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On Periodic Solutions of A Two-Neuron Network System with Sigmoidal Activation Functions

Cheng-Hsiung Hsu* and Suh-Yuh Yang†
Department of Mathematics, National Central University
Chung-Li 32054, Taiwan

Ting-Hui Yang‡
Department of Mathematics, Tamkang University
Tamsui, Taipei County 25137, Taiwan

Tzi-Sheng Yang§
Department of Mathematics, Tunghai University
Taichung 40704, Taiwan

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*E-mail: chhsu@math.ncu.edu.tw. The work of this author was partially supported by the National Science Council and the National Center for Theoretical Sciences of Taiwan.

†E-mail: syyang@math.ncu.edu.tw. The work of this author was partially supported by the National Science Council of Taiwan.

‡E-mail: thyang@math.tku.edu.tw. The work of this author was partially supported by the National Center for Theoretical Sciences of Taiwan.

§E-mail: tsyang@mail.thu.edu.tw. The work of this author was partially supported by the National Science Council of Taiwan.

Abstract

In this paper we study the existence, uniqueness and stability of periodic solutions for a two-neuron network system with or without external inputs. The system consists of two identical neurons, each possessing nonlinear feedback and connected to the other neuron via a nonlinear sigmoidal activation function. In the absence of external inputs but with appropriate conditions on the feedback and connection strengths, we prove the existence, uniqueness and stability of periodic solutions by using the Poincaré-Bendixson theorem together with Dulac's criterion. On the other hand, for the system with periodic external inputs, combining the techniques of the Liapunov function with the contraction mapping theorem, we propose some sufficient conditions for establishing the existence, uniqueness and exponential stability of the periodic solutions. Some numerical results are also provided to demonstrate the theoretical analysis.

Keywords. neural networks; periodic solutions; Poincaré-Bendixson theorem; Dulac's criterion; Liapunov functions; contraction mapping theorem

AMS subject classifications. 34A34; 34C25; 34D23

1. Introduction

This paper is concerned with the existence, uniqueness and stability of periodic solutions for a two-neuron network model which consists of two identical neurons, each possessing nonlinear feedback and connected to the other neuron via a nonlinear sigmoidal activation function. Assuming instantaneous updating of each neuron and communication between the neurons, the dynamics of this netlet is governed by the following system of first-order ordinary differential equations (cf. [8]):

$$\begin{aligned} x'(t) &= -x(t) + pf(x(t)) + sf(y(t)) + I_1(t), \\ y'(t) &= -y(t) + pf(y(t)) + rf(x(t)) + I_2(t), \end{aligned} \tag{1.1}$$

where x and y represent the voltages of the neurons; the real parameter p is the feedback strength, while r and s are the connection strengths; the nonlinear function f is the so-called activation function representing the output or firing rate; I_i, $i = 1, 2$, denote the external inputs to the neurons.

In the following we will consider the nonlinear activation function f of sigmoidal type. More specifically, we assume that f is a continuous piecewise smooth function possessing the following properties:

$$\begin{cases} f : \mathbb{R} \to \mathbb{R} \text{ is an odd function;} \\ f \text{ is differentiable at } x = 0 \text{ with } \lambda := f'(0) > 0; \\ f(\pm\infty) = \pm M, \text{where } M \text{ is a positive constant;} \\ f \text{ is concave downward on } (0, \infty). \end{cases} \tag{1.2}$$

To be more concrete, two typical examples of f are given by

$$\text{(i)} \quad f(x) = \tanh x \qquad \text{(ii)} \quad f(x) = \frac{1}{2}(|x+1| - |x-1|),$$

respectively (cf. Figure 1-1 and Figure 1-2), where the first one is the most popular choice in mathematical analysis due to the C^∞ smoothness (see, e.g., [6, 14, 16]), while the second one is only C^0 piecewise smooth that arises from the many engineering models of neural networks for the practicality of circuit implementation (see, e.g., [3, 4, 5]).

