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算子值 Gelfand-Levitan 方程修正模型及其應用(1/2)

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算子值 Gelfand-Levitan 方程修正模型及其應用 (1/2)

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一、中文摘要

在這篇文章裡, 我們藉由矩陣值的 Galfand-Levitan 方程來研究向量型 Sturm-Liouville 方程的譜反問題, 用這個方法, 我們證得向量型 Sturm-Liouville 方程的偶問題, Mixed-Data 問題的一些唯一定理 ...

關鍵詞：同譜問題、Sturm-Liouville 方程、譜反問題

Abstract

In this paper, we investigate inverse spectral problems for vectorial Sturm-Liouville equations via the matrix-valued Gelfand-Levitan equation. With this approach, we prove some uniqueness theorems for the even problem, mixed data problem and interior spectral data problem for vectorial Sturm-Liouville equations. ...

Keywords: Isospectral problems, Sturm-Liouville equations, inverse spectral problem

二、緣由與目的

The inverse spectral problems for scalar Sturm-Liouville equations is a quite interesting subject. One approach for these problems is via the Gelfand-Levitan equation. With the Gelfand-Levitan equation, one can construct all elements in isospectral set and prove some uniqueness properties. The other approach is by the Weyl M-function, it is quite powerful for us to prove the uniqueness theorems. But however, these two approaches are not well-developed for vectorial Sturm-Liouville equation as well as that for scalar Sturm-Liouville equations. Although, these are some researchers studied the inverse spectral problems for vectorial Sturm-Liouville equations by these two alternative methods, for examples, the reader can refer to (3), (5),(9) and (10). Unfortunately, neither the matrix-valued Gelfand-Levitan equation nor the generalized Weyl M-function for the vectorial Sturm-Liouville equation can completely solve the inverse spectral problem for vectorial cases.

三、結果與討論

Before go into details, we shall clarify some notations. Let $Q(x)$ be a continuous real symmetric $n\times n$ matrix-valued function on $[0,1]$, $\mathcal{A}$ and $\mathcal{B}$ be both real symmetric $n\times n$ matrices. A real number λ_0 is call an eigenvalue of multiplicity k of the following n-vectorial Sturm-Liouville equation

$$y^{''}(x)+(\lambda I_n-Q(x))y(x)=0,\ 0<x<1, \tag{0.1}$$

with boundary condition

$$\begin{cases} y^{'}(0)-\mathcal{A}y(0)=0, \\ y^{'}(1)+\mathcal{B}y(1)=0, \end{cases} \tag{0.2}$$

if the equation (0.1)-(0.2) has a non-trivial solution corresponding to $\lambda=\lambda_0$ and the geometric multiplicity of the corresponding eigenspace $E(\lambda_0)$ is k. From now on, we use the symbol $(Q,\mathcal{A},\mathcal{B})$ to denote the problem (0.1)-(0.2) and $\sigma(Q,\mathcal{A},\mathcal{B})=\cup_{n=1}^{\infty}\{(\lambda_n,k_n)\}$, $k_n=dim(E(\lambda_n))$, the spectral set of the problem $(Q,\mathcal{A},\mathcal{B})$, in particular, we use $\sigma_{\mathcal{D}}(Q)$ to denote the Dirichlet spectrum of the Dirichlet problem (Q,∞,∞) and $\sigma_{\mathcal{N}}(Q)$ the Neumann spectrum of the Neumann problem $(Q,I_n,0)$. Actually, if we let $\Phi(x,\lambda)$ denote the solution of the matrix-valued differential equation

$$Y^{''}(x,\lambda)+(\lambda I_n-Q(x))Y(x)=0,\ Y(0)=I_n,\ Y^{'}(0)=\mathcal{A}, \tag{0.3}$$

and $\mathcal{D}(\lambda)=\Phi^{'}(1,\lambda)+\mathcal{B}\Phi(1,\lambda)$. Then eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ of the problem $(Q,\mathcal{A},\mathcal{B})$ consist of the zeros of $\det\mathcal{D}(\lambda)$, and $dim(E(\lambda_n))=Nullity(\mathcal{D}(\lambda_n))$. While we talk about "isospectral problem " in scalar Sturm-Liouville equation, the definition is clear; but for the vectorial case, because of a technical reason, we have some restriction. Max Jodeit and B. M. Levitan formulated a definition as following

Definition 0.1 (see (3))**.** *Denote $Q_i(x)$ a continuous symmetric $n\times n$ matrix-valued function defined on $[0,1]$ and $\mathcal{A}_i$ and $\mathcal{B}_i$ two $n\times n$ real-valued symmetric matrices for $i=0,1$. We say the two problems $(Q_0,\mathcal{A}_0,\mathcal{B}_0)$ and $(Q_1,\mathcal{A}_1,\mathcal{B}_1)$ are isospectral problem if the following conditions are satisfied*

1. $\sigma(Q_0,\mathcal{A}_0,\mathcal{B}_0)=\sigma(Q_1,\mathcal{A}_1,\mathcal{B}_1)$,
2. *given an eigenfunction $\phi_0(x,\lambda)$ of problem $(Q_0,\mathcal{A}_0,\mathcal{B}_0)$, belonging to an eigenvalue λ, there exists an eigenfunction $\phi(x,\lambda)$ of problem $(Q_1,\mathcal{A}_1,\mathcal{B}_1)$, belonging to λ, such that*

   $$\phi_0(0,\lambda)=\phi_1(0,\lambda)\ (\phi^{'}(0,\lambda)=\phi_0^{'}(0,\lambda)\ \textit{for the Dirichlet problem }),$$

   and the converse also holds.

