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Intermediate Report on
Linear and nonlinear Perron-Frobenius theory

The following are three of the four problems (or directions for research) proposed in my three-year research project:

Problem 2. Let K be a proper cone and let $\pi(K)$ be the cone of all matrices A such that $AK \subseteq K$. Let $G(A)$ denote the digraph with vertex set equal to the set of all extreme rays of K and such that for any extreme rays F_1, F_2 of K, (F_1, F_2) is an arc if and only if $\Phi(AF_1) \supseteq F_2$, where $\Phi(S)$ denotes the face of K generated by S. We call $A \in \pi(K)$ K-primitive if there exists a positive integer m such that $A^m(K \backslash \{0\}) \subseteq \operatorname{int} K$; in this case, the smallest such m will be referred to as the K-exponent of A. For $A \in \pi(K)$, study the K-primitivity and K-exponent of A in terms of its associated digraph $G(A)$.

Problem 3. Improve/complete the recent work of Zaslavasky, McDonald and Naqvi on the characterization of the Jordan structure of seminonnegative (eventually) nonnegative matrices and the Jordan structure of the peripheral spectrum of a nonnegative matrix.

Problem 4. Continue some of my not completed work on the spectral theory of linear cone-preserving maps and study the open problems given in my review paper [T3].

(I am using the reference numbers in the present reference list, not those in the reference list of the original research project.)

In the past year I have spent most of my time working on Problem 4 and obtained some good results. Together with Raphael Loewy of Technion (Israel Institute of Technology), I have also carried out some nice investigations on Problem 2. About Problem 3 I have got some partial or related results. In the following sections I'll describe the progress made on each of these problems.

1. Progress on Problem 4

As a matter of fact, I have hardly touched upon the open problems I posed in my review paper [T5]. In the main I have been trying to complete the work of [T5], which had started in the early 1990s. The work is going to be the seventh of a sequence of papers (namely, [TW]. [T1], [TS1], [TS2], [TS3] and [T4]) on what I call the Geometric Spectral Theory of Positive Operators (in finite dimensions). As considerably more new material has been found, I intend to alter the title of [T5] to "On local Perron-Frobenius theory".

In this work I first focus on the problem of constructing closed, pointed (and if possible, also full, and hence proper)cones that are invariant under a given real (square) matrix A. For that matter, I also treat the question of when for a given real matrix A and a finite number of vectors $x_1, \ldots, x_k$ there exists a closed pointed cone C which contains the given vectors and is invariant under A.

The answer to the question of when a given real matrix leaves invariant a proper cone is known and is provided by the following:

THEOREM A. *Let A be an $n \times n$ real matrix. Then there exists a proper cone K in $\mathbb{R}^n$ such that $A \in \pi(K)$ if and only if A satisfies the following set of conditions:*

(a) *$\rho(A)$, the spectral radius of A, is an eigenvalue of A.*

(b) *For each eigenvalue λ of A with modulus $\rho(A)$, $\nu_\lambda(A) \leq \nu_{\rho(A)}(A)$, where $\nu_\lambda(A)$ denotes the index of λ as an eigenvalue of A.*

In order to characterize the Perron-Schaefer condition on A by a geometric property directly associated with A, for each nonnegative integer k, Schneider [Sch] introduced the *intrinsic cone* $w_k(A)$ of A, which consists of all nonnegative linear combinations of $A^k, A^{k+1}, \ldots$. (In fact, Schneider formulated his results in terms of a complex matrix. But since "cone" is a real concept, for conceptual clarity I prefer to use real matrices.) He obtained the following:

THEOREM B. *Let $A \in \mathcal{M}_n(\mathbb{R})$, and let k be a nonnegative integer. Then the cone $\operatorname{cl} w_k(A)$ is pointed if and only if A satisfies the Perron-Schaefer condition.*

In this work we show that there is a natural simple way to construct examples of invariant proper cones for a matrix that satisfies the Perron-Schaefer condition if we use a "local version" of Theorem B and the concept of A-cyclic cone.