It is well-known that the neural networks possess possibly three interesting types of dynamic behavior, namely, convergence, oscillation and chaotic behavior. The first dynamic behavior mainly concerns the stability of equilibrium points. For Hopfield-type neural networks, it was already known that when the connection strengths are symmetric, i.e., $r = s$ for two-neuron case, all trajectories tend to some equilibrium and hence exhibit no oscillations; see the pioneer work [8] of Hopfield in 1984. Consequently, in most of existing literature, sufficient stability conditions of equilibria for asymmetric connection weights and

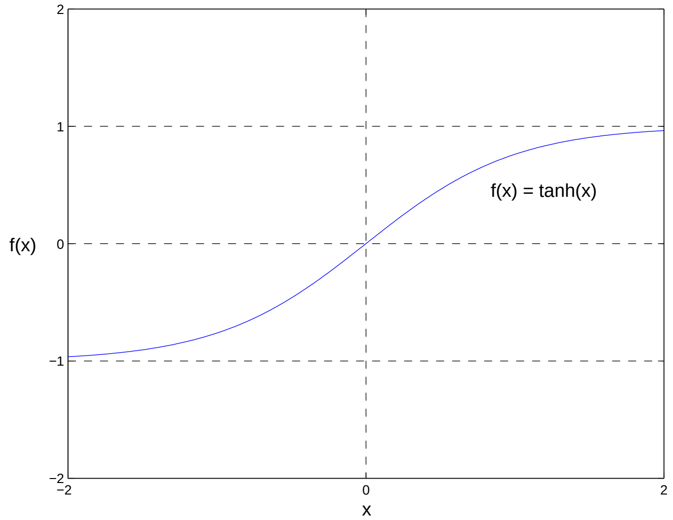

Figure 1-1: The sigmoidal activation function $f(x) = \tanh x$

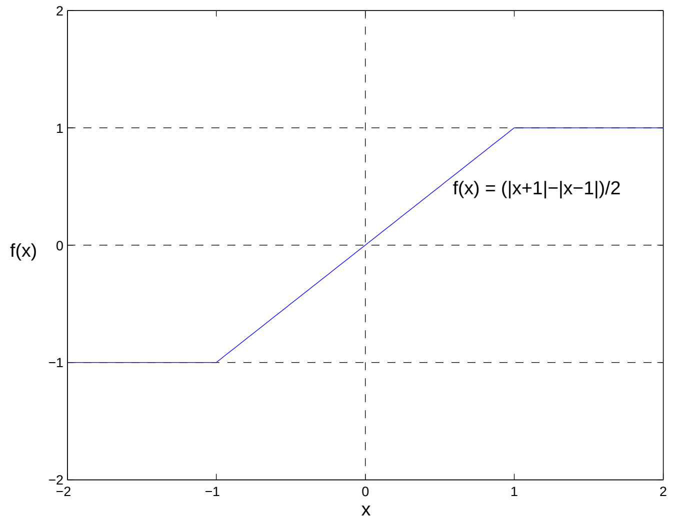

Figure 1-2: The sigmoidal activation function $f(x) = \frac{1}{2}(|x+1| - |x-1|)$

the mechanism for the onset of instability of equilibria are widely considered. See [1, 6, 10, 11, 12, 13, 14, 16, 17, 18, 19] and many references therein.

On the other hand, the understanding of the oscillatory and chaotic behavior of neural netlet such as system (1.1) are still not fully documented. The motivation for studying neural networks which exhibit limit cycle or chaotic behavior arise in neurobiology [6]. It is pointed out that limit cycle type attractors and chaotic attractors are possible even in simple neural networks of just two neurons. Thus, it is of great interest to understand the mechanism of neural networks which cause and sustain such periodic and chaotic activities. It is the purpose of this article to study the simple two-neuron model network (1.1) with or without external inputs capable of producing and sustaining temporal periodic behavior. For the numerical simulation of chaotic behavior of the two-neuron system with a single external periodic input, we refer the reader to [9] for more detailed discussions.

