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多元指數一般型 II 設限資料的完全統計推論與預測(2/2)

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中文摘要:

當存活資料或工業統計失敗資料是具有一元單參數或雙參數指數分配的一般型 II 設限資料時，文獻上並沒有適當的關鍵統計量的完全分配結果存在以供我們取得參數的完全區間估計或缺失值的預測區間．同樣的窘態也發生在多元單參數或雙參數指數分配的一般型 II 設限資料時．過去四十年，學者多以最大概似估計法處理這方面資料，但是往往須要特別程式解決，因而很多建立參數信賴區間及假設檢定研究問題多無法發展，結果多以逼近法來彌補．有鑑於此缺失,我們利用 Huffer and Lin (2001) 針對一般化留間隔線性組合的機率問題所提出的演算法，針對一元單參數或雙參數指數分配的一般型 II 設限樣本之最佳線性不偏估計式所形成的關鍵統計量計算出其實際的百分比數位置，進而建造出參數的完全區間區間或缺失值的預測區間．我們會將所得的結果和以最大概似估計式逼近法所建立區間區間或缺失值的預測區間相互比較．最後我們會呈現以所提出的推論計算方法所應用的實例.

計畫成果自評部分:

二年度所作研究已經獲得學術期刊接受.

1. Balakrishnan, N., Lin, C. T. and Chan, P. S. (2004). Exact inference and prediction for K-sample two-parameter exponential case under general Type-II censoring. Journal of Statistical Computation and Simulation. 74(12), 867–878. NSC-92-2118-M-032-001.
2. Balakrishnan, N. and Lin, C. T. (2005). Exact inference and prediction for k-sample exponential case under Type-II censoring. Journal of Statistical Computation and Simulation, 75(5), 315–331. NSC-93-2118-M-032-001.

Exact Inference and Prediction for K-Sample One-Parameter and Two-Parameter Exponential Case Under General Type-II Censoring

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Abstract

Exact inference and prediction intervals for the K-sample one-parameter and the two-parameter exponential case under general Type-II censored samples are derived using an algorithm of Huffer and Lin (2001). This approach provides a simple way to determine the exact percentage points of the pivotal quantities based on the best linear unbiased estimators in order to develop exact inference and prediction for the location and scale parameters as well as to construct exact prediction intervals for failure times unobserved in the i-th sample. Finally, we present an example to illustrate all the methods of inference developed in this paper.

Keywords and Phrases

Best linear unbiased estimator; exponential distribution; K-sample doubly Type-II censored sample; normalized spacings.

1 Introduction

Exponential distribution is a commonly used model in life-testing and reliability studies. The well-known property that the normalized spacings from an exponential distribution are independent and identically distributed as exponential allows the development of exact chi-square confidence intervals for the exponential scale parameter based on Type-II right censored samples. Unfortunately, such an exact distributional result can not be obtained for the best linear unbiased estimator (BLUE) of the scale parameter, or the location and scale parameters in the K-sample case under general Type-II censoring.

In this report, we develop exact confidence intervals for the one-parameter and the two-parameter cases, respectively, based on K-sample doubly Type-II censored exponential data by utilizing an algorithm of Huffer and Lin (2001). We also construct exact prediction intervals for failure times unobserved in the i-th sample. Finally, we illustrate all the methods of inference developed here.

