *Let G be a non-ableian p-group of order p^n containing a cyclic subgroup of index p.*

(i) *If p is an odd prime number, then G is isomorphic to $M(p^n)$; and*

(2) *If $p = 2$, then G is isomorphic to $M(2^n)$, $D(2^{n-1})$, $SD(2^{n-1})$, $Q(2^n)$ where $n \geq 3$ and*

$$M(p^n) =< \sigma, \tau : \ \sigma^{p^{n-1}} = \tau^p = 1, \ \tau^{-1}\sigma\tau = \sigma^{1+p^{n-2}} >$$

is the modular group,

$$D(2^{n-1}) =< \sigma, \tau : \ \sigma^{2^{n-1}} = \tau^2 = 1, \ \tau^{-1}\sigma\tau = \sigma^{-1} >$$

is the dihedral group,

$$SD(2^{n-1}) =< \sigma, \tau : \ \sigma^{2^{n-1}} = \tau^2 = 1, \ \tau^{-1}\sigma\tau = \sigma^{-1+2^{n-2}} >$$

is the quasi-dihedral group where $n \geq 4$ and

$$Q(2^n) =< \sigma, \tau : \ \sigma^{2^{n-1}} = \tau^4 = 1, \ \sigma^{2^{n-2}} = \tau^2, \ \tau^{-1}\sigma\tau = \sigma^{-1} >$$

is the generalized quaternion group

Thus we will concentrate on the rationality of $K(G)$ for $G = M(p^n),\ D(2^{n-1}),\ SD(2^{n-1}),\ Q(2^n)$ with the assumption that $[K(\zeta) : K] = 1$ or p where G is a group of exponent p^e and ζ is a primitive p^e-th root of unity. If $\zeta \in K$, then Theorem 1 follows from [Ka2, Theorem 1.5]. Hence we will assume that $[K(\zeta) : K] = p$. If p is an odd prime number, the condition $[K(\zeta) : K]$ implies that K contains a primitive p^{e-1}-th root of unity. If $p = 2$, the condition $[K(\zeta) : K] = 2$ implies that $\lambda(\zeta) = -\zeta,\ \pm\zeta^{-1}$ where λ is a generator of the Galois group of $K(\zeta)$ over K. (The case $\lambda(\zeta) = -\zeta$ is equivalent to that primitive 2^{e-1}-th root of unity belongs to K.) If K contains a primitive p^{e-1}-th root of unity, we will construct a faithful representation $G \longrightarrow GL(V)$ such that $\dim V = p^2$ and $K(V)$ is rational over K. For the remaining cases i.e. $p = 2$, we will add the root ζ to the ground field K and show that $K(G) = K(\zeta)(G)^{<\lambda>}$ is rational over K. In the case $p = 2$ we will construct various faithful representations according to the group $G = M(2^n),\ D(2^{n-1}),\ SD(2^{n-1}),\ Q(2^n)$ and the possible image $\lambda(\zeta)$ because it seems that a straightforward imitation of the case when K contains a primitive p^{e-1}-th root of unity will not lead to a complete solution.

The case when char $K = p > 0$ was taken care by the following theorem due to Kuniyoshi. Hence we will assume that char $K \neq p$.

Theorem 4. (Kuniyoshi [CK, Theorem 1.7]) *If char $K = p > 0$ and G is a finite p-group, then $K(G)$ is rational over K.*

We first prove that

Theorem 5. *Let p be any prime number, $G = M(p^n)$ the modular group of order p^n where $n \geq 3$ and K be any field containing a primitive p^{n-2}-th root of unity. Then $K(G)$ is rational over K.*

Using this theorem, we can prove Theorem 2.

The method used in the proof of Theorem 5 can be applied to other groups, e.g. $D(2^{n-1})$, $Q(2^n)$, $SD(2^{n-1})$. The following results will be used in the proof of Theorem 1.

Theorem 6. *Let $G = D(2^{n-1})$ or $Q(2^n)$ with $n \geq 4$. If K is a field containing a primitive 2^{n-2}-th root of unity, then $K(G)$ is rational over K.*

Theorem 7. *Let $G = SD(2^{n-1})$ with $n \geq 4$. If K is a field containing a primitive 2^{n-2}-th root of unity, then $K(G)$ is rational over K.*

This enables us to finish our proof of Theorem 1.

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