For any $A \in \mathcal{M}_n(\mathbb{R}), x \in \mathbb{R}^n$ and nonnegative integer k, we denote by $w_k(A, x)$ the convex cone $\operatorname{pos}\{A^k x, A^{k+1}x, \ldots\}$ and refer to it as an *A-cyclic cone*. It is easy to see that $\operatorname{cl} w_0(A, x)$ is an A-invariant closed cone containing x, and also that there is a closed, pointed A-invariant cone containing x if and only if the cone $\operatorname{cl} w_0(A, x)$ is pointed (then the latter cone is the smallest such A-invariant cone containing x). A fundamental question to ask is, when the cone $\operatorname{cl} w_0(A, x)$ is pointed. It turns out that the answer is provided by the concept of *local Perron-Schaefer condition*, which we introduced in [TS2]:

Given $A \in \mathcal{M}_n(\mathbb{C})$ and $0 \neq x \in \mathbb{C}^n$, we say A satisfies the *local Perron-Schaefer condition at* x if there is a generalized eigenvector y of A corresponding to $\rho_x(A)$ (the local spectral radius of A at x) that appears as a term in the representation of x as a sum of generalized eigenvectors of A, and moreover the order of the generalized eigenvector y is not less than that of any other generalized eigenvector that appears in the representation and corresponds to an eigenvalue with modulus $\rho_x(A)$.

I obtained the following:

Theorem 1.1. *Let $A \in \mathcal{M}_n(\mathbb{R})$ and let $0 \neq x \in \mathbb{R}^n$. Also let W_x denote the cyclic subspace of A relative to x. The following conditions are equivalent:*
(a) *A satisfies the local Perron-Schaefer condition at x.*
(b) *$A|_{W_x}$ satisfies the Perron-Schaefer condition.*
(c1) *For every nonnegative integer k, the convex cone $\operatorname{cl} w_k(A, x)$ is pointed.*
(c2) *For some nonnegative integer k, the convex cone $\operatorname{cl} w_k(A, x)$ is pointed.*
(d1) *For every nonnegative integerk, the convex cone $\operatorname{cl} w_k(A|_{W_x})$ is pointed.*
(d2) *For some nonnegative integer k, the convex cone $\operatorname{cl} w_k(A|_{W_x})$ is pointed.*
(e1) *There is a closed, pointed, convex cone C containing x such that $AC \subseteq C$.*
(e2) *There is a closed, pointed, convex cone C full in W_x such that $AC \subseteq C$.*

By considering the restriction map L_A on $\mathcal{M}_n(\mathbb{R})$ given by $L_A(X) = AX$, we recover Theorem B from Theorem 1.1.

I also obtained other theorems of the type similar to Theorem 1.1. More specifically, given a real matrix A, a nonzero vector x and a nonnegative integer k, I consider the problem of finding equivalent conditions for $w_k(A, x)$ (or its closure) to possess one or a combination of the following properties: (i) nonzero; (ii) pointed; (iii) not a linear subspace. The answers are given in terms of the representation of x as a sum of generalized eigenvectors of A; namely, whether in the representation there is a generalized eigenvector that corresponds to (i) a positive eigenvalue; (ii) a nonnegative eigenvalue; or (iii) zero and is of order greater than or equal to k.

The following observation is crucial:

Lemma 1.2. *Let $A \in \mathcal{M}_n(\mathbb{R})$ and let $x \in \mathbb{R}^n$. Suppose that A satisfies the local Perron-Schaefer condition at x. Then $E_{\rho_x(A)}^{(\nu_{\rho_x(A)}-1)}(A|_{W_x})x$ is the only eigenvector of A in $\operatorname{cl} w_0(A, x)$.*

Here we use $E_\lambda^{(k)}(A)$ to denote the component of A given by $E_\lambda^{(k)}(A) = (A - \lambda I)^k E_\lambda^{(0)}(A)$, where $E_\lambda^{(0)}(A)$ is the projection onto the generalized eigenspace of A corresponding to λ along the direct sum of generalized eigenspaces of A corresponding to other eigenvalues.

