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ON TRIPLE SOLUTIONS OF FOCAL BOUNDARY VALUE PROBLEMS ON TIME SCALE

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Abstract. We consider the following differential equation on a time scale $\mathbb{T}$

$$y^{\Delta\Delta}(t) + P(t, y(\sigma(t))) = 0, \quad t \in [a, b] \cap \mathbb{T}$$

together with focal boundary conditions

$$y(a) = 0, \qquad y^{\Delta}(\sigma(b)) = 0$$

where $a, b \in \mathbb{T}$ and $a < \sigma(b)$. By using two different fixed point theorems, criteria are established for the existence of triple positive solutions of the boundary value problem. We also include some examples to illustrate the results obtained.

Keywords : Time scale, fixed point theorems, boundary value problems, focal boundary conditions, positive solutions.

AMS Subject Classification : 34B15, 34B18, 39A10

1. INTRODUCTION

In this paper we shall consider the focal boundary value problem on a time scale $\mathbb{T}$

$$\begin{aligned} y^{\Delta\Delta}(t) + P(t, y(\sigma(t))) = 0, \quad t \in [a, b] \\ y(a) = 0, \qquad y^{\Delta}(\sigma(b)) = 0 \end{aligned} \tag{F}$$

where $a, b \in \mathbb{T}$ with $a < \sigma(b)$, and $P : [a, \sigma(b)] \times \mathbb{R} \to \mathbb{R}$ is continuous.

To understand (F), we recall some standard definitions as follows. The reader may refer to [1] for an introduction to the subject.

(a) Let $\mathbb{T}$ be a time scale, i.e., $\mathbb{T}$ is a closed subset of $\mathbb{R}$. We assume that $\mathbb{T}$ has the topology that it inherits from the standard topology on $\mathbb{R}$. Throughout, for any $a, b\ (> a)$, the interval $[a, b]$ is defined as $[a, b] = \{t \in \mathbb{T} \mid a \le t \le b\}$. Analogous notations for open and half-open intervals will also be used in the paper.

(b) For $t < \sup \mathbb{T}$ and $s > \inf \mathbb{T}$, the *forward jump operator* σ and the *backward jump operator* ρ are respectively defined by

$$\sigma(t) = \inf\{\tau \in \mathbb{T} \mid \tau > t\} \in \mathbb{T} \qquad \text{and} \qquad \rho(s) = \sup\{\tau \in \mathbb{T} \mid \tau < s\} \in \mathbb{T}.$$

(c) Fix $t \in \mathbb{T}$. Let $y : \mathbb{T} \to \mathbb{R}$. We define $y^\Delta(t)$ to be the number (if it exists) with the property that given $\epsilon > 0$, there is a neighborhood U of t such that for all $s \in U$,

$$|[y(\sigma(t)) - y(s)] - y^\Delta(t)[\sigma(t) - s]| < \epsilon|\sigma(t) - s|.$$

We call $y^\Delta(t)$ the *delta derivative* of $y(t)$. Define $y^{\Delta\Delta}(t)$ to be the delta derivative of $y^\Delta(t)$, i.e., $y^{\Delta\Delta}(t) = (y^\Delta(t))^\Delta$.

(d) If $F^\Delta(t) = f(t)$, then we define the integral

$$\int_a^t f(\tau)\Delta\tau = F(t) - F(a).$$

A solution y of (F) will be sought in $C[a, \sigma^2(b)]$, the space of continuous functions $\{y : [a, \sigma^2(b)] \to \mathbb{R}\}$. We say that y is a *positive solution* if $y(t) \geq 0$ for $t \in [a, \sigma^2(b)]$.

Boundary value problems have attracted a lot of attention in the recent literature, due mainly to the fact that they model many physical phenomena which include gas diffusion through porous media, nonlinear diffusion generated by nonlinear sources, thermal self-ignition of a chemically active mixture of gases in a vessel, catalysis theory, chemically reacting systems, adiabatic tubular reactor processes, as well as concentration in chemical or biological problems, just to name a few. In all these problems, only *positive* solutions are meaningful. Many papers have discussed the existence of single, double and triple positive solutions of boundary value problems on the real and discrete domains, we refer to [2–9] and the monographs [10, 11] which give a good documentary of the literature. A recent trend is to consider boundary value problems on time scales, which include the real and the discrete as special cases, see [12–17].

In the present work, *both* fixed point theorems of Leggett and Williams [18] as well as of Avery [19] are used to derive criteria for the existence of *triple positive solutions* of (F). In addition, estimates on the norms of these solutions are also provided. Not only that *new* results are obtained, we also discuss the relationship between the results in terms of generality, and illustrate the importance of the results through some examples. Moreover, it is noted that the boundary value problem (F) considered has a nonlinear term P which is *more general* than those discussed in the literature.

The paper is outlined as follows. In Section 2 we state the necessary definitions and fixed point theorems. Our main results and discussion are presented in Section 3. Finally, three examples are included in Section 4 as illustrations.

2. PRELIMINARIES

In this section we shall state some necessary definitions and the relevant fixed point theorems. Let B be a Banach space equipped with the norm $\|\cdot\|$.

Definition 2.1. Let C $(\subset B)$ be a nonempty closed convex set. We say that C is a *cone* provided the following conditions are satisfied:

(a) If $u \in C$ and $\alpha \geq 0$, then $\alpha u \in C$;

(b) If $u \in C$ and $-u \in C$, then $u = 0$.

Definition 2.2. Let C $(\subset B)$ be a cone. A map ψ is a *nonnegative continuous concave functional* on C if the following conditions are satisfied:

(a) $\psi : C \to \mathbb{R}^+ \cup \{0\}$ is continuous;

(b) $\psi(ty + (1-t)z) \geq t\psi(y) + (1-t)\psi(z)$ for all $y, z \in C$ and $0 \leq t \leq 1$.

Definition 2.3. Let C $(\subset B)$ be a cone. A map β is a *nonnegative continuous convex functional* on C if the following conditions are satisfied:

(a) $\beta : C \to \mathbb{R}^+ \cup \{0\}$ is continuous;

(b) $\beta(ty + (1-t)z) \leq t\beta(y) + (1-t)\beta(z)$ for all $y, z \in C$ and $0 \leq t \leq 1$.

Let γ, β, Θ be nonnegative continuous convex functionals on C and α, ψ be nonnegative continuous concave functionals on C. For nonnegative numbers w_i, $1 \leq i \leq 3$, we shall introduce the following notations:

$$
\begin{aligned}
&C(w_1) = \{u \in C \mid \|u\| < w_1\},\\
&C(\psi, w_1, w_2) = \{u \in C \mid \psi(u) \geq w_1 \text{ and } \|u\| \leq w_2\},\\
&P(\gamma, w_1) = \{u \in C \mid \gamma(u) < w_1\},\\
&P(\gamma, \alpha, w_1, w_2) = \{u \in C \mid \alpha(u) \geq w_1 \text{ and } \gamma(u) \leq w_2\},\\
&Q(\gamma, \beta, w_1, w_2) = \{u \in C \mid \beta(u) \leq w_1 \text{ and } \gamma(u) \leq w_2\},\\
&P(\gamma, \Theta, \alpha, w_1, w_2, w_3) = \{u \in C \mid \alpha(u) \geq w_1,\ \Theta(u) \leq w_2 \text{ and } \gamma(u) \leq w_3\},\\
&Q(\gamma, \beta, \psi, w_1, w_2, w_3) = \{u \in C \mid \psi(u) \geq w_1,\ \beta(u) \leq w_2 \text{ and } \gamma(u) \leq w_3\}.
\end{aligned}
$$

The following fixed point theorems are needed later. The first is usually called *Leggett-Williams' fixed point theorem*, and the second is known as the *five-functional fixed point theorem.*

Theorem 2.1. [18] Let C $(\subset B)$ be a cone, and $w_4 > 0$ be given. Assume that ψ is a nonnegative continuous concave functional on C such that $\psi(u) \le \|u\|$ for all $u \in \overline{C}(w_4)$, and let $S : \overline{C}(w_4) \to \overline{C}(w_4)$ be a continuous and completely continuous operator. Suppose that there exist numbers w_1, w_2, w_3 where $0 < w_1 < w_2 < w_3 \le w_4$ such that

(a) $\{u \in C(\psi, w_2, w_3) \mid \psi(u) > w_2\} \ne \emptyset$, and $\psi(Su) > w_2$ for all $u \in C(\psi, w_2, w_3)$;

(b) $\|Su\| < w_1$ for all $u \in \overline{C}(w_1)$;

(c) $\psi(Su) > w_2$ for all $u \in C(\psi, w_2, w_4)$ with $\|Su\| > w_3$.

