# 行政院國家科學委員會專題研究計畫　成果報告

## 卷積式微分方程單突出與複突出解之存在性(2/2)

計畫類別：個別型計畫
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計畫主持人：張慧京

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中文摘要:

我們將某卷積式微分方程中單峰解的存在性轉換為某算子的固定點. 利用計算 Frechet 導函數與中心流型定理來探討解的穩定性.

關鍵詞:

中心流型定理, 穩定性, Frechet 導函數

# Abstract

The Fréchet derivative of a certain operator was computed . Then the Center Manifold Theorey in a certain Banach space were used to analyze the stability of the one bump solution of a convolution type of equation.



Keyword: Certer Manifold Theory, stability, Fréchet derivative.


# 1 Report

In this project we study the stability of the one bump stationary solution of the following convolution type of equation :

$$\frac{\partial u(x,t)}{\partial t} = -u(x,t) + \int_{-\infty}^{\infty} w(x-y) f(u(y,t))\, dy + s(x,t) + h \tag{1.1}$$

This project is the continuation of the work that was reported in the mid-term report which showed the existence of the one bump solution of (1.1). We will summarize as follows:

It was shown in Amari [1] that if f(u) is the Heaviside function and the coupling function $w(x)$ satisfying the following hypothesis:

(H1) $w(x)$ is a continuous even function on $\mathbf{R}$ and $\int_{-\infty}^{\infty} w(y)dy$ is finite.

(H2) $w(x) > 0$ on an interval $(-\bar{x}, \bar{x})$ and $w(-\bar{x}) = w(\bar{x}) = 0$.

(H3) $w(x)$ is decreasing on $(0, \bar{x}]$.

(H4) $w(x) < 0$ on $(-\infty, \bar{x}) \cup (\bar{x}, \infty)$.

Then for any h satisfying $0 < W_\infty < -h < W_m$,where

$$W(x) = \int_0^x w(t)dt, \quad \forall x \in \mathbf{R} \tag{1.2}$$

and

$$W_m = \max_{x>0} W(x) \qquad \text{and} \qquad W_\infty = \lim_{x\to\infty} W(x),$$

there exist $a_1 < a_2$ such that

$$u_i(x) = W(x) - W(x - a_i) + h \tag{1.3}$$

$i = 1, 2$, are stationary solutions of (1.1) if and only if

$$W(a_i) = -h \tag{1.4}$$

$i = 1, 2$. Furthermore , it can be easily seen that for $i = 1, 2$,

$$u_i'(0) = -u_i'(a_i)$$

Let

$$R(u) = \{x|u(x) > 0\}$$

denote the excited region of the stationary solution $u$. A 1-bump solution is a solution whose excited region is a finite open interval, that is $R(u) = (b_1, b_2)$.

To study the stabilities of these two solutions, we formulate the problem as follows. Let

$$B = \{u|u : \mathbf{R} \to \mathbf{R}, u \text{ is } C^1 \text{ and } \|u\| < \infty\}$$

be the Banach space equipped with the sup norm, that is $\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty\}$, where $\|u\|_\infty = \sup_{x\in\mathbf{R}} |u(x)|$ and $\|u'\|_\infty = \sup_{x\in\mathbf{R}} |u'(x)|$. Let $w(x)$ be a $C^1$ function satisfying (H1)-(H4), define an operator $\mathcal{F}$ from $B$ to $B$ by

$$\mathcal{F}(u)(x) = \int_{-\infty}^{\infty} w(x - y) f(u(y)) dy + h \tag{1.5}$$

One can see that if $u$ is a stationary solution of (1.1), then $u$ satisfies $\mathcal{F}(u) = u$, and vice versa. To estimate the spectrum of the operator $\mathcal{F}$, we compute the Fréchet derivative of $\mathcal{F}$ at a 1-bump solution $u$ as follows.

**Lemma 1.1.** *Let $u \in B$ be a 1-bump function with excited region $R(u) = (0, a)$ and $\lim_{|x|\to\infty} u(x) = h$, $h < 0$, then $\mathcal{F}$ is Gateaux differentiable at $u$ and the Gateaux derivative of $\mathcal{F}$ at $u$ is given by*

$$d\mathcal{F}(u)v(x) = \frac{v(0)}{u'(0)}w(x) - \frac{v(a)}{u'(a)}w(x-a) \tag{1.6}$$

*for all $v \in B$ and $x \in \mathbf{R}$ .*

**Lemma 1.2.** *Let $u \in B$ be a 1-bump function with excited region $R(u) = (0, a)$. Suppose $u'(0) > 0$ and $u'(a) < 0$, then there exists $\delta > 0$ such that if $\|\theta - u\| < \delta$ , then $\theta$ is also a 1-bump function in $B$.*

**Theroem 1.3.** *$\mathcal{F}$ is Fréchet differentiable at $u$ and the Fréchet derivative of $\mathcal{F}$ at $u$ is equal to the Gateaux derivative of $\mathcal{F}$ at $u$.*

**Lemma 1.4.** *Let $u \in B$ be a 1-bump function of (1.1), then the spectrum of $D\mathcal{F}(u)$ is given by $\sigma(D\mathcal{F}(u)) = \{1, \frac{w(0)+w(a)}{w(0)-w(a)}\}$.*

In the following, we will use the center manifold theory to determine the stability of a one-bump solution. Recall the equation (1.1):

$$\frac{\partial u(x,t)}{\partial t} = -u(x,t) + \int_{-\infty}^{\infty} w(x-y)f(u(y,t))dy + s(x,t) + h$$

By letting $s(x,t) = 0$ and using the notation $\mathcal{F}$, this equation can be rewritten as:

$$\frac{\partial u(x,t)}{\partial t} = -u(x,t) + \mathcal{F}(u)$$

Let $u$ be a stationary smooth one-bump solution of $\mathcal{F}(u) = u$,then since the nonlinearity $\mathcal{F}$ is smooth in a small neighborhood of $u$ in the $C^1$ Banach space, thus the equation defines a smooth local flow in that neighborhood. Now observe that if $w(a) < 0$ then $\frac{w(0)+w(a)}{w(0)-w(a)} < 1$. This says that the spectreum of $-I + D\mathcal{F}(u)$ has one simple eigenvalue 0 and the rest has value less than 0. Therefore by applying the Center Manifold Theorem , there exists a one dimensional local center manifold through the one-bump solution. Furthermore, the local center manifold is asymptotically stable. Since any translate of $u$ is also a one-bump solution and the local center manifold contains every solution which remains in a small neighborhood for all time $t \in R$. This implies that the local center manifold is precisely the set of all translates of $u$ in a small neighborhood. This gives the following asymptotic phase theorem in terms of translates of $u$. It is interesting to compare our conclusion with the results in [Amari, 1977].

**Theroem 1.5.** *Let $u$ be a 1-bump solution of $\mathcal{F}(u) = u$ for which $R(u) = (0, a)$. If $w(a) > 0$, then every translate of $u$ has a one dimensional unstable leaf. Furthermore, this gives a smooth foliation of the center unstable manifold over the center manifold. If $w(a) < 0$, then $u$ is exponentially asymptotic stable with asymptotic phase, i.e., every solution whose initial data is near a translate $u_1$ of $u$ will converge exponentially to a translate $u_2$ of $u$ as $t \to \infty$ .*

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