Concerning the closed, pointed invariant cones of a given matrix, I obtained, among other results, the following:

Theorem 1.3. *Let $A \in \mathcal{M}_n(\mathbb{R})$ and let $x_1, \ldots x_k, k \geq 2$ be vectors of $\mathbb{R}^n$ and suppose that A satisfies the local Perron-Schaefer condition at $x_1, \ldots, x_k$. Then there exists a closed, pointed cone (or a proper cone) in $\mathbb{R}^n$ that contains $x_1, \ldots, x_k$ and is invariant under A if and only if the cone $\mathrm{pos}\{E_{\rho_{x_i}}^{(\nu_{\rho_{x_i}}-1)}(A|_{W_x})x_i : i = 1, \ldots, k\}$ is pointed.*

Theorem 1.4. *Let $A \in \mathbf{M}_n(\mathbb{R})$ satisfy the Perron-Schaefer condition. Let m be the largest geometric multiplicity of the eigenvalues of A. Then there exists a proper cone K in $\mathbb{R}^n$ invariant under A which is the sum of the closures of m A-cyclic cones, but there does not exist an A-invariant proper cone which is the sum of the closures of less than m A-cyclic cones.*

I have also considered the question of characterizing matrices A for which there is a proper cone K such that $A \in \mathrm{Aut}(K)$ (i.e., $AK = K$) and obtained the following partial results:

Theorem 1.5. *Let $A \in \mathbf{M}_n(\mathbb{R})$ and let $0 \neq x \in \mathbb{R}^n$. Then $\mathrm{cl}\, w_0(A, x)$ is pointed and $A|_{W_x} \in \mathrm{Aut}(\mathrm{cl}\, w_0(A, x))$ if and only if $A|_{W_x}$ is nonzero, diagonalizable (over $\mathbb{C}$), all eigenvalues of $A|_{W_x}$ are of the same moduli and $\rho_x(A)$ is an eigenvalue of $A|_{W_x}$.*

Theorem 1.6. *Let $A \in \mathbf{M}_n(\mathbb{R})$ and let $x \in \mathbb{R}^n$. Consider the following conditions*:

(a) *$A|_{W_x}$ is nonsingular and the cone $\mathrm{cl}\,\mathrm{pos}\{(A|_{W_x})^i : i = 0, \pm 1, \pm 2, \ldots\}$ is pointed (and hence $A|_{W_x}$ is an automorphism of the cone).*

(b) *There exists a proper cone C in W_x containing x such that $A|_{W_x} \in \mathrm{Aut}(C)$.*

(c) *There exists a closed, pointed cone C containing x such that $A|_{\mathrm{span}\, C} \in \mathrm{Aut}(C)$.*

(d) *$A|_{W_x}$ is nonsingular, $A|_{W_x}$ and $(A|_{W_x})^{-1}$ both satisfy the Perron- Schaefer condition.*

(e) *Let $x = x_1 + \ldots + x_k$ be the representation of A as a sum of generalized eigenvectors of A corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ respectively. Then $\lambda_1, \ldots, \lambda_k$ are all nonzero, and there exist i, j such that $\lambda_i = \rho_x(A), \mathrm{ord}_A(x_i) = ord_A(x)$ and $\lambda_j = (\rho_x(A^{-1}))^{-1}, \mathrm{ord}_A(x_j) = \max\{\mathrm{ord}_A(x_l) : |\lambda_l| = \lambda_j\}$.*

Then (a), (b) *and* (c) *are equivalent, and so are* (d) *and* (e), *and furthermore we have* (a) $\Rightarrow$ (d).

I also obtained analogous results for the class of cross-positive matrices.

Given a proper cone K in $\mathbb{R}^n$ and an $n \times n$ real matrix A, we say A is *cross-positive* on K if we have $\langle y, Az \rangle \geq 0$ for all $y \in K$ and $z \in K^*$ such that $\langle y, z \rangle = 0$. Schneider and Vidyasagar [SV] and, independently, Elsner ([Els1], [Els2]), have proved theorems of Perron-Frobenius type for cross-positive matrices. They also obtained several equivalent conditions for cross-positive matrices, one of which says that $A \in \Sigma(K)$ if and only if A is exponentially K-nonnegative, i.e., exp(tA) $\in \pi(K)$ for all $t \geq 0$.