Then, S has (at least) three fixed points $u^1,\ u^2$ and u^3 in $\overline{C}(w_4)$. Furthermore, we have

$$\begin{aligned} &u^1 \in C(w_1),\ u^2 \in \{u \in C(\psi, w_2, w_4) \mid \psi(u) > w_2\} \\ \text{and}\quad &u^3 \in \overline{C}(w_4) \backslash \left(C(\psi, w_2, w_4) \cup \overline{C}(w_1)\right). \end{aligned} \tag{2.1}$$

Theorem 2.2. [19] Let C $(\subset B)$ be a cone. Assume that there exist positive numbers w_5, M, nonnegative continuous convex functionals γ, β, Θ on C, and nonnegative continuous concave functionals α, ψ on C, with

$$\alpha(u) \le \beta(u) \qquad \text{and} \qquad \|u\| \le M\gamma(u)$$

for all $u \in \overline{P}(\gamma, w_5)$. Let $S : \overline{P}(\gamma, w_5) \to \overline{P}(\gamma, w_5)$ be a continuous and completely continuous operator. Suppose that there exist nonnegative numbers $w_i,\ 1 \le i \le 4$ with $0 < w_2 < w_3$ such that

(a) $\{u \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5) \mid \alpha(u) > w_3\} \ne \emptyset$, and $\alpha(Su) > w_3$ for all $u \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5)$;

(b) $\{u \in Q(\gamma, \beta, \psi, w_1, w_2, w_5) \mid \beta(u) < w_2\} \ne \emptyset$, and $\beta(Su) < w_2$ for all $u \in Q(\gamma, \beta, \psi, w_1, w_2, w_5)$;

(c) $\alpha(Su) > w_3$ for all $u \in P(\gamma, \alpha, w_3, w_5)$ with $\Theta(Su) > w_4$;

(d) $\beta(Su) < w_2$ for all $u \in Q(\gamma, \beta, w_2, w_5)$ with $\psi(Su) < w_1$.

Then, S has (at least) three fixed points $u^1,\ u^2$ and u^3 in $\overline{P}(\gamma, w_5)$. Furthermore, we have

$$\beta(u^1) < w_2, \quad \alpha(u^2) > w_3, \quad \text{and} \quad \beta(u^3) > w_2 \quad \text{with} \quad \alpha(u^3) < w_3. \tag{2.2}$$

For clarity, we shall list some conditions with respect to the boundary value problem (F) that are needed later. Note that in these conditions, $y^\sigma = y \circ \sigma$, $B = C[a, \sigma^2(b)]$, and the sets $\tilde{K}$ and K are given by

$$\tilde{K} = \left\{ y \in B \;\middle|\; y(t) \ge 0 \text{ for } t \in [a, \sigma^2(b)] \right\}$$

and

$$K = \left\{ y \in \tilde{K} \;\middle|\; y(t) > 0 \text{ for some } t \in [a, \sigma^2(b)] \right\} = \tilde{K} \backslash \{0\}.$$

(C1) Assume that

$$P(t, y^\sigma) \geq 0, \quad y \in \tilde{K}, \ t \in [a, \sigma(b)] \qquad \text{and} \qquad P(t, y^\sigma) > 0, \quad y \in K, \ t \in [a, \sigma(b)].$$

(C2) There exist continuous functions f, μ, ν with $f : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ and $\mu, \nu : [a, \sigma(b)] \to \mathbb{R}^+ \cup \{0\}$ such that

$$\mu(t) f(y^\sigma) \leq P(t, y^\sigma) \leq \nu(t) f(y^\sigma), \quad y \in \tilde{K}, \ t \in [a, \sigma(b)].$$

(C3) There exists a number $0 < c \leq 1$ such that

$$\mu(t) \geq c\nu(t), \ t \in [a, \sigma(b)].$$

3. MAIN RESULTS

Let the Banach space

$$B = \left\{ y \ \middle| \ y \in C[a, \sigma^2(b)] \right\} \tag{3.1}$$

be equipped with norm

$$\|y\| = \sup_{t \in [a, \sigma^2(b)]} |y(t)|. \tag{3.2}$$

To apply the fixed point theorems in Section 2, we need to define an operator $S : B \to B$ so that a solution y of the boundary value problem (F) is a fixed point of S, i.e., $y = Sy$. For this, let $G(t, s)$ be the Green's function of the boundary value problem

$$\begin{aligned} -y^{\Delta\Delta}(t) = 0, \quad t \in [a, b] \\ y(a) = 0, \qquad y^\Delta(\sigma(b)) = 0. \end{aligned} \tag{3.3}$$

If y is a solution of (F), then it can be represented as

$$y(t) = \int_a^{\sigma(b)} G(t, s) P(s, y(\sigma(s))) \Delta s, \quad t \in [a, \sigma^2(b)].$$

Hence, we shall define the operator $S : B \to B$ by

$$Sy(t) = \int_a^{\sigma(b)} G(t, s) P(s, y(\sigma(s))) \Delta s, \quad t \in [a, \sigma^2(b)]. \tag{3.4}$$

It is clear that a fixed point of the operator S is a solution of (F).

Our first lemma gives the properties of the Green's function $G(t, s)$ which will be used later.

Lemma 3.1. [15] It is known that

(a) $G(t,s) = \begin{cases} t-a, & t \le s \\ \sigma(s)-a, & \sigma(s) \le t; \end{cases}$

(b) $0 \le G(t,s) \le G(\sigma(s),s), \quad (t,s) \in [a,\sigma^2(b)] \times [a,\sigma(b)];$

(c) for fixed m such that $a < m < \sigma^2(b)$, we have

$$G(t,s) \ge MG(\sigma(s),s), \quad (t,s) \in [m,\sigma^2(b)] \times [a,\sigma(b)]$$

where the constant $0 < M < 1$ is given by

$$M = \frac{m-a}{\sigma^2(b)-a}.$$

As an example, when $m = \frac{1}{4}[3a+\sigma(b)]$, we have $M = \frac{1}{4}\frac{\sigma(b)-a}{\sigma^2(b)-a}$.

Lemma 3.2. The operator S defined in (3.4) is continuous and completely continuous.

Proof. From Lemma 3.1(a), we have $G(t,s) \in C[a,\sigma^2(b)],\ t \in [a,\sigma^2(b)]$ and the map $t \to G(t,s)$ is continuous from $[a,\sigma^2(b)]$ to $C[a,\sigma^2(b)]$. This together with $P : [a,\sigma(b)] \times \mathbb{R} \to \mathbb{R}$ is continuous ensures (as in [12]) that S is continuous and completely continuous. □

Next, we define a cone in B as

$$C = \Big\{ y \in B \;\Big|\; y^{\Delta\Delta}(t) \le 0 \text{ for } t \in [a,b],\ y(t) \text{ is nondecreasing for } t \in [a,\sigma^2(b)], \\ \text{and } y(t) \ge 0 \text{ for } t \in [a,\sigma^2(b)]. \Big\} \tag{3.5}$$

Note that $C \subseteq \tilde{K}$. Moreover, a fixed point of S obtained in C will be a *positive solution* of the boundary value problem (F).

Lemma 3.3. Let (C1) hold. Then, the operator S maps C into itself.