The main results for the present paper can be summarized as follows. For the system (1.1) with $rs < 0$ but in the absence of external inputs, applying the Poincaré-Bendixson theorem we prove the existence of periodic solutions under appropriate conditions on the feedback and connection strengths $[r, p, s]$. In addition, based on the stability analysis of the unique equilibrium point with Dulac's criterion, we further prove the uniqueness of periodic solutions for some suitable feedback and connection strengths with C^2 activation function $f(x)$.

In contrast, for the system with periodic external inputs, if the ratio ω_1/ω_2 of the periods of the inputs is rational then combining the techniques of the Liapunov function with the contraction mapping theorem, we are able to establish the existence, uniqueness and exponential stability of the periodic solution when, roughly speaking, the sum of absolute values of the feedback and connection strengths $[r, p, s]$ is small enough (cf. (5.1)). However, if the sum of absolute values of the strengths is large, numerical evidences show that it may still have periodic solutions for the system. Finally, we present a numerical example showing the existence of the so-called *quasi-periodic* solutions when the ratio of the periods of the inputs is irrational.

This paper is organized as follows. In section 2, we give some sufficient conditions for the uniqueness of equilibrium point and investigate its stability for system without external inputs. We then prove the existence of periodic solutions in section 3. If the activation function f is of class C^2, the uniqueness of periodic solutions is established in section 4. In section 5, the existence, uniqueness, and exponential stability of periodic solutions of system (1.1) with periodic external inputs are derived. In each case, numerical results are also provided to demonstrate the theoretical analysis.

2. Sufficient conditions for unique equilibrium

In this and next two sections, we shall always assume that $I_i(t) \equiv 0$ for $i = 1, 2$. We study some symmetric properties of solutions of system (1.1), and give some sufficient conditions for the uniqueness of equilibrium and investigate its stability when $rs < 0$. The uniqueness and stability of equilibrium will play crucial roles in our analysis for establishing the existence of periodic solutions.

Proposition 2.1. *Let $I_i(t) \equiv 0$ for $i = 1, 2$.*

(i) *If $(x(t), y(t))$ is a solution of (1.1) then $(-x(t), -y(t))$ is also a solution of (1.1).*

(ii) *If $(x(t), y(t))$ is a solution of (1.1) with feedback and connection strengths $[r, p, s]$ then $(-x(t), y(t))$ is also a solution of (1.1) with feedback and connection strengths $[-r, p, -s]$.*

Proof. The proof is straightforward. □

According to part (ii) of Proposition 2.1, the dynamics of solutions of system (1.1) with feedback and connection strengths $[r, p, s]$ is uniquely determined by the dynamics of solutions of system (1.1) with feedback and connection strengths $[-r, p, -s]$, and *vice versa*. Therefore, in the following we may always consider that the strengths $[r, p, s]$ satisfy $s < 0 < r$ with $r + s \geq 0$.

By inspection, we see that $(0, 0)$ is an equilibrium of system (1.1). We now investigate its local stability by examining the eigenvalues of the Jacobian matrix of the corresponding linearized first-order system at $(0, 0)$.

Proposition 2.2. *Let $I_i(t) \equiv 0$ for $i = 1, 2$. If $s < 0 < r$ and $p\lambda < 1$ (resp., $p\lambda > 1$) then the equilibrium $(0, 0)$ is locally stable (resp., unstable).*

Proof. By elementary computations, the characteristic polynomial of the Jacobian matrix of (1.1) linearized at $(0, 0)$ is given by

$$P(x) = x^2 + 2(1 - p\lambda)x + (1 - p\lambda)^2 - rs\lambda^2.$$

Thus the zeros of $P(x)$ are $-1 + p\lambda \pm \sqrt{rs}\lambda$, and then the assertions follow immediately. This completes the proof. □

