行政院國家科學委員會專題研究計畫　成果報告

Vonoroi 圖、Delaunay 剖分、Quick Hull 算法 和一般性剖分算法之比較

計畫類別：個別型計畫
計畫編號：NSC94-2115-M-032-012-
執行期間：94 年 08 月 01 日至 95 年 07 月 31 日
執行單位：淡江大學數學系

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報告類型：精簡報告

處理方式：本計畫可公開查詢

中　華　民　國 95 年 11 月 1 日

Comparison of Regular Triangulation to Vonoroi Diagrams, Delaunay Triangulation, and Quick Hull Algorithm

Voronoi Diagram

Let S be a set of n points in $\mathbb{R}^d$. Voronoi diagram is a polyhedral partition of $\mathbb{R}^d$: for every $p \in S$, $\mathrm{vo}(p)$ is a polyhedron containing p (not necessarily bounded), and the distance between its point x and p is no longer than the distance between x and any point in $S \backslash \{p\}$, i.e.

$$\mathrm{vo}(p) := \left\{ x \in \mathbb{R}^d \middle| \mathrm{dist}(x,p) \leq \mathrm{dist}(x,q)\ \forall\, q \in S \backslash \{p\} \right\},$$

$\mathrm{vo}(p)$ is called *Voronoi cell* of p, its vertices is called the *Voronoi vertices* of S. The union of boundaries of all Voronoi cells of S form the *Voronoi diagram*, a partition of $\mathbb{R}^d$.

For any $v \in \mathbb{R}^d$, define the *nearest neighbor set of v in S* by

$$\mathrm{nb}(S,v) := \left\{ p \in S \backslash \{v\} \middle| \mathrm{dist}(v,p) \leq \mathrm{dist}(v,q)\ \forall\, q \in S \backslash \{v\} \right\}$$

Clearly, $\mathrm{nb}(S,v)$ contains only finitely many points of S. Equivalently, We can also define that a Voronoi vertex v is the one whose nearest neighbor set $\mathrm{nb}(S,v)$ contains maximal points of S.

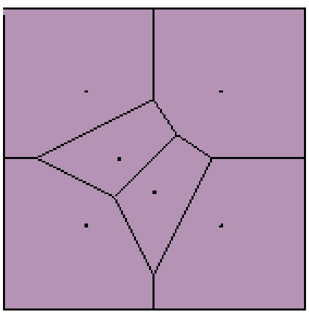

Figure 1: An example of 2-dimensional Voronoi Diagram

Delaunay Triangulation

Let S be a set of n points in $\mathbb{R}^d$. S gives a Voronoi diagram. Let v be a Voronoi vertex of S. Then $\text{conv}(\text{nb}(S, v))$ is called a *Delaunay cell* of v, contained in $\text{conv}(S)$. All the Delaunay cells of the Voronoi vertices of S, providing a partition of $\text{conv}(S)$, is called *Delaunay complex*。

Note that a Delaunay complex is not necessarily a triangulation (simplicial subdivision) of $\text{conv}(S)$. This occurs only when the number of points in S is great enough ($> d$) and points are distributed random enough (so-called *in general position*, this implies that no more than $d+1$ points on the sphere $S^{d-1} \subset \mathbb{R}^d$), $\text{nb}(S, v)$ will be affinely independent set of $d + 1$ points, and hence $\text{conv}(\text{nb}(S, v))$ a d-dimensional simplex.