Proof. Let $y \in C$. From (3.4), Lemma 3.1(b) and (C1) we have

$$Sy(t) = \int_a^{\sigma(b)} G(t,s)P(s,y(\sigma(s)))\Delta s \ge 0,\ t \in [a,\sigma^2(b)]. \tag{3.6}$$

Next, it is clear from (3.4) and (C1) that

$$(Sy)^{\Delta\Delta}(t) = -P(t,y(\sigma(t))) \le 0,\ t \in [a,b]. \tag{3.7}$$

It follows that $(Sy)^{\Delta}(t)$ is nonincreasing for $t \in [a,\sigma(b)]$. Hence, we have for $t \in [a,\sigma(b)]$,

$$(Sy)^{\Delta}(t) \ge (Sy)^{\Delta}(\sigma(b)) = \int_a^{\sigma(b)} G^{\Delta}(\sigma(b),s)P(s,y(\sigma(s)))\Delta s = 0.$$

This implies that $Sy(t)$ is nondecreasing for $t \in [a,\sigma^2(b)]$. Together with (3.6) and (3.7), we have shown that $Sy \in C$. □

Remark 3.1. If (C1) and (C2) hold, then it follows from (3.4) and Lemma 3.1(b) that for $y \in \tilde{K}$ and $t \in [a, \sigma^2(b)]$,

$$\int_a^{\sigma(b)} G(t,s)\mu(s) f(y(\sigma(s)))\Delta s \leq Sy(t) \leq \int_a^{\sigma(b)} G(t,s)\nu(s) f(y(\sigma(s)))\Delta s. \tag{3.8}$$

Moreover, using (3.6), (3.8) and Lemma 3.1(b), we obtain for $t \in [a, \sigma^2(b)]$,

$$|Sy(t)| = Sy(t) \leq \int_a^{\sigma(b)} G(t,s)\nu(s) f(y(\sigma(s)))\Delta s \leq \int_a^{\sigma(b)} G(\sigma(s),s)\nu(s) f(y(\sigma(s)))\Delta s.$$

Therefore, we have

$$\|Sy\| \leq \int_a^{\sigma(b)} G(\sigma(s),s)\nu(s) f(y(\sigma(s)))\Delta s. \tag{3.9}$$

Let $a < m < \sigma^2(b)$ be fixed such that

$$\eta = \min\{t \in \mathbb{T} \mid t \geq m\}$$

exists and satisfies $m \leq \eta < \sigma(b)$. Also, let $\tau_1, \tau_2, \tau_3, \tau_4 \in [a, \sigma^2(b)]$ be fixed with $\tau_3 > \tau_2$ and $\tau_4 > \tau_1$. For subsequent results, we define the following constants:

$$\begin{aligned}
q &= \int_a^{\sigma(b)} G(\sigma^2(b),s)\nu(s)\Delta s = \int_a^{\sigma(b)} [\sigma(s)-a]\nu(s)\Delta s,\\
r &= \int_\eta^{\sigma(b)} G(\eta,s)\mu(s)\Delta s = \int_\eta^{\sigma(b)} (\eta-a)\mu(s)\Delta s,\\
d_1 &= \int_{\tau_2}^{\rho(\tau_3)} G(\tau_2,s)\mu(s)\Delta s = \int_{\tau_2}^{\rho(\tau_3)} (\tau_2-a)\mu(s)\Delta s,\\
d_2 &= \int_{\tau_1}^{\sigma(b)} G(\sigma^2(b),s)\nu(s)\Delta s = \int_{\tau_1}^{\sigma(b)} [\sigma(s)-a]\nu(s)\Delta s,\\
d_3 &= \int_a^{\tau_1} G(\sigma^2(b),s)\nu(s)\Delta s = \int_a^{\tau_1} [\sigma(s)-a]\nu(s)\Delta s,\\
d_4 &= \int_{\tau_1}^{\rho(\tau_4)} G(\tau_4,s)\nu(s)\Delta s = \int_{\tau_1}^{\rho(\tau_4)} [\sigma(s)-a]\nu(s)\Delta s,\\
d_5 &= \int_a^{\tau_1} G(\tau_4,s)\nu(s)\Delta s + \int_{\rho(\tau_4)}^{\sigma(b)} G(\tau_4,s)\nu(s)\Delta s\\
&= \int_a^{\tau_1} [\sigma(s)-a]\nu(s)\Delta s + \int_{\rho(\tau_4)}^{\sigma(b)} (\tau_4-a)\nu(s)\Delta s.
\end{aligned} \tag{3.10}$$

Lemma 3.4. Let (C1) and (C2) hold, and assume

(C4) the function $[\sigma(s)-a]\nu(s)$ is nonzero for some $s \in [a, \sigma(b))$.

Suppose that there exists a number $d > 0$ such that for $0 \le x \le d$,

$$f(x) < \frac{d}{q}. \tag{3.11}$$

Then,

$$S(\overline{C}(d)) \subseteq C(d) \subset \overline{C}(d). \tag{3.12}$$

Proof. Let $y \in \overline{C}(d)$. Then, it follows that $0 \le y(s) \le d$ for $s \in [a, \sigma^2(b)]$. This implies

$$0 \le y(\sigma(s)) \le d, \;\; s \in [a, \sigma(b)]. \tag{3.13}$$

Noting Sy ($\in C$) is nondecreasing and (3.13), we apply (3.8), (C4), (3.11) and (3.10) to get the following for $t \in [a, \sigma^2(b)]$:

$$\begin{aligned} Sy(t) \;\le\; Sy(\sigma^2(b)) \;&\le\; \int_a^{\sigma(b)} G(\sigma^2(b), s)\nu(s) f(y(\sigma(s)))\Delta s \\ &= \int_a^{\sigma(b)} [\sigma(s)-a]\nu(s) f(y(\sigma(s)))\Delta s \\ &< \int_a^{\sigma(b)} [\sigma(s)-a]\nu(s)\, \frac{d}{q}\, \Delta s \\ &= q\, \frac{d}{q} \;=\; d. \end{aligned}$$

This implies $\|Sy\| < d$. Coupling with the fact that $Sy \in C$ (Lemma 3.3), we have $Sy \in C(d)$. The conclusion (3.12) is now immediate. $\square$

The next lemma is similar to Lemma 3.4 and its proof is omitted.

Lemma 3.5. Let (C1) and (C2) hold. Suppose that there exists a number $d > 0$ such that for $0 \le x \le d$,

$$f(x) \le \frac{d}{q}.$$

Then,

$$S(\overline{C}(d)) \subseteq \overline{C}(d).$$

We are now ready to establish existence criteria for three positive solutions. Our first result employs Theorem 2.1.

Theorem 3.1. Let (C1)–(C4) hold, and assume

(C5) the function $\mu(s)$ is nonzero for some $s \in [\eta, \sigma(b))$.

Suppose that there exist numbers w_1, w_2, w_3 with

$$0 < w_1 < w_2 < \frac{w_2}{Mc} \le w_3$$

such that the following hold:

(P) $f(x) < \frac{w_1}{q}$ for $0 \le x \le w_1$;

(Q) one of the following holds:

(Q1) $\limsup_{x\to\infty} \frac{f(x)}{x} < \frac{1}{q}$;

(Q2) there exists a number d ($\ge w_3$) such that $f(x) \le \frac{d}{q}$ for $0 \le x \le d$;

(R) $f(x) > \frac{w_2}{r}$ for $w_2 \le x \le w_3$.