We should note that, according to definition 0.1, problems $(Q_0, \mathcal{A}_0, \mathcal{B}_0)$ and $(U^*Q_0U, U^*\mathcal{A}_0U, U^*\mathcal{B}_0U)$, where U is an unitary, maybe not isospectral problems, although these two problems can be viewed almost the same one. The following is an example.

Example 0.1. *Let* $Q_0(x) = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, $U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ *and* $Q_1 = U^*Q_0U$. *Then the Neumann problems* $(Q_0, I_n, 0)$ *and* $(Q_1, I_n, 0)$ *have the same Neumann eigenvalues*

$$\sigma_N(Q_0) = \sigma_N(Q_1) = \{n^2\pi^2 + 1, n^2\pi^2 + 3\}_{n=0}^{\infty}$$

but the eigenfunctions of problem $(Q_0, I_n, 0)$ *are of the form*

$$\{(1, 0)^T \cos(n\pi x),\ (0, 1)^T \cos(n\pi x)\}_{n=1}^{\infty}$$

whereas the eigenfunctions of problems are of the form

$$\{(1/\sqrt{2}, 1/\sqrt{2})^T \cos(n\pi x),\ (1/\sqrt{2}, -1/\sqrt{2})^T \cos(n\pi x)\}_{n=1}^{\infty}.$$

Hence, problems $(Q_0, I_n, 0)$ *and* $(Q_1, I_n, 0)$ *are not isospectral problems according to definition 0.1.*

This example suggest that, to investigate all the triples $(Q, \mathcal{B}, \mathcal{B})$ with the same prescribed spectrum, we shall have one more extensive definition, we may formulate the definition as following

Definition 0.2. *Let* $Q_i(x)$, $\mathcal{A}_i$ *and* $\mathcal{B}_i$, $i = 1, 2$, *are as that above. We say* $(Q_0, \mathcal{A}_0, \mathcal{B}_0)$ *and* $(Q_1, \mathcal{A}_1, \mathcal{B}_1)$ *are isospectral if* $\sigma(Q_0, \mathcal{A}_0, \mathcal{B}_0) = \sigma(Q_1, \mathcal{A}_1, \mathcal{B}_1)$.

Generally, definition 0.1 is different from definition 0.2, but if all the eigenvalues of the problems $(Q_i, \mathcal{A}_i, \mathcal{B}_i)$ are of full multiplicity n, then the two definitions coincide, in this case, the techniques developed by Jodeit and Levitan can be applied to inverse spectral problems for vectorial Sturm-Liouville equations. In the last section of this paper, we will use this approach to prove the following theorems

Theorem 3.1. *Let* $Q(x)$ *be a continuous symmetric even* $n \times n$ *matrix-valued function defined on* $[0, 1]$ *and* $\mathcal{A} \in M_n(\mathbb{R})$*isymmetric. Suppose that all eigenvalues of the problem are of full multiplicity. Then* $Q(x)$ *is uniquely determined by its spectrum* $\sigma(Q, \mathcal{A}, \mathcal{A})$.

Theorem 3.2. *Let* $Q(x)$ *be a continuous* $n \times n$ *matrix-valued function defined on* $[0, 1]$, $\mathcal{A}$ *and* $\mathcal{B}$ *be both symmetric* $n \times n$ *matrices over* $\mathbb{R}$.. *Suppose that* $Q(x)$ *is prescribed on* $[1/2, 1]$. *Then* $Q(x)$ *is uniquely determined by* $\sigma(Q, \mathcal{A}, \mathcal{B})$.

Proof. Let $Q_0(x) = q(x)I_n$ such that $\sigma_{\mathcal{D}}(Q_0) = \sigma_{\mathcal{D}}(Q) = \{(\lambda_k, n)\}_{k=1}^{\infty}$. Let $Y_Q(x,\lambda)$ be as that define in (**??**) and (**??**), then

$$Y_{Q_0}(x,\lambda) = y_q(x,\lambda)I_n,$$

where $y_q(x,\lambda)$ is the solution of

$$y''(x,\lambda) + (\lambda - q(x))y_q(x,\lambda) = 0,\ y_q(0,\lambda) = 0,\ y_q'(0,\lambda) = 1. \tag{3.4}$$

We denote $\mathcal{F}(x,t) = \sum_{k=1}^{\infty}\sum_{i=1}^{n}(\frac{1}{\alpha_k^2} - \frac{1}{\alpha_{k,0}^2})y_{0,i}(x,\lambda_k)y_{0,i}^*(t,\lambda_k)$, where

$$y_{0,j}(x,\lambda_k) = Y_{Q_0}(x,\lambda_k)e_j,\ \alpha_{k,0}^2 = \|y_{0,j}(x,\lambda_k)\|_2^2$$

and $\alpha_n^2 = \|Y_Q(x,\lambda_k)e_j\|_2^2$, for $j = 1,2,3,\ldots,n$. then as we mentioned in section 2, there exists a unique $\mathcal{K}(x,t)$ which satisfies

$$\mathcal{K}(x,t) + \mathcal{F}(x,t) + \int_0^x \mathcal{K}(x,s)\mathcal{F}(s,t)\,ds = 0,$$

and

$$Y_Q(x,\lambda) = Y_{Q_0}(x,\lambda) + \int_0^x \mathcal{K}(x,t)Y_{Q_0}(t,\lambda)\,dt. \tag{3.5}$$

Since $\mathcal{F}(x,t)$ is a diagonal matrix-valued function, $\mathcal{K}(x,t)$ is also diagonal. Hence $Q(x) = 2\frac{d}{dx}\mathcal{K}(x,x) + Q_0(x)$ is also diagonal. then, By the uniqueness theorem for scalar even Sturm-Liouville equation, we conclude $Q(x) = Q_0(x) = q(x)I_n$. □

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