2 Exact Inference and Prediction for the One-Parameter Case

We suppose that K independent doubly Type-II censored samples are available from an exponential distribution with probability density function $f(y;\sigma) = e^{-y/\sigma}/\sigma$, $y \geq 0$, $\sigma > 0$. Denote the total number of observations in the i-th sample by n_i, and the vector of order statistics observed from the i-th sample by $\boldsymbol{Y}_i = (Y_{i(r_i+1:n_i)}, Y_{i(r_i+2:n_i)}, \ldots, Y_{i(n_i-s_i:n_i)})^T$, $1 \leq r_i + 1 \leq n_i - s_i \leq n_i$, $i = 1, 2, \ldots, K$. Then, the vector $\boldsymbol{Y} = (\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_K)^T$ has its mean vector as $\boldsymbol{\mu} = E(\boldsymbol{Y}) = \sigma(\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_K)^T$ and variance-covariance matrix as $Var(\boldsymbol{Y}) = \sigma^2\boldsymbol{\Sigma} = \sigma^2\ \mathrm{Diag}(\boldsymbol{\Sigma}_1, \ldots, \boldsymbol{\Sigma}_K)$, where $\boldsymbol{\mu}_i = (\mu_{i(r_i+1:n_i)}, \ldots, \mu_{i(n_i-s_i:n_i)})^T$ with $\mu_{i(\ell:n_i)} = \sum_{j=1}^{\ell} 1/(n_i - j + 1)$, $1 \leq \ell \leq n_i$, and

$$\boldsymbol{\Sigma}_i = \begin{pmatrix} \sigma_{i(r_i+1,r_i+1:n_i)} & \sigma_{i(r_i+1,r_i+2:n_i)} & \cdots & \sigma_{i(r_i+1,n_i-s_i:n_i)} \\ \sigma_{i(r_i+1,r_i+2:n_i)} & \sigma_{i(r_i+2,r_i+2:n_i)} & \cdots & \sigma_{i(r_i+2,n_i-s_i:n_i)} \\ \vdots & \vdots & \cdots & \vdots \\ \sigma_{i(r_i+1,n_i-s_i:n_i)} & \sigma_{i(r_i+2,n_i-s_i:n_i)} & \cdots & \sigma_{i(n_i-s_i,n_i-s_i:n_i)} \end{pmatrix}$$

with $\sigma_{i(\ell,\ell:n_i)} = \sigma_{i(\ell,q:n_i)} = \sum_{j=1}^{\ell} 1/(n_i - j + 1)^2$, $1 \leq \ell \leq q \leq n_i$.

We can then apply the Gauss-Markov theorem to obtain the BLUE of σ. Specifically, we will minimize the expression of the generalized variance

$$(\boldsymbol{Y} - \boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{Y} - \boldsymbol{\mu}) = \sum_{i=1}^{K}(\boldsymbol{Y}_i - \sigma\boldsymbol{\mu}_i)^T\boldsymbol{\Sigma}_i^{-1}(\boldsymbol{Y}_i - \sigma\boldsymbol{\mu}_i)$$

with respect to σ, which yields the BLUE of σ as

$$\sigma^* = \frac{\sum_{i=1}^{K} \boldsymbol{\mu}_i^T\boldsymbol{\Sigma}_i^{-1}\boldsymbol{Y}_i}{\sum_{i=1}^{K} \boldsymbol{\mu}_i^T\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\mu}_i} = \frac{\sum_{i=1}^{K} w_i\sigma_i^*}{\sum_{i=1}^{K} w_i}, \tag{1}$$

where

$$\sigma_i^* = \frac{\boldsymbol{\mu}_i^T\boldsymbol{\Sigma}_i^{-1}\boldsymbol{Y}_i}{\boldsymbol{\mu}_i^T\boldsymbol{\Sigma}_i^{-1}\boldsymbol{\mu}_i} = \frac{1}{w_i}\sum_{j=r_i+1}^{n_i-s_i} a_{ij}Y_{i(j:n_i)}$$

with

$$a_{ij} = \begin{cases} \left[\displaystyle\sum_{\ell=n_i-r_i}^{n_i} \frac{1}{\ell}\right] \Big/ \left[\displaystyle\sum_{\ell=n_i-r_i}^{n_i} \frac{1}{\ell^2}\right] - (n_i - r_i - 1) & \text{for } j = r_i + 1, \\ 1 & \text{for } r_i + 2 \le j \le n_i - s_i - 1, \\ s_i + 1 & \text{for } j = n_i - s_i, \end{cases}$$

and

$$w_i = \boldsymbol{\mu}_i^T \boldsymbol{\Sigma}_i^{-1} \boldsymbol{\mu}_i = (n_i - r_i - s_i - 1) + \frac{\left[\displaystyle\sum_{\ell=n_i-r_i}^{n_i} \frac{1}{\ell}\right]^2}{\displaystyle\sum_{\ell=n_i-r_i}^{n_i} \frac{1}{\ell^2}} \quad \text{for } \; i = 1, \ldots, K.$$