Then, the boundary value problem (F) has (at least) three positive solutions $y^1, y^2, y^3 \in C$ such that

$$\begin{aligned} &\|y^1\| = y^1(\sigma^2(b)) < w_1; \qquad y^2(t) > w_2,\ t \in [\eta, \sigma^2(b)]; \\ &\|y^3\| = y^3(\sigma^2(b)) > w_1 \qquad \text{and} \qquad \min_{t\in[\eta,\sigma^2(b)]} y^3(t) = y^3(\eta) < w_2. \end{aligned} \tag{3.14}$$

Proof. We shall employ Theorem 2.1. First, we shall prove that condition (Q) implies the existence of a number w_4 where $w_4 \ge w_3$ such that

$$S(\overline{C}(w_4)) \subseteq \overline{C}(w_4). \tag{3.15}$$

Suppose that (Q2) holds. Then, by Lemma 3.5 we immediately have (3.15) where we pick $w_4 = d$. Suppose now that (Q1) is satisfied. Then, there exist $N > 0$ and $\epsilon < \frac{1}{q}$ such that

$$\frac{f(x)}{x} < \epsilon, \quad x > N. \tag{3.16}$$

Define $M_0 = \max_{0\le x\le N} f(x)$. In view of (3.16), it is clear that the following holds for all $x \in \mathbb{R}$,

$$f(x) \le M_0 + \epsilon\, x. \tag{3.17}$$

Now, pick the number w_4 so that

$$w_4 > \max\left\{ w_3,\ M_0 \left(\frac{1}{q} - \epsilon \right)^{-1} \right\}. \tag{3.18}$$

Let $y \in \overline{C}(w_4)$. Then, $0 \le y(s) \le w_4$ for $s \in [a, \sigma^2(b)]$. This implies

$$0 \le y(\sigma(s)) \le w_4,\ s \in [a, \sigma(b)]. \tag{3.19}$$

Then, using (3.8), (3.17), (3.19) and (3.18) we find for $t \in [a, \sigma^2(b)]$,

$$\begin{aligned} Sy(t) \ \le \ Sy(\sigma^2(b)) \ &\le \ \int_a^{\sigma(b)} G(\sigma^2(b), s)\nu(s) f(y(\sigma(s)))\Delta s \\ &\le \ \int_a^{\sigma(b)} G(\sigma^2(b), s)\nu(s)[M_0 + \epsilon\, y(\sigma(s))]\Delta s \\ &\le \ \int_a^{\sigma(b)} G(\sigma^2(b), s)\nu(s)[M_0 + \epsilon\, w_4]\Delta s \\ &= \ q(M_0 + \epsilon\, w_4) \\ &< \ q\left[w_4\left(\frac{1}{q} - \epsilon\right) + \epsilon\, w_4\right] \ = \ w_4. \end{aligned}$$

This leads to $\|Sy\| < w_4$ and so $Sy \in C(w_4) \subset \overline{C}(w_4)$. Thus, (3.15) follows immediately.

Let $\psi : C \to \mathbb{R}^+ \cup \{0\}$ be defined by

$$\psi(y) = \min_{t \in [\eta, \sigma^2(b)]} y(t) = y(\eta). \tag{3.20}$$

Clearly, ψ is a nonnegative continuous concave functional on C and $\psi(y) \le \|y\|$ for all $y \in C$.

We shall verify that condition (a) of Theorem 2.1 is satisfied. First, we note that

$$y^*(t) = \frac{w_2 + w_3}{2} \in \{ y \in C(\psi, w_2, w_3) \mid \psi(y) > w_2 \}.$$

Thus, $\{y \in C(\psi, w_2, w_3) \mid \psi(y) > w_2\} \ne \emptyset$. Next, let $y \in C(\psi, w_2, w_3)$. Then, $w_2 \le \psi(y) \le \|y\| \le w_3$ provides $w_2 \le y(s) \le w_3$ for $s \in [\eta, \sigma^2(b)]$, which leads to

$$w_2 \le y(\sigma(s)) \le w_3, \ \ s \in [\eta, \sigma(b)]. \tag{3.21}$$

Noting (3.21), we apply (3.8), (3.21), (C5), (R) and (3.10) to get

$$\begin{aligned} \psi(Sy) \ &= \ \min_{t \in [\eta, \sigma^2(b)]} (Sy)(t) \ = \ Sy(\eta) \\ &\ge \ \int_a^{\sigma(b)} G(\eta, s)\mu(s) f(y(\sigma(s)))\Delta s \\ &\ge \ \int_\eta^{\sigma(b)} G(\eta, s)\mu(s) f(y(\sigma(s)))\Delta s \\ &= \ \int_\eta^{\sigma(b)} (\eta - a)\mu(s) f(y(\sigma(s)))\Delta s \\ &> \ \int_\eta^{\sigma(b)} (\eta - a)\mu(s)\ \frac{w_2}{r}\ \Delta s \\ &= \ r\,\frac{w_2}{r} \ = \ w_2. \end{aligned}$$

Therefore, we have shown that $\psi(Sy) > w_2$ for all $y \in C(\psi, w_2, w_3)$.

Next, condition (b) of Theorem 2.1 is fulfilled since by Lemma 3.4 and condition (P), we have $S(\overline{C}(w_1)) \subseteq C(w_1)$.

Finally, we shall show that condition (c) of Theorem 2.1 holds. Let $y \in C(\psi, w_2, w_4)$ with $\|Sy\| > w_3$. Using (3.8), Lemma 3.1(c), (C3) and (3.9), we get

$$\begin{aligned}
\psi(Sy) &= \min_{t\in[\eta,\sigma^2(b)]}(Sy)(t) \;=\; Sy(\eta) \\
&\geq \int_a^{\sigma(b)} G(\eta,s)\mu(s)f(y(\sigma(s)))\Delta s \\
&\geq \int_a^{\sigma(b)} MG(\sigma(s),s)c\nu(s)f(y(\sigma(s)))\Delta s \\
&\geq Mc\|Sy\| \\
&> Mcw_3 \;\geq\; Mc\,\frac{w_2}{Mc} \;=\; w_2.
\end{aligned}$$

Hence, we have proved that $\psi(Sy) > w_2$ for all $y \in C(\psi, w_2, w_4)$ with $\|Sy\| > w_3$.

It now follows from Theorem 2.1 that the boundary value problem (F) has (at least) three *positive* solutions $y^1, y^2, y^3 \in \overline{C}(w_4)$ satisfying (2.1). It is easy to see that here (2.1) reduces to (3.14). □

We shall now employ Theorem 2.2 to give other existence criteria. In applying Theorem 2.2 it is possible to choose the functionals and constants in many different ways. We shall present two results to show the arguments involved. In particular the first result is a generalization of Theorem 3.1.

Theorem 3.2. Let (C1)–(C3) hold. Assume there exist $\tau_1, \tau_2, \tau_3 \in [a, \sigma^2(b)]$ with

$$a \leq \tau_1 \leq \eta \leq \tau_2 < \rho(\tau_3) \leq \sigma(b) \tag{3.22}$$

such that

(C6) the function $\mu(s)$ is nonzero for some $s \in [\tau_2, \rho(\tau_3))$;

(C7) the function $[\sigma(s) - a]\nu(s)$ is nonzero for some $s \in [\tau_1, \sigma(b))$.

Suppose that there exist numbers $w_i,\ 2 \leq i \leq 5$ with

$$0 < w_2 < w_3 < \frac{w_3}{Mc} \leq w_4 \leq w_5 \qquad \text{and} \qquad w_2 > \frac{d_3}{q}\, w_5$$

such that the following hold:

(P) $f(x) < \frac{1}{d_2}\left(w_2 - \frac{w_5 d_3}{q}\right)$ for $0 \leq x \leq w_2$;

(Q) $f(x) \leq \frac{w_5}{q}$ for $0 \leq x \leq w_5$;

(R) $f(x) > \frac{w_3}{d_1}$ for $w_3 \le x \le w_4$.

Then, the boundary value problem (F) has (at least) three positive solutions $y^1, y^2, y^3 \in \overline{C}(w_5)$ such that

$$\begin{aligned} &\max_{t\in[\tau_1,\sigma^2(b)]} y^1(t) = y^1(\sigma^2(b)) = \|y^1\| < w_2; \qquad y^2(t) > w_3,\ t \in [\tau_2,\tau_3]; \\ &\max_{t\in[\tau_1,\sigma^2(b)]} y^3(t) = y^3(\sigma^2(b)) = \|y^3\| > w_2 \qquad \text{and} \qquad \min_{t\in[\tau_2,\tau_3]} y^3(t) = y^3(\tau_2) < w_3. \end{aligned} \tag{3.23}$$

Proof. In the context of Theorem 2.2, we define the following functionals on C :

$$\begin{aligned} &\gamma(y) = \|y\| = y(\sigma^2(b)), \\ &\psi(y) = \min_{t\in[\eta,\sigma^2(b)]} y(t) = y(\eta), \\ &\beta(y) = \Theta(y) = \max_{t\in[\tau_1,\sigma^2(b)]} y(t) = y(\sigma^2(b)), \\ &\alpha(y) = \min_{t\in[\tau_2,\tau_3]} y(t) = y(\tau_2). \end{aligned} \tag{3.24}$$

First, we shall show that the operator S maps $\overline{P}(\gamma, w_5)$ into $\overline{P}(\gamma, w_5)$. Let $x \in \overline{P}(\gamma, w_5) = \overline{C}(w_5)$. Then, we have $0 \le x \le w_5$. Together with (Q) and Lemma 3.5, we get $S(\overline{C}(w_5)) \subseteq \overline{C}(w_5)$, or equivalently $S : \overline{P}(\gamma, w_5) \to \overline{P}(\gamma, w_5)$.