The following result, which is the analogue of Theorem A for cross-positive matrices, is obtained by Elsner ([Els 1], Satz 4.1):

THEOREM C. *For an $n \times n$ real matrix A, let $\xi(A)$ denote the spectral abscissa of A. The following conditions are equivalent*:

(a) $\xi(A) \in \sigma(A)$ *and* $\nu_{\xi(A)}(A) \geq \nu_\lambda(A)$ *for all* $\lambda \in \sigma(A)$ *with* $\Re\lambda = \xi(A)$.

(b) *There exists a proper cone K in $\mathbb{R}^n$ such that A is exponentially K-nonnegative.*

(By the spectral abscissa of A we mean the maximum of the real part of the eigenvalues of A.)

We refer to condition (a) of the preceding theorem as the ESV-condition.

We say $A \in \mathcal{M}_n(\mathbb{R})$ satisfies the *local ESV condition* at x if in the representation of x as a sum of generalized eigenvectors of A there is a generalized eigenvectors y corresponding to $\xi_x(A)$, the spectral abscissa of $A|_{W_x}$, and moreover the order of y is not less than that of any generalized eigenvector in the representation that corresponds to an eigenvalue with real part equal to $\xi_x(A)$.

I obtained, in particular, the following analogue of Theorem 1.1:

Theorem 1.7. *Let $A \in \mathcal{M}_n(\mathbb{R})$ and let $O \neq x \in \mathbb{R}^n$. The following conditions are equivalent*:

(a) *A satisfies the local ESV condition.*

(b) *$A|_{W_x}$ satisfies the ESV condition.*

(c) *The cone* $\mathrm{cl}(pos\{e^{tA}x : t \geq 0\})$ *is pointed.*

(d) *There exists a closed, pointed cone C containing x and full in W_x such that $A|_{W_x}$ is cross-positive on C.*

(e) *There exists a closed, pointed cone C full in W_x such that $A|_{W_x}$ is cross-positive on C.*

In the study of the geometric spectral theory of positive linear operators (in finite dimensions) we have introduced in [TS2] the concept of spectral pair of a vector relative to a general square complex matrix and also the concept of the spectral pair of a face relative to a K-nonnegative matrix. The spectral pair has proved to be a useful concept in the study of the spectral theory of a cone-preserving map. Now I have observed that if we consider spectral abscissa in place of spectral radius, then we can also introduce the concept of abscissa pair of a vector relative to a square complex matrix and also the concept of abscissa pair of a face relative to a matrix cross-positive on K. I also noticed that many of the results on the spectral theory of a cone-preserving map have analogues for cross-positive matrices.

I believe more work can be covered under the title "On local Perron-Frobenius theory". For instance, the present work may throw light on known results about cone reachability problem (see [BNS, Chapter 6]) or on the work of Friedland and Schneider [FR], Rothblum [Rot], and Denardo and Rothblum [DR] on asymptotic growth rate of powers of a nonnegative matrix. Further investigations will be carried out.

2. Progress on Problem 2

My work in connection with Problem 2 can be divided into two parts, the first part being done on my own and the second part with Raphael Loewy.

The first part of the work is concerned with aspects of the digraph for a cone-preserving of a more general nature. In particular, we examine the relation between the digraph $G(A)$ of A and the face $\Phi(A)$ of $\pi(K)$ generated by A. Below is a list of the major results I have obtained:

Theorem 2.1. *Let K be a proper cone, and let $A, B \in \pi(K)$. Consider the following conditions:*

1. $\Phi(A) \subseteq \Phi(B)$.
2. *$G(A)$ is a subdigraph of $G(B)$.*
3. *For all $x \in \operatorname{Ext} K$, $\Phi(Ax) \subseteq \Phi(Bx)$.*
4. *For all $x \in K$, $\Phi(Ax) \subseteq \Phi(Bx)$.*
5. *A belongs to the intersection of all simple faces of $\pi(K)$ that contain B.*
6. $\mathrm{cl}_{\pi(K)}(\Phi(A)) \subseteq \mathrm{cl}_{\pi(K)}(\Phi(B))$.