Let us denote $S_{i(j)} = (n_i - j + 1)[Y_{i(j:n_i)} - Y_{i(j-1:n_i)}]$ for $j = 1, 2, \ldots, n_i - s_i$ and $i = 1, 2, \ldots, K$ with the convention that $Y_{i(0:n_i)} = 0$. Then, the BLUE of σ in (1) can be rewritten as

$$\sigma^* = \frac{\displaystyle\sum_{i=1}^{K} \sum_{j=1}^{n_i - s_i} c_{ij} S_{i(j)}}{W}, \tag{2}$$

where

$$c_{ij} = \begin{cases} \dfrac{1}{(n_i - j + 1)} \displaystyle\sum_{\ell=r_i+1}^{n_i-s_i} a_{i\ell} & \text{for } j = 1, \ldots, r_i + 1, \\ \dfrac{1}{(n_i - j + 1)} \displaystyle\sum_{\ell=j}^{n_i-s_i} a_{i\ell} & \text{for } j = r_i + 2, \ldots, n_i - s_i, \end{cases} \tag{3}$$

and

$$W = \sum_{i=1}^{K} w_i. \tag{4}$$

Using the properties of normalized spacings and independence of the samples, we can apply the algorithm of Huffer and Lin (2001) with $Z_{i(j)} = S_{i(j)}/\sigma$ to determine the exact value of t satisfying

$$P\left(\frac{\sigma^*}{\sigma} > t\right) = P\left(\frac{\displaystyle\sum_{i=1}^{K} \sum_{j=1}^{n_i-s_i} c_{ij} Z_{i(j)}}{W} > t\right) = \alpha \tag{5}$$

for any specified value of α.

In the special case when all the K-samples are Type-II right-censored (i.e., $r_i = 0$ for $i = 1, \ldots, K$), we have $w_i = n_i - s_i$ and

$$\frac{(n_i - s_i)\sigma_i^*}{\sigma} = \frac{\sum_{j=1}^{n_i-s_i} Y_{i(j:n_i)} + s_i Y_{i(n_i-s_i:n_i)}}{\sigma} = \sum_{j=1}^{n_i-s_i} Z_{i(j)},$$

which readily yields that $2(n_i - s_i)\sigma_i^*/\sigma$ has exactly a chi-square distribution with $2(n_i - s_i)$ degrees of freedom, and thus $2\sum_{i=1}^{K}(n_i - s_i)\sigma_i^*/\sigma$ has exactly a chi-square distribution with $2\sum_{i=1}^{K}(n_i - s_i)$ degrees of freedom.

By similar arguments, we can construct exact prediction intervals for the failure time unobserved in the i-th sample, viz. $Y_{i(h:n_i)}$ for $n_i - s_i < h \le n_i$, $i = 1, \ldots, K$, by finding the exact value of t such that

$$\begin{aligned} & P\left(\frac{Y_{i(h:n_i)} - Y_{i(n_i-s_i:n_i)}}{\sigma^*} > t\right) \\ = \;& P\left(\sum_{j=n_i-s_i+1}^{h} \frac{1}{n_i - j + 1}\, Z_{i(j)} - \frac{t}{W}\sum_{i=1}^{K}\sum_{j=1}^{n_i-s_i} c_{ij}\; Z_{i(j)} > 0\right) = \alpha \end{aligned} \tag{6}$$

with c_{ij} and W as defined in (3) and (4). Thus, for a specified α, we can determine values of t_1 and t_2 such that

$$P\left(\frac{Y_{i(h:n_i)} - Y_{i(n_i-s_i:n_i)}}{\sigma^*} > t_1\right) = \frac{\alpha}{2} \quad \text{and} \quad P\left(\frac{Y_{i(h:n_i)} - Y_{i(n_i-s_i:n_i)}}{\sigma^*} > t_2\right) = 1 - \frac{\alpha}{2};$$

then, an exact $100(1-\alpha)\%$ prediction interval for $Y_{i(h:n_i)}$ is $(Y_{i(n_i-s_i:n_i)} + t_2\sigma^*, Y_{i(n_i-s_i:n_i)} + t_1\sigma^*)$.