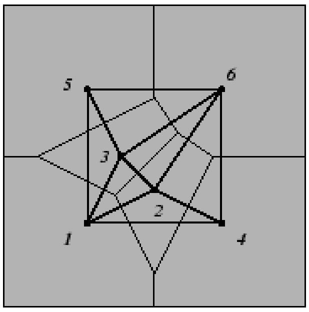

Figure 2: An example of 2-dimensional Delaunay Triangulation

Constructing a Voronoi Diagram

Embed every point $p \in S$ into $\mathbb{R}^{d+1}$ ($p = (p_1, \cdots, p_d) \in \mathbb{R}^d \hookrightarrow \hat{p} = (p_1, \cdots, p_d, p_{d+1}) \in \mathbb{R}^{d+1}$) with lifting value p_{d+1} 爲 $p_1^2 + \cdots + p_d^2$, i.e., every point of the lifted S, $\hat{S} = \{\hat{p} \in \mathbb{R}^{d+1} \big| \, p \in S\}$, is on the paraboloid $\mathscr{P} : x_{d+1} = x_1^2 + \cdots + x_d^2$. This can be check easily that hyperplane $\mathscr{H}_{\hat{p}}$ tangent to the paraboloid $\mathscr{P}$ at $\hat{p}$ is

$$\mathscr{H}_{\hat{p}} : x_{d+1} = \sum_{i=1}^{d} 2p_i x_i - \sum_{i=1}^{d} p_i.$$

Since $\mathscr{P}$ is the graph of a convex function, the half space

$$\mathscr{H}_{\hat{p}}^{+} : x_{d+1} \geq \sum_{i=1}^{d} 2p_i x_i - \sum_{i=1}^{d} p_i$$

contains the *epigraph* of $\mathscr{P}$:

$$\mathscr{P}^+ : x_{d+1} \geq x_1^2 + \cdots + x_d^2$$

The intersection of these n half spaces $\mathscr{H}_{\hat{p}}^+$ determines a system of inequalities $Ax \leq b$

$$\hat{Q} := \bigcap_{\hat{p} \in \hat{S}} \mathscr{H}_{\hat{p}}^+, \subset \mathscr{P}^+,$$

is an unbounded polyhedron, and a lower hull of $\hat{Q}$ (a d-face of $\hat{Q}$) projected back to $\mathbb{R}^d$, gives a Voronoi cell.

Constructing a Delaunay Complex

The same notations above in Voronoi Diagram construction, conv($\hat{S}$) contained in epigraph $\mathscr{P}^+$, is also an unbounded polyhedron, and a lower hull of conv($\hat{S}$) projected back to $\mathbb{R}^d$, gives a Delaunay cell.

Some words to these problems

Because Delaunay complex is completely determined by the Voronoi vertices, Delaunay Triangulation must be closely related to Voronoi Diagram, in fact, Delaunay complex is a dual of Voronoi diagram. This duality is defined and can be seen in linear programming. For us, resolving a Delaunay Triangulation problem is easier than Voronoi Diagram

1. $\mathscr{P}^+$ is a convex set, $\hat{S}$ is on $\mathscr{P} = \partial(\mathscr{P}^+)$, and every point of $\hat{S}$ must be a vertex of conv($\hat{S}$);

2. conv($\hat{S}$) can only described by its vertices, on the other hand, $\hat{Q}$ is represented by a system of linear inequalities. Though these two problems are dual to each other, under the circumstance of points of S in general position, every lower hull of conv($\hat{S}$) is d-simplex, very neat, however, lower hulls of $\hat{Q}$ can have different numbers of vertices. This means that it is hard to check if the diagram is obtained correctly.

3. The reason why we are interested in Delaunay Triangulation is because that this subdivision give every simplex a better shape, i.e. no obtuse angle in any simplex. However, when applying the lifting by parabolic

function, the inner normals of lower hulls are becoming hardly distinguished, i.e. lower hulls are not to close to the axis of symmetry of the paraboloid. Our overhead process is to scale and to shift the point set, then apply our one point test to the lifted set, simpler than mixed volume computation — the relation table is not “crossing-supports”, simply on one set, and built up simultaneously while pivoting in the linear programming model.

Unfortunately, we failed to find a good way to apply Quick Hull Algorithm to Delaunay triangulation this time, and coding in C++ is not ready yet. Lots more efforts should be done in the future.