Conditions (b)–(e) *are equivalent and they always imply condition* (f) *and are implied by condition* (a).

Theorem 2.2. *Let K be a proper cone, and let $A, B \in \pi(K)$.*

(i) *If* $\mathrm{cl}_{\pi(K)}(\Phi(A)) \subseteq \mathrm{cl}_{\pi(K)}(\Phi(B))$ *and A is K-primitive, then B is also K-primitive and we have $\gamma(B) \leq \gamma(A)$.*

(ii) *If* $\mathrm{cl}_{\pi(K)}(\Phi(A)) = \mathrm{cl}_{\pi(K)}(\Phi(B))$, *then A and B are both K-primitive or both not K-primitive, and if they both are, then $\gamma(A) = \gamma(B)$.*

Theorem 2.3. *For a proper cone K, the following conditions are equivalent*:

(a) $d_{\pi(K)}$ *is injective.*

(b) *For any $A, B \in \pi(K)$, conditions* (a)-(g) *of Theorem 1 are equivalent.*

(c) *For any $A, B \in \pi(K)$, conditions* (a)-(g) *of Corollary 1 are equivalent.*

(d) d_K *is injective (or bijective), and each face of $\pi(K)$ can be written as an intersection of simple faces.*

(e) *Each face of $\pi(K)$ other than $\pi(K)$ itself can be written as an intersection of maximal faces.*

Theorem 2.4. *Let K be a proper cone. In order that for any $A, B \in \pi(K)$, we have*

$$G(A) = G(B) \quad \textit{iff} \quad G(A^T) = G(B^T)$$

it is necessary and sufficient that the duality operator d_K be bijective (which is the case if K is polyhedral).

Theorem 2.5. *Let K be a proper cone and let $A \in \pi(K)$. The following are equivalent statements*:

(a) *A is K-irreducible.*

(b) *The following conditions are both satisfied*:

(i) *For any final strong component $G(A)$, the join of all extreme rays which form the vertices of $\mathcal{C}$ is K.*

(ii) *For any $x \in \mathrm{Ext}\, K$, if the vertex $\Phi(x)$ has no access to a final strong component of $G(A)$, then the cone generated by all vertices of $G(A)$ which have access from $\Phi(x)$ intersects* $\mathrm{int}\, K$.

(c) *For any $x \in \mathrm{Ext}\, K$, the join of all extreme rays which have access from $\Phi(x)$ equals K.*

Theorem 2.6. *Let K be a polyhedral cone and let $A \in \pi(K)$. In order that A is K-primitive, it is necessary and sufficient that for any final strong component $\mathcal{C}$ of $G(A)$, either $\mathcal{C}$ is a primitive digraph and the join of all extreme rays which form the vertices of $\mathcal{C}$ is K, or $\mathcal{C}$ has index of imprimitivity greater than 1 and the join of all extreme rays in one (or, each) of the sets of imprimitivity of $\mathcal{C}$ is K.*

Theorem 2.7. *Let $K_1,\ K_2$ be proper cones each with a bijective duality operator. Then K_1 and K_2 are combinatorial equivalent if and only if there exists a bijection $\varphi : \mathcal{E}(K_1) \longrightarrow \mathcal{E}(K_2)$ such that for any positive integer p and any $E_1, \ldots, E_p \in \mathcal{E}(K_1)$, $E_1 \vee \cdots \vee E_p = K_1$ if and only if $\varphi(E_1) \vee \cdots \vee \varphi(E_p) = K_2$.*

I intend to write a paper on my own with title "Digraphs for cone-preserving maps revisited" on the above work.

The second part of my work on Problem 2 has been carried out jointly with Raphael Loewy and focuses on the exponents of K-primitive matrices. (We intend to write a joint paper entitled "On exponents of K-primitive matrices".) We have found that many of the classical results on the exponents of primitive nonnegative matrices (see, for instance, Section 3.5 of [BR]) have analogues in the cone-preserving map setting, but in this more general setting the situation is often more delicate/complicated.