Next, we shall prove that condition (a) of Theorem 2.2 is fulfilled. It is noted that

$$y^*(t) = \frac{w_3 + w_4}{2} \in \{y \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5) \mid \alpha(y) > w_3\}$$

and hence $\{y \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5) \mid \alpha(y) > w_3\} \neq \emptyset$. Now, let $y \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5)$. Then, by definition we have $\alpha(y) \ge w_3$ and $\Theta(y) \le w_4$ which imply $w_3 \le y(s) \le w_4$ for $s \in [\tau_2, \tau_3]$. Thus, we have

$$w_3 \le y(\sigma(s)) \le w_4,\ s \in [\tau_2, \rho(\tau_3)]. \tag{3.25}$$

Noting (3.25), we apply (3.8), (C6), (R) and (3.10) to obtain

$$\begin{aligned} \alpha(Sy) &= \min_{t\in[\tau_2,\tau_3]} (Sy)(t) = Sy(\tau_2) \\ &\ge \int_a^{\sigma(b)} G(\tau_2, s)\mu(s) f(y(\sigma(s)))\Delta s \\ &\ge \int_{\tau_2}^{\rho(\tau_3)} G(\tau_2, s)\mu(s) f(y(\sigma(s)))\Delta s \\ &= \int_{\tau_2}^{\rho(\tau_3)} (\tau_2 - a)\mu(s) f(y(\sigma(s)))\Delta s \\ &> \int_{\tau_2}^{\rho(\tau_3)} (\tau_2 - a)\mu(s)\ \frac{w_3}{d_1}\ \Delta s \\ &= d_1\ \frac{w_3}{d_1} = w_3. \end{aligned}$$

Hence, $\alpha(Sy) > w_3$ for all $y \in P(\gamma,\Theta,\alpha,w_3,w_4,w_5)$.

We shall now verify that condition (b) of Theorem 2.2 is satisfied. Let w_1 be such that $0 < w_1 < w_2$. We note that

$$y^*(t) = \frac{w_1+w_2}{2} \in \{y \in Q(\gamma,\beta,\psi,w_1,w_2,w_5) \mid \beta(y) < w_2\}.$$

Hence, $\{y \in Q(\gamma,\beta,\psi,w_1,w_2,w_5) \mid \beta(y) < w_2\} \neq \emptyset$. Next, let $y \in Q(\gamma,\beta,\psi,w_1,w_2,w_5)$. Then, we have $\beta(y) \le w_2$ and $\gamma(y) \le w_5$ which provide

$$\begin{aligned} &0 \le y(s) \le w_2,\ s \in [\tau_1,\sigma^2(b)] \quad \text{or} \quad 0 \le y(\sigma(s)) \le w_2,\ s \in [\tau_1,\sigma(b)];\\ &0 \le y(s) \le w_5,\ s \in [a,\sigma^2(b)] \quad \text{or} \quad 0 \le y(\sigma(s)) \le w_5,\ s \in [a,\sigma(b)]. \end{aligned} \tag{3.26}$$

Noting (3.8), (3.26), (C7), (P), (Q) and (3.10), we find

$$\begin{aligned} \beta(Sy) &= \max_{t\in[\tau_1,\sigma^2(b)]}(Sy)(t) \;=\; Sy(\sigma^2(b))\\ &\le \int_a^{\sigma(b)} G(\sigma^2(b),s)\nu(s)f(y(\sigma(s)))\Delta s\\ &= \int_a^{\sigma(b)} [\sigma(s)-a]\nu(s)f(y(\sigma(s)))\Delta s\\ &= \int_a^{\tau_1} [\sigma(s)-a]\nu(s)f(y(\sigma(s)))\Delta s + \int_{\tau_1}^{\sigma(b)} [\sigma(s)-a]\nu(s)f(y(\sigma(s)))\Delta s\\ &< \int_a^{\tau_1} [\sigma(s)-a]\nu(s)\Delta s\ \frac{w_5}{q} + \int_{\tau_1}^{\sigma(b)} [\sigma(s)-a]\nu(s)\Delta s\ \frac{1}{d_2}\left(w_2 - \frac{w_5 d_3}{q}\right)\\ &= d_3\ \frac{w_5}{q} + d_2\ \frac{1}{d_2}\left(w_2 - \frac{w_5 d_3}{q}\right) \;=\; w_2. \end{aligned}$$

Therefore, $\beta(Sy) < w_2$ for all $y \in Q(\gamma,\beta,\psi,w_1,w_2,w_5)$.

Next, we shall show that condition (c) of Theorem 2.2 is met. Using Lemma 3.1(b), we observe that for $y \in C$,

$$\begin{aligned} \Theta(Sy) &= \max_{t\in[\tau_1,\sigma^2(b)]}(Sy)(t) \;=\; Sy(\sigma^2(b))\\ &\le \int_a^{\sigma(b)} G(\sigma^2(b),s)\nu(s)f(y(\sigma(s)))\Delta s\\ &\le \int_a^{\sigma(b)} G(\sigma(s),s)\nu(s)f(y(\sigma(s)))\Delta s. \end{aligned} \tag{3.27}$$

Moreover, (C3) and Lemma 3.1(c) yield for $y \in C$,

$$\begin{aligned}
\alpha(Sy) &= \min_{t\in[\tau_2,\tau_3]}(Sy)(t) \;=\; Sy(\tau_2) \\
&\geq \int_a^{\sigma(b)} G(\tau_2,s)\mu(s)f(y(\sigma(s)))\Delta s \\
&\geq \int_a^{\sigma(b)} G(\tau_2,s)c\nu(s)f(y(\sigma(s)))\Delta s \\
&\geq Mc\int_a^{\sigma(b)} G(\sigma(s),s)\nu(s)f(y(\sigma(s)))\Delta s. \qquad (3.28)
\end{aligned}$$

A combination of (3.27) and (3.28) gives

$$\alpha(Sy)\geq Mc\;\Theta(Sy),\quad y\in C. \qquad (3.29)$$

Let $y\in P(\gamma,\alpha,w_3,w_5)$ with $\Theta(Sy)>w_4$. Then, it follows from (3.29) that

$$\alpha(Sy)\geq Mc\;\Theta(Sy)>Mcw_4\geq Mc\,\frac{w_3}{Mc}=w_3.$$

Thus, $\alpha(Sy)>w_3$ for all $y\in P(\gamma,\alpha,w_3,w_5)$ with $\Theta(Sy)>w_4$.

Finally, we shall prove that condition (d) of Theorem 2.2 is fulfilled. Let $y\in Q(\gamma,\beta,w_2,w_5)$ with $\psi(Sy)<w_1$. Then, we have $\beta(y)\leq w_2$ and $\gamma(y)\leq w_5$ which give (3.26). Using (3.8), (3.26), (C7), (P), (Q) and (3.10), we get as in an earlier part $\beta(Sy)<w_2$ for all $y\in Q(\gamma,\beta,w_2,w_5)$ with $\psi(Sy)<w_1$.

It now follows from Theorem 2.2 that the boundary value problem (F) has (at least) three *positive* solutions $y^1,y^2,y^3\in\overline{P}(\gamma,w_5)=\overline{C}(w_5)$ satisfying (2.2). It is clear that (2.2) reduces to (3.23) immediately. □

Consider the case when

$$\tau_1=a,\qquad \tau_2=\eta,\qquad \rho(\tau_3)=\sigma(b)\text{ or equivalently }\tau_3=\sigma^2(b).$$

Then, it is clear that

$$d_1=r,\qquad d_2=q,\qquad d_3=0. \qquad (3.30)$$

In this case Theorem 3.2 yields the following corollary.