3 Exact Inference and Prediction for the Two-Parameter Case

We suppose that K independent doubly Type-II censored samples are available from an exponential distribution with probability density function $f(y; \mu, \sigma) = \frac{1}{\sigma}e^{-(y-\mu)/\sigma}$, $y \ge \mu$, $\sigma > 0$. Denote the total number of observations in the i-th sample by n_i, and the vector of order statistics observed from the i-th sample by $\boldsymbol{Y}_i = (Y_{i(r_i+1:n_i)}, Y_{i(r_i+2:n_i)}, \ldots, Y_{i(n_i-s_i:n_i)})^T$, $1 \le r_i + 1 \le n_i - s_i \le n_i$, $i = 1, 2, \ldots, K$; that is, r_i smallest and s_i largest order statistics have been censored in the i-th sample. Then, the vector $\boldsymbol{Y} = (\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_K)^T$ has its mean vector as $\boldsymbol{\mu} = E(\boldsymbol{Y}) = (\mu\mathbf{1} + \sigma\boldsymbol{\alpha}_1, \mu\mathbf{1} + \sigma\boldsymbol{\alpha}_2, \ldots, \mu\mathbf{1} + \sigma\boldsymbol{\alpha}_K)^T$

and variance-covariance matrix as $Var(\boldsymbol{Y}) = \sigma^2 \boldsymbol{\Sigma} = \sigma^2 \ \text{Diag}(\boldsymbol{\Sigma}_1, \ldots, \boldsymbol{\Sigma}_K)$, where $\boldsymbol{\alpha}_i = (\mu_{i(r_i+1:n_i)}, \ldots, \mu_{i(n_i-s_i:n_i)})^T$ with $\mu_{i(\ell:n_i)} = \sum_{j=1}^{\ell} 1/(n_i - j + 1)$, $r_i + 1 \le \ell \le n_i - s_i$, and

$$\boldsymbol{\Sigma}_i = \begin{pmatrix} \sigma_{i(r_i+1,r_i+1:n_i)} & \sigma_{i(r_i+1,r_i+2:n_i)} & \cdots & \sigma_{i(r_i+1,n_i-s_i:n_i)} \\ \sigma_{i(r_i+1,r_i+2:n_i)} & \sigma_{i(r_i+2,r_i+2:n_i)} & \cdots & \sigma_{i(r_i+2,n_i-s_i:n_i)} \\ \vdots & \vdots & \cdots & \vdots \\ \sigma_{i(r_i+1,n_i-s_i:n_i)} & \sigma_{i(r_i+2,n_i-s_i:n_i)} & \cdots & \sigma_{i(n_i-s_i,n_i-s_i:n_i)} \end{pmatrix}$$

with $\sigma_{i(\ell,\ell:n_i)} = \sigma_{i(\ell,q:n_i)} = \sum_{j=1}^{\ell} 1/(n_i - j + 1)^2$, $r_i + 1 \le \ell \le q \le n_i - s_i$; see, for example, David (1981) and Arnold, Balakrishnan and Nagaraja (1992).