We first answer in the affirmative the following conjecture posed by Steve Kirkland in 1999 at the 8th ILAS Conference held in Barcelona:

Conjecture If K is a polyhedral proper cone with m extreme rays, then for any K-primitive matrix $A \in \pi(K), \gamma(A) \leq m^2 - 2m + 2$, where $\gamma(A)$ denotes the exponent of A, i.e., the least positive integer p such that $A^p(K \backslash \{0\}) \subseteq \operatorname{int} K$.

In fact, we have obtained much better results. For a proper cone K, denote by $\gamma(K)$ the maximum of the exponents of K-primitive matrices if it exists. (We know $\gamma(K)$ is defined for every polyhedral cone K, but for some nonpolyhedral cones K, $\gamma(K)$ may not exist). After careful detailed analysis, we are able to show that when K is a polyhedral cone with m extreme rays, then $\gamma(K)$ is less than or equal to $m^2 - 3m + 3$ or $m^2 - 3m + 2$, depending on whether m is odd or even, and also that the bound(s) are attained for special kind of minimal cones.

We believe the problem of determining the exact value of $\gamma(K)$ for a given proper cone K is a worthwhile but difficult problem. The relatively easier and somewhat related problem of determining the critical exponent of a normed linear space has been considered by other people and elementary algebraic geometry has been employed as a tool in the investigation (see, for instance, [BL, Chapter 2, Section 6]). Here, we have considered in particular the problem of determining $\gamma(K_n)$ for the n-dimensional ice-cream cone K_n. Making use of the results of [LS], we are able to show that $\gamma(K_n)$ is equal to one of the two numbers n or $n-1$. Further work will be needed to determine the exact value of $\gamma(K_n)$.

I suspect our cone-theoretic approach can yield an easier alternative solution of the following conjecture posed by Hartwig and Neumann [HN] in 1993: If A is an $n \times n$ nonnegative primitive matrix whose minimal polynomial has degree m, then $\gamma(A) \leq m^2 - 2m + 2$. The conjecture was settled in the affirmative by Shen [she] in 1995. The proofs given in [HN] (for a partial result towards their conjecture) and in

[she] are graph-theoretic and involve lengthy tedious arguments. But the concept of exponent is not graph-theoretic, so I believe there is a cone-theoretic solution.

3. Progress on Problem 4

As I have said in my research project proposal, I want to complete the work of [McD], [NM1], [NM2], [ZM] and [ZT] (by McDonald, Naqvi, Zaslavsky and me) on eventually nonnegative matrices. They have used complicated matrix-theoretic methods, but I intend to use my favorite geometric (cone-theoretic) approach. So far I have made progress on three items, which may or may not help in the final solution of the problem. But at least I have got some good by-products in this investigation.

First, I am now able to to provide a conceptual geometric (operator-theoretic) proof for the result which is the starting point of the recent work on eventually nonnegative matrices, namely, the characterization of the Jordan form of an irreducible m-cyclic matrix whose mth power is permutationally similar to the direct sum of m eventually positive matrices. The characterization first appeared in [ZT], my joint paper with Boris Zaslavasky. The proof given there takes 10 pages and involves hard lengthy computations. My initial success gives some hope for my geometric approach.

Next, I have found a nice equivalent condition for the so-called extended Tam-Schneider condition, which was introduced by McDonald in [Mc], where she characterized the part of the Jordan form of a nonnegative matrix that is pertaining to its peripheral eigenvalues. After rewriting, the definition of the condition can be given as follows:

Definition. Let $\mathcal{J}$ be a collection of Jordan blocks all of whose eigenvalues have modulus one. Let m be the size of the largest Jordan block in $\mathcal{J}$. We say $\mathcal{J}$ satisfies the *extended Tam-Schneider condition* provided that there exist m collections of Jordan blocks $\mathcal{J}_1, \ldots \mathcal{J}_m$, each satisfying the Tam-Schneider condition and with $\mathcal{J}_m = \mathcal{J}$, such that for $j = 2, \ldots, m, \mathcal{J}_{j-1}$ can be obtained from $\mathcal{J}_j$ in the following way: Reduce the size by one for each maximal Jordan block and a select number of non-maximal Jordan blocks (and keeping the remaining blocks) such that the geometric spectrum of the subcollection of Jordan blocks whose sizes are reduced by one is the union of certain complete sets of roots of unity. (By the *geometric spectrum* of a collection of Jordan blocks we mean the multi-set consisting of the eigenvalues of the collection, the multiplicity of each element being equal to the number of Jordan blocks in the collection associated with that element.)

Let $\mathcal{J}$ be a collection of Jordan blocks. For any $\lambda \in \sigma(\mathcal{J})$ we denote by $\eta(\lambda)$ the height characteristic of the direct sum of Jordan blocks in $\mathcal{J}$ associated with λ.

Here is my equivalent condition:

Theorem 3.1. *Let $\mathcal{J}$ be a collection of Jordan blocks all of whose eigenvalues have modulus 1. Suppose that $\mathcal{J}$ satisfies the Perron-Schaefer condition and let m be the index of 1 in $\mathcal{J}$. Then $\mathcal{J}$ satisfies the extended Tam-Schneider condition if and only if for each $\lambda \in \sigma(\mathcal{J})$ there exists an m-tuple $\ell(\lambda) = (\ell_1(\lambda), \dots, \ell_m(\lambda))$ of nonnegative integers such that $\ell(\lambda) \preceq \eta(\lambda)^*$, and furthermore for $j = 1, \dots, m$, the multi-set $A_j := \{\lambda, \dots, \lambda \ (l_j(\lambda) \text{ times}), \ \lambda \in \sigma(\mathcal{J})\}$ is the union of certain complete sets of roots of unity.*

Third, using the operator-theoretic approach, I have solved a special case of the famous Carlson Problem and also got some new interesting by-products in the process. The famous Carlson Problem (see [Car]) asks for the relation between the Jordan form of a triangular block matrix and those of its square diagonal blocks. Clearly, it is somewhat related to the Inverse Elementary Divisor Problem for Nonnegative Matrices (or for eventually nonnegative matrices). Actually, the Carlson problem had been solved; it is a consequence of the known solution for the famous "$A = B + C$" problem, which asks for the relation between three n-tuples of real numbers (arranged in nonincreasing order) so that they are respectively the spectra of Hermitian matrices A, B and C. The solution of the latter problem is complicated; it relies on advanced tools such as algebraic geometry, Schubert calculus, etc (see [Ful]). It seems worthwhile to look for an elementary solution, but conceivably this is very hard. One might start with looking for an elementary solution of the relatively easier Carlson Problem first. I have chosen to take an operator-theoretic approach instead of the commonly adopted matrix-theoretic approach (cf. [JS] and [JSE]). For the 2×2 triangular block case of the Carlson Problem, I am now able to offer a geometric solution for the one-to-many subcase, i.e. when the Jordan form of the first diagonal block has only one elementary Jordan block while that of the second diagonal block may have more than one elementary diagonal block. (I believe my solution would shed light on the work of [JS] and [JSE].) Here is my answer:

Theorem 3.2. *Let A be a 2×2 block matrix of the form*

$$\begin{bmatrix} J_q(0) & \mathbf{O} \\ * & J_{p_1}(0) \oplus J_{p_2}(0) \oplus \cdots J_{p_t}(0) \end{bmatrix},$$