Corollary 3.1. Let (C1)–(C3) hold, and assume

(C6)$'$ the function $\mu(s)$ is nonzero for some $s\in[\eta,\sigma(b))$;

(C7)$'$ the function $[\sigma(s)-a]\nu(s)$ is nonzero for some $s\in[a,\sigma(b))$.

Suppose that there exist numbers $w_i,\ 2\leq i\leq 5$ with

$$0<w_2<w_3<\frac{w_3}{Mc}\leq w_4\leq w_5$$

such that the following hold:

(P) $f(x) < \frac{w_2}{q}$ for $0 \le x \le w_2$;

(Q) $f(x) \le \frac{w_5}{q}$ for $0 \le x \le w_5$;

(R) $f(x) > \frac{w_3}{r}$ for $w_3 \le x \le w_4$.

Then, the boundary value problem (F) has (at least) three positive solutions $y^1, y^2, y^3 \in \overline{C}(w_5)$ such that

$$\|y^1\| = y^1(\sigma^2(b)) < w_2; \qquad y^2(t) > w_3,\ t \in [\eta, \sigma^2(b)];$$
$$\|y^3\| = y^3(\sigma^2(b)) > w_2 \qquad \text{and} \qquad \min_{t\in[\eta,\sigma^2(b)]} y^3(t) = y^3(\eta) < w_3.$$

Remark 3.2. Corollary 3.1 is actually Theorem 3.1. Hence, Theorem 3.2 is more general than Theorem 3.1.

Another application of Theorem 2.2 yields the next result.

Theorem 3.3. Let (C1)–(C3) hold. Assume there exist numbers $\tau_1, \tau_2, \tau_3, \tau_4 \in [a, \sigma^2(b)]$ with

$$a < \eta \le \tau_1 \le \tau_2 < \rho(\tau_3) \le \sigma(b) \qquad \text{and} \qquad \tau_1 < \rho(\tau_4) \le \sigma(b) \tag{3.31}$$

such that (C6) holds, and

(C8) the function $[\sigma(s) - a]\nu(s)$ is nonzero for some $s \in [\tau_1, \rho(\tau_4))$.

Suppose that there exist numbers $w_i,\ 1 \le i \le 5$ with

$$0 < w_1 \le w_2 \cdot Mc < w_2 < w_3 < \frac{w_3}{Mc} \le w_4 \le w_5 \qquad \text{and} \qquad w_2 > \frac{d_5}{q}\, w_5$$

such that the following hold:

(P) $f(x) < \frac{1}{d_4}\left(w_2 - \frac{w_5 d_5}{q}\right)$ for $w_1 \le x \le w_2$;

(Q) $f(x) \le \frac{w_5}{q}$ for $0 \le x \le w_5$;

(R) $f(x) > \frac{w_3}{d_1}$ for $w_3 \le x \le w_4$.

Then, the boundary value problem (F) has (at least) three positive solutions $y^1, y^2, y^3 \in \overline{C}(w_5)$ such that

$$\begin{gathered}\max_{t\in[\tau_1,\tau_4]} y^1(t) = y^1(\tau_4) < w_2; \qquad y^2(t) > w_3,\ t \in [\tau_2, \tau_3];\\ \max_{t\in[\tau_1,\tau_4]} y^3(t) = y^3(\tau_4) > w_2 \qquad \text{and} \qquad \min_{t\in[\tau_2,\tau_3]} y^3(t) = y^3(\tau_2) < w_3.\end{gathered} \tag{3.32}$$

Proof. In the context of Theorem 2.2, we define the following functionals on C :

$$\begin{aligned}\gamma(y) &= \|y\| = y(\sigma^2(b)),\\ \psi(y) &= \min_{t\in[\tau_1,\tau_4]} y(t) = y(\tau_1),\\ \beta(y) &= \max_{t\in[\tau_1,\tau_4]} y(t) = y(\tau_4),\\ \alpha(y) &= \min_{t\in[\tau_2,\tau_3]} y(t) = y(\tau_2),\\ \Theta(y) &= \max_{t\in[\tau_2,\tau_3]} y(t) = y(\tau_3).\end{aligned} \tag{3.33}$$

First, using (Q) it can be shown (as in the proof of Theorem 3.2) that $S : \overline{P}(\gamma, w_5) \to \overline{P}(\gamma, w_5)$.

Next, using (R) and a similar argument as in the proof of Theorem 3.2, we can verify that condition (a) of Theorem 2.2 is fulfilled.

Now, we shall check that condition (b) of Theorem 2.2 is satisfied. Clearly,

$$y^*(t) = \frac{w_1 + w_2}{2} \in \left\{ y \in Q(\gamma, \beta, \psi, w_1, w_2, w_5) \;\middle|\; \beta(y) < w_2 \right\}$$

and so $\{y \in Q(\gamma, \beta, \psi, w_1, w_2, w_5) \mid \beta(y) < w_2\} \neq \emptyset$. Let $y \in Q(\gamma, \beta, \psi, w_1, w_2, w_5)$. Then, we have $\psi(y) \geq w_1$, $\beta(y) \leq w_2$ and $\gamma(y) \leq w_5$ which imply

$$\begin{aligned} &w_1 \leq y(s) \leq w_2,\ s \in [\tau_1, \tau_4] \quad \text{or} \quad w_1 \leq y(\sigma(s)) \leq w_2,\ s \in [\tau_1, \rho(\tau_4)];\\ &0 \leq y(s) \leq w_5,\ s \in [a, \sigma^2(b)] \quad \text{or} \quad 0 \leq y(\sigma(s)) \leq w_5,\ s \in [a, \sigma(b)]. \end{aligned} \tag{3.34}$$

Using (3.8), (3.34), (C8), (P), (Q) and (3.10), we find

$$\begin{aligned} \beta(Sy) &= \max_{t\in[\tau_1,\tau_4]} (Sy)(t) \;=\; Sy(\tau_4)\\ &\leq \int_a^{\sigma(b)} G(\tau_4, s)\nu(s) f(y(\sigma(s)))\Delta s\\ &= \int_{\tau_1}^{\rho(\tau_4)} G(\tau_4, s)\nu(s) f(y(\sigma(s)))\Delta s\\ &\quad + \int_a^{\tau_1} G(\tau_4, s)\nu(s) f(y(\sigma(s)))\Delta s + \int_{\rho(\tau_4)}^{\sigma(b)} G(\tau_4, s)\nu(s) f(y(\sigma(s)))\Delta s\\ &= \int_{\tau_1}^{\rho(\tau_4)} [\sigma(s) - a]\nu(s) f(y(\sigma(s)))\Delta s\\ &\quad + \int_a^{\tau_1} [\sigma(s) - a]\nu(s) f(y(\sigma(s)))\Delta s + \int_{\rho(\tau_4)}^{\sigma(b)} (\tau_4 - a)\nu(s) f(y(\sigma(s)))\Delta s\\ &< \int_{\tau_1}^{\rho(\tau_4)} [\sigma(s) - a]\nu(s)\Delta s\ \frac{1}{d_4}\left(w_2 - \frac{w_5 d_5}{q}\right) \end{aligned}$$

$$
\begin{aligned}
& + \left\{ \int_a^{\tau_1} [\sigma(s) - a]\nu(s)\Delta s + \int_{\rho(\tau_4)}^{\sigma(b)} (\tau_4 - a)\nu(s)\Delta s \right\} \frac{w_5}{q} \\
= \;\; & d_4 \, \frac{1}{d_4} \left(w_2 - \frac{w_5 d_5}{q} \right) + d_5 \, \frac{w_5}{q} \;\; = \;\; w_2.
\end{aligned}
$$

Therefore, $\beta(Sy) < w_2$ for all $y \in Q(\gamma, \beta, \psi, w_1, w_2, w_5)$.