We minimize the generalized variance $W = (\boldsymbol{Y} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{Y} - \boldsymbol{\mu})$ to obtain the BLUEs of μ and σ as

$$\mu^* = \frac{\left(\sum_{i=1}^K \boldsymbol{\alpha}_i^T \boldsymbol{\Sigma}_i^{-1} \boldsymbol{\alpha}_i\right)\left(\sum_{i=1}^K \mathbf{1}^T \boldsymbol{\Sigma}_i^{-1} \boldsymbol{Y}_i\right) - \left(\sum_{i=1}^K \mathbf{1}^T \boldsymbol{\Sigma}_i^{-1} \boldsymbol{\alpha}_i\right)\left(\sum_{i=1}^K \boldsymbol{\alpha}_i^T \boldsymbol{\Sigma}_i^{-1} \boldsymbol{Y}_i\right)}{\left(\sum_{i=1}^K \mathbf{1}^T \boldsymbol{\Sigma}_i^{-1} \mathbf{1}\right)\left(\sum_{i=1}^K \boldsymbol{\alpha}_i^T \boldsymbol{\Sigma}_i^{-1} \boldsymbol{\alpha}_i\right) - \left(\sum_{i=1}^K \mathbf{1}^T \boldsymbol{\Sigma}_i^{-1} \boldsymbol{\alpha}_i\right)^2} \tag{7}$$

and

$$\sigma^* = \frac{\left(\sum_{i=1}^K \mathbf{1}^T \boldsymbol{\Sigma}_i^{-1} \mathbf{1}\right)\left(\sum_{i=1}^K \boldsymbol{\alpha}_i^T \boldsymbol{\Sigma}_i^{-1} \boldsymbol{Y}_i\right) - \left(\sum_{i=1}^K \mathbf{1}^T \boldsymbol{\Sigma}_i^{-1} \boldsymbol{\alpha}_i\right)\left(\sum_{i=1}^K \mathbf{1}^T \boldsymbol{\Sigma}_i^{-1} \boldsymbol{Y}_i\right)}{\left(\sum_{i=1}^K \mathbf{1}^T \boldsymbol{\Sigma}_i^{-1} \mathbf{1}\right)\left(\sum_{i=1}^K \boldsymbol{\alpha}_i^T \boldsymbol{\Sigma}_i^{-1} \boldsymbol{\alpha}_i\right) - \left(\sum_{i=1}^K \mathbf{1}^T \boldsymbol{\Sigma}_i^{-1} \boldsymbol{\alpha}_i\right)^2}. \tag{8}$$

Let us denote $N = \sum_{i=1}^K n_i, R = \sum_{i=1}^K r_i, S = \sum_{i=1}^K s_i, Q = \sum_{i=1}^K q_i, T = \sum_{i=1}^K t_i$ and $U = \sum_{i=1}^K u_i$, where $q_i = \left(\sum_{\ell=n_i-r_i}^{n_i} \frac{1}{\ell}\right)^2 / \left(\sum_{\ell=n_i-r_i}^{n_i} \frac{1}{\ell^2}\right)$, $t_i = \left(\sum_{\ell=n_i-r_i}^{n_i} \frac{1}{\ell}\right) / \left(\sum_{\ell=n_i-r_i}^{n_i} \frac{1}{\ell^2}\right)$, and $u_i = 1/\left(\sum_{\ell=n_i-r_i}^{n_i} \frac{1}{\ell^2}\right)$ for $i = 1, 2, \ldots, K$. Observe that $t_i^2 = q_i u_i$ for $i = 1, 2, \ldots, K$. Since $\boldsymbol{\alpha}_i^T \boldsymbol{\Sigma}_i^{-1} \boldsymbol{\alpha}_i = n_i - r_i - s_i - 1 + q_i$, $\mathbf{1}^T \boldsymbol{\Sigma}_i^{-1} \mathbf{1} = u_i$, and $\mathbf{1}^T \boldsymbol{\Sigma}_i^{-1} \boldsymbol{\alpha}_i = t_i$, we readily have $A_1 = \sum_{i=1}^K \boldsymbol{\alpha}_i^T \boldsymbol{\Sigma}_i^{-1} \boldsymbol{\alpha}_i = N - R - S - K + Q$, $A_2 = \sum_{i=1}^K \mathbf{1}^T \boldsymbol{\Sigma}_i^{-1} \boldsymbol{\alpha}_i = T$, $A_3 = \sum_{i=1}^K \mathbf{1}^T \boldsymbol{\Sigma}_i^{-1} \mathbf{1} = U$, and $D = A_1 A_3 - A_2^2$. Note that D is positive since $T^2 = \left(\sum_{i=1}^K t_i\right)^2 = \left(\sqrt{q_i}\sqrt{u_i}\right)^2 \le QU$ by Cauchy-Schwarz inequality. Also, we find that

