行政院國家科學委員會專題研究計畫　成果報告

A Study on Some Polynomial Systems Associated with a Certain Family of Differential Equations

計畫類別：個別型計畫
計畫編號：NSC93-2115-M-032-011-
執行期間：93年08月01日至94年07月31日
執行單位：淡江大學數學系

計畫主持人：陳功宇
共同主持人：方仁駿

報告類型：精簡報告

處理方式：本計畫可公開查詢

中　華　民　國 94年10月31日

行政院國家科學委員會專題研究計劃成果報告

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計劃類別：☑ 個別計劃　　□ 整合型計劃

計劃編號：NSC93-2115-M-032-011

執行期間：93 年 8 月 1 日至 94 年 7 月 31 日

個別型計劃：計劃主持人：陳功宇

共同主持人 :方仁駿

整合型計劃：總計劃主持人：
子計劃主持人：

註：整合型計劃總報告與子計劃成果報告請分開編印各成一冊
，彙整一起繳送國科會。

處理方式：□可立即對外提供參考

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☑兩年後可對外提供參考

(必要時，本會得展延發表時限)

執行單位：淡江大學數學系

中華民國　94 年 10 月 31 日

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中文摘要

設 f=(f_j),1≤j≤k　為非零多項式序列且 deg(f_j)≤j.我們考慮下列微分方程式

$$(*)\ T(f)(y)=\sum_{j=1}^{k} f_j y^{(j)} = a_n y.$$

本計畫在研究

(1) 找出a_n與 f 的充要條件使得(*)對於所有 n 都恰有一首一 n 次多項式解 y=P_n(x,f)≠0(或 T(f)有一組簡單的多項式集特徵函數)並且找出它們的 Rodrigues 公式與一些生成函數.

(2) 當(1)成立時那些會是正交多項式並且找出它們的權值函數

(3) 當 $g_\alpha \to$f, T(f)(y) = a_ny 與 T(g_α)(y)=$b_n(\alpha)$y 都各有一組簡單的首一多項式(simple set of polynomials)解集時找出這些多項式集的一些極限關係

關鍵詞:. 生成函數, 正交多項式, Rodrigues 公式

Abstract

Let f=(f_j),1≤j≤k be sequence of nonzero polynomials and deg(f_j)≤j. We consider the following differential equations.

$$(*)\ T(f)(y)=\sum_{j=1}^{k} f_j y^{(j)} = a_n y.$$

The project studies the following:

(1) Find necessary and sufficient conditions on a_n and f such that (*) has exactly one nonzero solution y=P_n(x,f)≠0 of monic polynomials of degree n for all n. In this case, we want

to find their Rodrigue ” s formula and generating functions.

(2) If (1) is true, which are orthogonal polynomials and then find the corresponding weight function.

(3) If $g_\alpha \to$ f, T(f)(y) = a_ny and T(g_α)(y)=$b_n(\alpha)$y have exactly one nonzero solution y=P_n(x,f) and y=P_n(x, g_α) of monic polynomials of degree n for all n, then find their limit relation.

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key words: generating function, orthogonal polynomial, Rodrigue ” s formula.

Our result is the following

Main results

Theorem.

Let f and g be nonzero polynomials with $\deg f \le 2$ and $\deg g \le 1$.

Then $\phi_n(x) = C_n f(x)\exp(-\int \frac{g(x)dx}{f(x)})D^n\left[f^{n-1}(x)\exp(\int \frac{g(x)dx}{f(x)})\right]$ is a polynomial solution of differential equation $f(x)y'' + g(x)y' + \left[\frac{n}{2}(1-n)f''(x) - ng'(x)\right]y = 0$, $\deg \phi_n \le n$. Moreover,

(1). If $f(x) = (x-\alpha)(x-\beta)$, where $\alpha, \beta \in R$, and $\alpha \ne \beta$, then $\{\phi_n(x)\}_{n\ge 0}^{\infty}$ is a sequence of orthogonal polynomials of degree n with weight function $w(x) = \exp\left(\int \frac{g(x)f'(x)}{f(x)}dx\right)$ on the interval (α, β) and the following generating function hold.

$$\sum_{n=0}^{\infty} \frac{\phi_n(x)}{n!}t^n$$

$$= f(x)\exp(-h(x))[\frac{\exp h(\varsigma)}{f(\varsigma)} \cdot \frac{1}{t(\alpha+\beta) - \sqrt{t^2(\alpha^2 - 4\alpha\beta + \beta^2) + 2(\alpha\beta + \alpha + \beta - 2x)t + 1}}$$

$$+ \frac{\exp h(\eta)}{f(\eta)} \cdot \frac{1}{t(\alpha+\beta) + \sqrt{t^2(\alpha^2 - 4\alpha\beta + \beta^2) + 2(\alpha\beta + \alpha + \beta - 2x)t + 1}}]$$

where

$$\varsigma = \frac{(\alpha+\beta)t+1+\sqrt{t^2(\alpha^2-4\alpha\beta+\beta^2)+2(\alpha\beta+\alpha+\beta-2x)t+1}}{2t}$$

$$\eta = \frac{(\alpha+\beta)t+1-\sqrt{t^2(\alpha^2-4\alpha\beta+\beta^2)+2(\alpha\beta+\alpha+\beta-2x)t+1}}{2t}$$

and $h(z) = \int g(z)dz$.

(2). If $f(x)=1$ and $g(x)=px+q$, then $\{\phi_n(x)\}$ is a sequence of orthogonal polynomials of degree n with weight function $w(x)=\exp\left(\int \frac{g(x)f'(x)}{f(x)}dx\right)$ on the interval $(-\infty,\infty)$ if $p<0$ and $\{\phi_n(x)\}$ is not orthogonal if $p \geq 0$. Moreover, $\{\phi_n(x)\}$ has the following generating function $\sum_{n=0}^{\infty}\frac{\phi_n(x)}{n!}t^n = \exp(h(x+t)-h(x))$, where $h(z)=\int g(z)dz$.

(3). If $f(x)=x$ and $g(x)=px+q$, where $p>0$, $q>0$, then $\{\phi_n(x)\}$ is a sequence of orthogonal polynomials of degree n with weight function $w(x)=\exp\left(\int \frac{g(x)f'(x)}{f(x)}dx\right)$ on the interval $(0,\infty)$. Moreover, $\{\phi_n(x)\}$ has the following generating function $\sum_{n=0}^{\infty}\frac{\phi_n(x)}{n!}t^n = \exp\left(h(\frac{x}{1-t})-h(x)\right)$.

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