Next, we shall show that condition (c) of Theorem 2.2 is met. We observe that, by (3.8) and Lemma 3.1(b), for $y \in C$,

$$
\begin{aligned}
\Theta(Sy) \;\; & = \;\; \max_{t \in [\tau_2, \tau_3]} (Sy)(t) \;\; = \;\; Sy(\tau_3) \\
& \le \;\; \int_a^{\sigma(b)} G(\tau_3, s)\nu(s) f(y(\sigma(s)))\Delta s \\
& \le \;\; \int_a^{\sigma(b)} G(\sigma(s), s)\nu(s) f(y(\sigma(s)))\Delta s. \qquad (3.35)
\end{aligned}
$$

Moreover, using (3.8), (C3) and Lemma 3.1(c), we obtain (3.28) for $y \in C$. A combination of (3.28) and (3.35) yields (3.29). Following a similar argument as in the proof of Theorem 3.2, we get $\alpha(Sy) > w_3$ for all $y \in P(\gamma, \alpha, w_3, w_5)$ with $\Theta(Sy) > w_4$.

Finally, we shall prove that condition (d) of Theorem 2.2 is fulfilled. As in (3.35), by (3.8) and Lemma 3.1(b), we see that for $y \in C$,

$$
\beta(Sy) = \max_{t \in [\tau_1, \tau_4]} (Sy)(t) = Sy(\tau_4) \le \int_a^{\sigma(b)} G(\sigma(s), s)\nu(s) f(y(\sigma(s)))\Delta s. \qquad (3.36)
$$

On the other hand, it follows from (3.8), (C3) and Lemma 3.1(c) that for $y \in C$,

$$
\begin{aligned}
\psi(Sy) \;\; & = \;\; \min_{t \in [\tau_1, \tau_4]} (Sy)(t) \;\; = \;\; Sy(\tau_1) \\
& \ge \;\; \int_a^{\sigma(b)} G(\tau_1, s)\mu(s) f(y(\sigma(s)))\Delta s \\
& \ge \;\; \int_a^{\sigma(b)} G(\tau_1, s) c\nu(s) f(y(\sigma(s)))\Delta s \\
& \ge \;\; Mc \int_a^{\sigma(b)} G(\sigma(s), s)\nu(s) f(y(\sigma(s)))\Delta s. \qquad (3.37)
\end{aligned}
$$

A combination of (3.36) and (3.37) gives

$$
\psi(Sy) \ge Mc \; \beta(Sy), \quad y \in C. \qquad (3.38)
$$

Let $y \in Q(\gamma, \beta, w_2, w_5)$ with $\psi(Sy) < w_1$. Then, (3.38) leads to

$$
\beta(Sy) \le \frac{1}{Mc} \; \psi(Sy) < \frac{1}{Mc} \; w_1 \le \frac{1}{Mc} \; w_2 \cdot Mc = w_2.
$$

Thus, $\beta(Sy) < w_2$ for all $y \in Q(\gamma, \beta, w_2, w_5)$ with $\psi(Sy) < w_1$.

It now follows from Theorem 2.2 that the boundary value problem (F) has (at least) three *positive* solutions $y^1, y^2, y^3 \in \overline{P}(\gamma, w_5) = \overline{C}(w_5)$ satisfying (2.2). Furthermore, (2.2) reduces to (3.32) immediately. $\square$

4. EXAMPLES

To show the usefulness of the results obtained in Section 3, we shall provide three examples to illustrate Theorems 3.1–3.3, respectively. Throughout, we consider the time scale

$$\mathbb{T} = \{2^k \mid k \in \mathbb{Z}\} \cup \{0\}.$$

Example 4.1. Consider the boundary value problem (F) with $a = 2$, $b = 2^5 = 32$ and the nonlinear term

$$P(t,x) = f(x) = \begin{cases} \dfrac{w_1}{2q}, & 0 \le x \le w_1 \\ l(x), & w_1 \le x \le w_2 \\ \dfrac{1}{2}\left(\dfrac{d}{q} + \dfrac{w_2}{r}\right), & x \ge w_2 \end{cases} \tag{4.1}$$

where $l(x)$ satisfies

$$l''(x) = 0, \qquad l(w_1) = \frac{w_1}{2q}, \qquad l(w_2) = \frac{1}{2}\left(\frac{d}{q} + \frac{w_2}{r}\right) \tag{4.2}$$

and w_i's and d are as in the context of Theorem 3.1.

Taking $m = \eta = 16$ and the functions $\mu = \nu \equiv 1$ (this implies $c = 1$), by direct computation we have

$$M = \frac{1}{9}, \qquad q = 2604, \qquad r = 672.$$

Hence, the w_i's and d are numbers satisfying the relation

$$0 < w_1 < w_2 < \frac{w_2}{Mc} = 9w_2 \le w_3 \le d. \tag{4.3}$$

We shall check the conditions of Theorem 3.1. First, it is clear that (C1)–(C5) are fulfilled. Next, condition (P) is obviously satisfied. Noting that $\frac{w_2}{r} < \frac{d}{q}$ (or $d > \frac{q}{r}\, w_2 = 3.875 w_2$), we find for $0 \le x \le d$,

$$f(x) \le \max\{l(w_1), l(w_2)\} = l(w_2) = \frac{1}{2}\left(\frac{d}{q} + \frac{w_2}{r}\right) < \frac{1}{2}\left(\frac{d}{q} + \frac{d}{q}\right) = \frac{d}{q}.$$

Thus, condition (Q2) is met. Finally, (R) is satisfied since for $w_2 \le x \le w_3$ we have

$$f(x) = \frac{1}{2}\left(\frac{d}{q} + \frac{w_2}{r}\right) > \frac{1}{2}\left(\frac{w_2}{r} + \frac{w_2}{r}\right) = \frac{w_2}{r}.$$

By Theorem 3.1, the boundary value problem (F) with $a = 2,\ b = 32$ and (4.1)–(4.3) has (at least) three positive solutions $y^1, y^2, y^3 \in C$ such that

$$\begin{aligned} &\|y^1\| = y^1(128) < w_1; && y^2(t) > w_2,\ t \in [16, 128]; \\ &\|y^3\| = y^3(128) > w_1 \quad \text{and} && \min_{t\in[16,128]} y^3(t) = y^3(16) < w_2. \end{aligned} \tag{4.4}$$

To illustrate further, fix

$$w_1 = 1, \qquad w_2 = 2, \qquad d = 19 \qquad \text{and} \qquad \text{any } w_3 \text{ such that } 9w_2 \le w_3 \le d.$$

Clearly, (4.3) is fulfilled. We find that the boundary value problem (F) with $a = 2,\ b = 32$, (4.1), (4.2) indeed has three positive solutions $y^1, y^2, y^3 \in C$ that satisfy (4.4). They are tabulated as follows:

t	2	4	8	16	32	64	128
$y^1(t)$	0	0.0238	0.0699	0.1559	0.3034	0.5000	0.5000
$y^2(t)$	0	0.6104	1.8296	4.1307	8.0754	13.3350	13.3350
$y^3(t)$	0	0.0418	0.1238	0.2816	0.5728	1.0568	1.0568

Example 4.2. Consider the boundary value problem (F) with $a = 2,\ b = 2^5 = 32$ and the nonlinear term

$$P(t,x) = f(x) = \begin{cases} \dfrac{1}{2d_2}\left(w_2 - \dfrac{w_5 d_3}{q}\right), & 0 \le x \le w_2 \\ l(x), & w_2 \le x \le w_3 \\ \dfrac{1}{2}\left(\dfrac{w_3}{r} + \dfrac{w_3}{d_1}\right), & x \ge w_3 \end{cases} \tag{4.5}$$

where $l(x)$ satisfies

$$l''(x) = 0, \qquad l(w_2) = \frac{1}{2d_2}\left(w_2 - \frac{w_2 d_3}{q}\right), \qquad l(w_3) = \frac{1}{2}\left(\frac{w_3}{r} + \frac{w_3}{d_1}\right) \tag{4.6}$$

and w_i's are as in the context of Theorem 3.2.