$$(\mathbf{1}^T \boldsymbol{\Sigma}_i^{-1})_j = \begin{cases} u_i & \text{for } j = r_i + 1, \\ 0 & \text{for } r_i + 2 \le j \le n_i - s_i, \end{cases}$$

and

$$(\boldsymbol{\alpha}_i^T \boldsymbol{\Sigma}_i^{-1})_j = \begin{cases} t_i - (n_i - r_i - 1) & \text{for } j = r_i + 1, \\ 1 & \text{for } r_i + 2 \le j \le n_i - s_i - 1, \\ s_i + 1 & \text{for } j = n_i - s_i, \end{cases}$$

which readily yield

$$\sum_{i=1}^K \mathbf{1}^T \boldsymbol{\Sigma}_i^{-1} \boldsymbol{Y}_i = u_1 Y_{1(r_1+1:n_1)} + u_2 Y_{2(r_2+1:n_2)} + \ldots + u_K Y_{K(r_K+1:n_K)}$$

and

$$\sum_{i=1}^{K} \boldsymbol{\alpha}_i^T \boldsymbol{\Sigma}_i^{-1} \boldsymbol{Y}_i = \sum_{i=1}^{K} \left\{ [t_i - (n_i - r_i - 1)] Y_{i(r_i+1:n_i)} + \sum_{j=r_i+2}^{n_i-s_i-1} Y_{i(j:n_i)} + (s_i+1) Y_{i(n_i-s_i:n_i)} \right\}.$$

Substituting all these expressions into (7) and (8), we obtain the BLUEs μ^* and σ^* as

$$\begin{aligned} \mu^* &= \frac{\sum_{i=1}^{K} [(N-R-S-K+Q)u_i - Tt_i] Y_{i(r_i+1:n_i)}}{D} \\ &\quad - \frac{T \sum_{i=1}^{K} [\sum_{j=r_i+2}^{n_i-s_i-1} Y_{i(j:n_i)} + (s_i+1) Y_{i(n_i-s_i:n_i)} - (n_i - r_i - 1) Y_{i(r_i+1:n_i)}]}{D} \end{aligned} \quad (9)$$

and

$$\begin{aligned} \sigma^* &= \frac{U \sum_{i=1}^{K} \left\{ [t_i - (n_i - r_i - 1)] Y_{i(r_i+1:n_i)} + \sum_{j=r_i+2}^{n_i-s_i-1} Y_{i(j:n_i)} + (s_i+1) Y_{i(n_i-s_i:n_i)} \right\}}{D} \\ &\quad - \frac{T \sum_{i=1}^{K} u_i Y_{i(r_i+1:n_i)}}{D}. \end{aligned} \quad (10)$$

Denote $S_{i(j)} = (n_i - j + 1)\left(Y_{i(j:n_i)} - Y_{i(j-1:n_i)}\right)$ for $j = 1, 2, \ldots, n_i - s_i$ and $i = 1, 2, \ldots, K$, with the convention that $Y_{i(0:n_i)} = \mu$. By the spcaings property mentioned earlier in Section 1, we readily have these $S_{i(j)}$'s to be i.i.d. exponential (with scale paramter σ) random variables. Then, the BLUEs of μ and σ in (9) and (10) can be simplified and rewritten as

$$\mu^* = \mu + \frac{\sum_{i=1}^{K} \sum_{j=1}^{r_i+1} \frac{(N-R-S-K+Q)u_i - Tt_i}{n_i-j+1} S_{i(j)} - T \sum_{i=1}^{K} \sum_{j=r_i+2}^{n_i-s_i} S_{i(j)}}{D}$$

and

$$\sigma^* = \frac{U \sum_{i=1}^{K} \sum_{j=r_i+2}^{n_i-s_i} S_{i(j)} - \sum_{i=1}^{K} \sum_{j=1}^{r_i+1} \frac{Tu_i - Ut_i}{n_i-j+1} S_{i(j)}}{D}.$$