Next, we are going to give some sufficient conditions for the uniqueness of equilibrium of system (1.1). Notice that an equilibrium $(\bar{x}, \bar{y})$ of (1.1) must satisfy the following system of equations:

$$\begin{aligned} \bar{x} - pf(\bar{x}) &= sf(\bar{y}), \\ \bar{y} - pf(\bar{y}) &= rf(\bar{x}). \end{aligned} \tag{2.1}$$

Proposition 2.3. *Let $I_i(t) \equiv 0$ for $i = 1, 2$. If $s < 0 < r$ and $p\lambda < 1$ then $(0,0)$ is the unique equilibrium of* (1.1)

Proof. Let $(\bar{x}, \bar{y})$ be a solution of (2.1). By virtue of properties (1.2) of f, we have

$$f(x) \le \lambda x \text{ (resp., } \ge \lambda x) \quad \text{for } x \ge 0 \text{ (resp., } \le 0). \tag{2.2}$$

We first assume that $p \ge 0$ (the case of $p < 0$ can be achieved in a similar way). If $\bar{x} \ge 0$ then

$$sf(\bar{y}) = \bar{x} - pf(\bar{x}) \ge \bar{x} - p\lambda\bar{x} = \bar{x}(1 - p\lambda) \ge 0,$$

which implies $\bar{y} \le 0$ and, in addition,

$$rf(\bar{x}) = \bar{y} - pf(\bar{y}) \le \bar{y} - p\lambda\bar{y} = \bar{y}(1 - p\lambda) \le 0.$$

Consequently, $\bar{x} \le 0$, and hence $\bar{x} = 0$ and $\bar{y} = 0$. On the other hand, if $\bar{x} \le 0$ then similar arguments show that $\bar{x} = \bar{y} = 0$ again. Therefore, $(0,0)$ is the unique equilibrium. This completes the proof. □

Remark 2.1. *According to* Proposition 2.2*, the unique equilibrium* $(0,0)$ *in* Proposition 2.3 *is locally stable. However, if the activation function f is of class C^1, then one can further prove that it is actually globally asymptotically stable. The underlying ideas are based on the Poincaré-Bendixson theorem with Dulac's criterion as follows. We first prove that any solution of* (1.1) *is bounded as $t \to +\infty$ (see* Section 3) *and, for $f \in C^1(\mathbb{R})$, we can verify the divergence of the vector field of* (1.1) *satisfying*

$$\begin{aligned}
&\nabla \cdot (-x + pf(x) + sf(y), -y + pf(y) + rf(x)) \\
&= -1 + pf'(x) - 1 + pf'(y) \\
&\le \begin{cases} -2(1 - p\lambda) & \text{if } p \ge 0 \\ -2 & \text{if } p < 0 \end{cases} \\
&< 0 \qquad \text{for all } (x, y) \in \mathbb{R}^2.
\end{aligned}$$

Therefore, Dulac's criterion ensures that there has no periodic solution. It follows from the Poincaré-Bendixson theorem that the unique equilibrium $(0,0)$ *is globally asymptotically stable.* □

In contrast, if the conditions in Proposition 2.3 fail then there may have equilibria other than $(0,0)$. For example, in the case of $p\lambda > 1$, some other conditions are needed for ensuring the uniqueness of equilibrium. For further details, we first introduce the following technical lemma (cf. Figure 2.2).

Lemma 2.1. *Let $I_i(t) \equiv 0$ for $i = 1, 2$ and $h(x) := x - pf(x)$. Assume that $s < 0 < r$ and $p\lambda > 1$, then there exist constants $B > A > 0$ such that*

(i) *The continuous function* $h(\cdot)$ *is negative on* $(0,B)$ *and positive on* (B,∞) *with* $h(B)=0$.

(ii) *The minimum value of* $h(x)$ *for* $x\in(0,\infty)$ *occurs at* $x=A$.