Taking $m = \eta = 16,\ \tau_1 = 4,\ \tau_2 = 32,\ \tau_3 = 128$ and the functions $\mu = \nu \equiv 1$ (this implies $c = 1$), by direct computation we have

$$M = \frac{1}{9}, \qquad q = 2604, \qquad r = 672, \qquad d_1 = 960, \qquad d_2 = 2600, \qquad d_3 = 4.$$

Hence, the w_i's are numbers satisfying the relation

$$0 < w_2 < w_3 < \frac{w_3}{Mc} = 9w_3 \le w_4 \le w_5 < \frac{q}{d_3}\, w_2 = 651 w_2. \tag{4.7}$$

We shall check the conditions of Theorem 3.2. Clearly, (C1)–(C3), (C6) and (C7) are fulfilled. Next, condition (P) is obviously satisfied. Since $r < d_1 < d_2$ and $\frac{w_3}{r} < \frac{w_5}{q}$ (or $w_5 > \frac{q}{r}\, w_3 = 3.875 w_3$), we find for $0 \le x \le w_5$,

$$f(x) \le \max\{l(w_2), l(w_3)\} = l(w_3) = \frac{1}{2}\left(\frac{w_3}{r} + \frac{w_3}{d_1}\right) < \frac{1}{2}\left(\frac{w_3}{r} + \frac{w_3}{r}\right) = \frac{w_3}{r} < \frac{w_5}{q}.$$

Hence, condition (Q) is met. Finally, (R) is satisfied since for $w_3 \le x \le w_4$ we have

$$f(x) = \frac{1}{2}\left(\frac{w_3}{r} + \frac{w_3}{d_1}\right) > \frac{1}{2}\left(\frac{w_3}{d_1} + \frac{w_3}{d_1}\right) = \frac{w_3}{d_1}.$$

It follows from Theorem 3.2 that the boundary value problem (F) with $a = 2,\ b = 32$ and (4.5)–(4.7) has (at least) three positive solutions $y^1, y^2, y^3 \in \overline{C}(w_5)$ such that

$$\begin{aligned} &\max_{t\in[4,128]} y^1(t) = y^1(128) = \|y^1\| < w_2; \qquad y^2(t) > w_3,\ t \in [32,128]; \\ &\max_{t\in[4,128]} y^3(t) = y^3(128) = \|y^3\| > w_2 \qquad \text{and} \qquad \min_{t\in[32,128]} y^3(t) = y^3(32) < w_3. \end{aligned} \tag{4.8}$$

As an example, fix

$$w_2 = 1, \qquad w_3 = 2, \qquad w_5 = 19 \qquad \text{and} \qquad \text{any } w_4 \text{ such that } 9w_3 \le w_4 \le w_5.$$

Clearly, (4.7) holds. Indeed, the boundary value problem (F) with $a = 2,\ b = 32$, (4.5), (4.6) has three positive solutions $y^1, y^2, y^3 \in \overline{C}(19)$ that satisfy (4.8). They are tabulated as follows:

t	2	4	8	16	32	64	128
$y^1(t)$	0	0.0232	0.0680	0.1516	0.2950	0.4862	0.4862
$y^2(t)$	0	0.2850	0.8535	1.9845	3.9273	6.5178	6.5178
$y^3(t)$	0	0.0443	0.1313	0.2994	0.6118	1.1408	1.1408

Remark 4.1. In Example 4.2, we see that for $w_3 \le x \le w_4$,

$$f(x) = \frac{1}{2}\left(\frac{w_3}{r} + \frac{w_3}{d_1}\right) < \frac{1}{2}\left(\frac{w_3}{r} + \frac{w_3}{r}\right) = \frac{w_3}{r}.$$

Thus, condition (R) of Corollary 3.1 is *not* satisfied. Recalling that Corollary 3.1 is actually Theorem 3.1, Example 4.2 illustrates the case when Theorem 3.2 is applicable but not Theorem 3.1. Hence, this example shows that Theorem 3.2 is indeed more general than Theorem 3.1.

Example 4.3. Consider the boundary value problem (F) with $a = 2,\ b = 2^7 = 128$ and the nonlinear term

$$P(t,x) = f(x) = \begin{cases} \frac{1}{2d_4}\left(w_2 - \frac{w_5 d_5}{q}\right), & 0 \le x \le w_2 \\ l(x), & w_2 \le x \le w_3 \\ \frac{1}{2}\left(\frac{w_3}{d_1} + \frac{w_5}{q}\right), & x \ge w_3 \end{cases} \tag{4.9}$$

where $l(x)$ satisfies

$$l''(x) = 0, \qquad l(w_2) = \frac{1}{2d_4}\left(w_2 - \frac{w_5 d_5}{q}\right), \qquad l(w_3) = \frac{1}{2}\left(\frac{w_3}{d_1} + \frac{w_5}{q}\right) \tag{4.10}$$

and w_i's are as in the context of Theorem 3.3.

Taking $m = \eta = 16,\ \tau_1 = 32,\ \tau_2 = 32,\ \tau_3 = 128,\ \tau_4 = 512$ and the functions $\mu = \nu \equiv 1$ (this implies $c = 1$), by direct computation we have

$$M = \frac{7}{255}, \qquad q = 43180, \qquad d_1 = 960, \qquad d_4 = 42560, \qquad d_5 = 620.$$

Hence, the w_i's are numbers satisfying the relation

$$0 < w_1 \le w_2 \cdot Mc = \frac{7w_2}{255} < w_2 < w_3 < \frac{w_3}{Mc} = \frac{255w_3}{7} \le w_4 \le w_5 < \frac{q}{d_5}\ w_2 = \frac{2159w_2}{31}. \tag{4.11}$$

Moreover, we impose

$$w_5 > \frac{q}{d_1}\ w_3 = \frac{2159}{48}\ w_3. \tag{4.12}$$

We shall check the conditions of Theorem 3.3. It is obvious that (C1)–(C3), (C6) and (C8) are satisified. Next, condition (P) is obviously fulfilled. Noting that $d_1 < d_4$ and (4.12), we find for $0 \le x \le w_5$,

$$f(x) \le \max\{l(w_2), l(w_3)\} = l(w_3) = \frac{1}{2}\left(\frac{w_3}{d_1} + \frac{w_5}{q}\right) < \frac{1}{2}\left(\frac{w_5}{q} + \frac{w_5}{q}\right) = \frac{w_5}{q}.$$

Hence, condition (Q) is met. Finally, (R) is satisfied since for $w_3 \le x \le w_4$, we have by (4.12)

$$f(x) = \frac{1}{2}\left(\frac{w_3}{d_1} + \frac{w_5}{q}\right) > \frac{1}{2}\left(\frac{w_3}{d_1} + \frac{w_3}{d_1}\right) = \frac{w_3}{d_1}.$$

By Theorem 3.3 the boundary value problem (F) with $a = 2,\ b = 128$ and (4.9)–(4.12) has (at least) three positive solutions $y^1, y^2, y^3 \in \overline{C}(w_5)$ such that

$$\begin{aligned} &\max_{t\in[32,512]} y^1(t) = y^1(512) = \|y^1\| < w_2; \qquad y^2(t) > w_3,\ t \in [32,128]; \\ &\max_{t\in[32,512]} y^3(t) = y^3(512) = \|y^3\| > w_2 \qquad \text{and} \qquad \min_{t\in[32,128]} y^3(t) = y^3(32) < w_3. \end{aligned} \tag{4.13}$$

As a further illustration, fix

$$w_2 = 37, \qquad w_3 = 38, \qquad w_5 = 1710 \qquad \text{and}$$

$$\text{any } w_1,\ w_4 \text{ such that } 0 < w_1 \le \frac{7w_2}{255}, \quad \frac{255w_3}{7} \le w_4 \le w_5.$$

Clearly, (4.11) and (4.12) hold. The boundary value problem (F) with $a = 2,\ b = 32$, (4.9), (4.10) indeed has three positive solutions $y^1, y^2, y^3 \in \overline{C}(1710)$ that satisfy (4.13). They are tabulated as follows:

t	2	4	8	16	32	64	128	256	512
$y^1(t)$	0	0.0743	0.2217	0.5118	1.0733	2.1215	3.9183	6.3141	6.3141
$y^2(t)$	0	19.9552	59.8644	138.4160	290.4512	574.2502	1060.7629	1709.4464	1709.4464
$y^3(t)$	0	0.3161	0.9471	2.2045	4.7005	9.6176	19.1523	37.0239	37.0239

Remark 4.2. Examples 4.1–4.3 illustrate Theorems 3.1–3.3 well, particularly the estimates on the norms of the three solutions (see (3.14), (3.23) and (3.32)). We remark that examples on a general $\mathbb{T}$ can be similarly constructed.

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