The variances and covariance of the BLUEs μ^* and σ^* are obtained as

$$\begin{aligned} Var(\mu^*) &= \frac{U\sigma^2}{D}, \\ Cov(\mu^*, \sigma^*) &= \frac{-T\sigma^2}{D}. \end{aligned}$$

Using the property of normalized spacings and independence of the samples, we can apply the algorithm outlined in Section 2 to determine the exact value of d satisfying

$$P\left(\frac{\sigma^*}{\sigma} > d\right) = P\left(\frac{U \sum_{i=1}^{K} \sum_{j=r_i+2}^{n_i-s_i} Z_{i(j)} - \sum_{i=1}^{K} \sum_{j=1}^{r_i+1} \frac{Tu_i - Ut_i}{n_i-j+1} Z_{i(j)}}{D} > d\right) = \alpha, \quad (11)$$

and

$$
\begin{aligned}
& P\left(\frac{\mu^*-\mu}{\sigma^*} > d\right) \\
= \; & P\left(\sum_{i=1}^{K}\sum_{j=1}^{r_i+1}\frac{(N-R-S-K+Q+Td)u_i-(T+Ud)t_i}{n_i-j+1}Z_{i(j)}\right. \\
& \left. -\sum_{i=1}^{K}\sum_{j=r_i+2}^{n_i-s_i}(T+Ud)Z_{i(j)} > 0\right) = \alpha \qquad (12)
\end{aligned}
$$

for any $0 < \alpha < 1$, where $Z_{i(j)}$'s are i.i.d. standard exponential random variables for $i = 1, \ldots, K$ and $j = 1, \ldots, n_i - s_i$.

In a similar manner, we can construct exact prediction intervals for a failure time unobserved in the i-th sample, viz. $Y_{i(h:n_i)}$ for $n_i - s_i < h \leq n_i$, $i = 1, \ldots, K$, by finding the exact value of d such that

$$
\begin{aligned}
\alpha &= P\left(\frac{Y_{i(h:n_i)}-Y_{i(n_i-s_i:n_i)}}{\sigma^*} > d\right) \\
&= P\left(\sum_{j=n_i-s_i+1}^{h}\frac{1}{n_i-j+1}Z_{i(j)}\right. \\
&\quad \left. -\frac{d}{D}\left[U\sum_{i=1}^{K}\sum_{j=r_i+2}^{n_i-s_i}Z_{i(j)}-\sum_{i=1}^{K}\sum_{j=1}^{r_i+1}\frac{Tu_i-Ut_i}{n_i-j+1}Z_{i(j)}\right] > 0\right). \qquad (13)
\end{aligned}
$$

Thus, for a specified α, we can determine values of d_1 and d_2 such that

$$
P\left(\frac{Y_{i(h:n_i)}-Y_{i(n_i-s_i:n_i)}}{\sigma^*} > d_1\right) = \frac{\alpha}{2} \quad \text{and} \quad P\left(\frac{Y_{i(h:n_i)}-Y_{i(n_i-s_i:n_i)}}{\sigma^*} > d_2\right) = 1-\frac{\alpha}{2};
$$

then, an exact $100(1-\alpha)\%$ prediction interval for the unobserved failure time $Y_{i(h:n_i)}$ is given by $(Y_{i(n_i-s_i:n_i)} + d_2\sigma^*, Y_{i(n_i-s_i:n_i)} + d_1\sigma^*)$.