where $t, p_1, \dots p_t, q$ are positive integers. The possible Jordan forms of A include $J_q(0) \oplus J_{p_1}(0) \oplus \cdots \oplus J_{p_t}$, together with all the possible Jordan matrices that can be obtained in *the following manner. Choose* ℓ (≥ 1) *distinct integers* $i_1, \ldots, i_\ell$ *from* $\{1, \ldots, t\}$ *such that the corresponding* p_{i_j}*s satisfy* $p_{i_{j+1}} - p_{i_j} \geq 2$ *for* $j = 1, \ldots \ell - 1$ *and for which the following condition can be met: there exist integers* $a_i, b_i, i = 1, \ldots, \ell$ *such that* $0 \leq a_1 < a_2 < \cdots < a_\ell \leq q, 1 \leq b_1 < b_2 < \cdots < b_\ell$ *and* $p_{i_k} = a_k + b_k$ *for* $k = 1, \ldots \ell$. *Then form the Jordan matrix* $J_{a_1}(0) \oplus J_{a_2+b_1}(0) \oplus \cdots \oplus J_{a_\ell+b_{\ell-1}}(0) \oplus J_{q+b_\ell}(0) \oplus \bigoplus_j J_{p_j}(0)$, *where the* j *in the last summand runs through all integers from* 1 *to* t *that are different from* $i_1, \ldots, i_\ell$.

In the process of establishing the above result, I determine for a given $n \times n$ complex matrix A and a given vector $u \in \mathbb{C}^n$, a minimal A-reducing subspace that contains u. (A subspace of $\mathbb{C}^n$ is said to be *A-reducing* if it is A-invariant and it has a complementary subspace which is also A-invariant.) The answer is provided by part(iii) of the next result.

Theorem 3.3. *Let* A *be an* $n \times n$ *nilpotent complex matrix and let* u *be a nonzero vector in* $\mathbb{C}^n$. *Suppose that for* $i = 0, 1, \ldots, \mathrm{ht}(u) - 2$ *the value of* $\mathrm{dpth}(A^{i+1}u) - \mathrm{dpth}(A^i u)$ *is different from* 1 *for* s *values of* i. *Then*:

(i) *u can be represented as* $y_1 + \ldots + y_{s+1}$ *for some vectors* $y_1, \ldots, y_{s+1}$, *each belonging to a saturated marked Jordan chain, such that* $\mathrm{ht}(y_1) < \mathrm{ht}(y_2) < \cdots < \mathrm{ht}(y_{s+1})$ *and* $\mathrm{dpth}(y_1) < \mathrm{dpth}(y_2) < \cdots < \mathrm{dpth}(y_{s+1})$.

(ii) *Suppose* $s \geq 1$ *and* $\mathrm{dpth}(A^{i+1}u) - \mathrm{dpth}(A^i u)$ *is different from* 1 *for* $i = j_1 < \cdots < j_s$. *Then we have* $\mathrm{ht}(y_k) = j_k + 1$ *for* $k = 1, \ldots, s$ *and* $\mathrm{ht}(y_{s+1}) = \mathrm{ht}(u)$. *Also, the values of* $\mathrm{dpth}(y_k)$*s are given by*: $\mathrm{dpth}(y_1) = \mathrm{dpth}(u)$ *and* $\mathrm{dpth}(y_{k+1}) = \mathrm{dpth}(y_k) + \mathrm{dpth}(A^{j_k+1}u) - \mathrm{dpth}(A^{j_k}u) - 1$ *for* $k = 1, \ldots, s$.

(iii) *If we choose a saturated marked Jordan chain containing* y_i *for each* i, *then the* $s + 1$ *Jordan chains together form a Jordan basis for a minimal* A*-reducing subspace that contains* u.

As can be seen, the solution depends on the concepts of height and depth of a vector with respect to a nilpotent matrix. These concept were known and had been used by Bru, Rodman and Schneider in their paper [BRS]. As by-products of these investigations I am also able to characterize when a given A-invariant subspace is marked or strongly marked, thus completing the work of Bru, Rodman and Schneider in this direction. (An A-invariant subspace is said to be *marked* if it has a Jordan basis which can be extended to a Jordan basis for the whole space; it is *strongly marked* if each of its Jordan bases can be extended to a Jordan basis for the whole space. In [BRS] a criterion is found for a Jordan basis of a given invariant subspace to be extendable to a Jordan basis of the whole space.)

I have not yet settled the problem of determining for a given matrix A and a given finite number of vectors in the underlying space a minimal A-reducing subspace that contains each of the given vectors. This seems to be the next step I should do.

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