4 Numerical Illustration

For the purpose of the illustration, we only present the location and scale parameter exponential case. Consider Nelson's data (1982, Ch. 10, Table 4.1) which give 60 times to breakdown in minutes of an insulating fluid subjected to high voltage stress. The failure times were observed in the form of six groups each with ten insultating fluids. For the purpose of illustrating the methods of inference detailed in the previous sections, we consider the following six doubly Type-II censored samples with $(n_i, r_i, s_i), i = 1, \ldots, 6$, taken respectively to be $(10, 2, 1), (10, 1, 1), (10, 1, 1), (10, 1, 1), (10, 1, 2), (10, 1, 2)$:

Group 1	–	–	1.54	1.70	1.82	1.89	2.17	2.24	4.03	–
Group 2	–	0.18	0.55	0.66	0.71	1.30	1.63	2.17	2.75	–
Group 3	–	0.64	0.82	0.93	1.08	1.99	2.06	2.15	2.57	–
Group 4	–	0.06	0.50	0.70	1.17	2.80	3.57	3.72	3.82	–
Group 5	–	0.78	0.80	1.08	1.13	2.44	3.17	5.55	–	–
Group 6	–	1.49	1.56	2.10	2.12	3.83	3.97	5.13	–	–

For the above data, we find the maximum likelihood estimates of μ and σ from to be $\hat{\mu} = -0.016$ and $\hat{\sigma} = 2.449$, respectively. Furthermore, through a Monte Carlo simulation study, the variance-covariance matrix of these MLEs is determined to be

$$Var\begin{pmatrix} \hat{\mu} \\ \hat{\sigma} \end{pmatrix} = \hat{\sigma}^2 \begin{pmatrix} 0.0081 & -0.00737 \\ -0.00737 & 0.02536 \end{pmatrix} = \begin{pmatrix} 0.04858 & -0.04420 \\ -0.04420 & 0.15210 \end{pmatrix}.$$

Now, upon using the asymptotic normality of the MLEs, we find an approximate 95% confidence interval for μ (based on MLEs) to be $\left[\hat{\mu} - 1.96\sqrt{0.04858}, \hat{\mu} + 1.96\sqrt{0.04858}\right] = [-0.416, 0.385]$, and an approximate 95% confidence interval for σ (based on MLEs) to be $\left[\hat{\sigma} - 1.96\sqrt{0.15210}, \hat{\sigma} + 1.96\sqrt{0.15210}\right] = [1.685, 3.213]$.

Next, we obtain the BLUEs of μ and σ as $\mu^* = 0.23812$ and $\sigma^* = 2.17461$ from (9) and (10). Applying recursions of Huffer and Lin (2001) to $\boldsymbol{A}$ in (11), we find $P\left(\frac{\sigma^*}{\sigma} > 0.71215\right) = 0.975$ and $P\left(\frac{\sigma^*}{\sigma} > 1.33555\right) = 0.025$ using which we obtain an exact 95% confidence interval for σ to be $\left[\frac{\sigma^*}{1.33555}, \frac{\sigma^*}{0.71215}\right] = [1.62825, 3.05358]$.

Next, applying recursions again to $\boldsymbol{A}$ in (12), we find $P\left(\frac{\mu^*-\mu}{\sigma^*} > -0.11249\right) = 0.975$ and $P\left(\frac{\mu^*-\mu}{\sigma^*} > 0.18210\right) = 0.025$ using which we obtain an exact 95% confidence interval for μ to be $[-0.15788, 0.48274]$.

It is important to mention that the confidence intervals based on BLUEs are exact while those based on MLEs are approximate. Yet, they are close numerically; furthermore, both confidence intervals for μ include 0, indicating that one may as well use a one-parameter exponential for the data at hand.

References

Huffer, F. W. and Lin, C. T. (2001). Computing the joint distribution of general linear combinations of spacings or exponential variates, *Statistica Sinica*, **11**, 1141–1157.

Nelson, W. (1982). *Applied Life Data Analysis*, New York: John Wiley & Sons.