(iii) *If* $(\bar{x},\bar{y})$ *is an equilibrium of* (1.1) *then we have*

$$f(\bar{y})\le\frac{A-pf(A)}{s}\qquad\text{for }\bar{x}\ge 0,$$
$$f(\bar{x})\ge\frac{A-pf(A)}{r}\qquad\text{for }\bar{y}\ge 0.$$

Proof. Since the activation function f is piecewise smooth, the concavity of f on $(0,\infty)$ implies that f' is monotonic non-increasing on $(0,\infty)$ except for finitely many points where f' does not exist. As a consequence of $f'(\infty)=0$, we have $f'(x)\ge 0$ on $(0,\infty)$ whenever it exists. Let

$$A:=\sup\{x>0|f'(x)>\frac{1}{p}\}\quad\text{and}\quad C:=\inf\{x>0|f'(x)<\frac{1}{p}\}.\tag{2.3}$$

Then we obtain $0<A\le C<\infty$ since $f'(0)=\lambda>\frac{1}{p}$ and $f'(\infty)=0<\frac{1}{p}$. By the concavity of f, we can verify that $h'(x)<0$ on $(0,A)$, $h'(x)>0$ on (C,∞), and $h'(x)=0$ on (A,C). Since $h(0)=0$, it follows that $h(x)$ has a unique zero on $(0,\infty)$, denoted by B, and $h(A)$ is the minimum value of $h(x)$ for $x\in(0,\infty)$. Hence, we have the results of part (i) and part (ii). Moreover, by (i) and (ii), we have

$$f(\bar{y})=\frac{h(\bar{x})}{s}\le\frac{A-pf(A)}{s}\qquad\text{for }\bar{x}\ge 0,\tag{2.4}$$

and

$$f(\bar{x})=\frac{h(\bar{y})}{r}\ge\frac{A-pf(A)}{r}\qquad\text{for }\bar{y}\ge 0.\tag{2.5}$$

This completes the proof. □

With the help of Lemma 2.1, we obtain the following sufficient condition for the uniqueness of equilibrium when $s<0<r$ and $p\lambda>1$.

Theorem 2.1. *Let* $I_i(t)\equiv 0$ *for* $i=1,2$, $s<0<r$ *and* $p\lambda>1$. *If*

$$\min\{r,-s\}\ge\frac{p(pf(A)-A)}{B}\tag{2.6}$$

then $(0,0)$ *is the unique equilibrium of* (1.1)*, and it is unstable.*

Proof. According to Lemma 2.1, (2.6) is equivalent to

$$f(B)\ge\frac{A-pf(A)}{s}\quad\text{and}\quad f(B)\ge\frac{pf(A)-A}{r}.\tag{2.7}$$

First, if $(\bar{x}, \bar{y})$ is an equilibrium of (1.1) with $\bar{x} \geq 0$ and $\bar{y} \geq 0$, then part (iii) of Lemma 2.1 implies that

$$f(\bar{y}) \leq \frac{A - pf(A)}{s} \leq f(B).$$

Since f is monotonically non-decreasing, we have $0 \leq \bar{y} \leq B$ which implies $h(\bar{y}) \leq 0$. Hence,

$$\bar{y} - pf(\bar{y}) \leq 0 \leq rf(\bar{x}),$$

and we can conclude that $\bar{x} = 0$ and $\bar{y} = 0$ immediately. Secondly, if $\bar{x} \leq 0$ and $\bar{y} \geq 0$, by part (iii) of Lemma 2.1 again, we have

$$f(\bar{x}) \geq \frac{A - pf(A)}{r} \geq -f(B) = f(-B).$$

Similarly, we obtain $-B \leq \bar{x} \leq 0$ and $h(\bar{x}) \geq 0$. Thus,

$$\bar{x} - pf(\bar{x}) \geq 0 \geq sf(\bar{y}),$$

which forces $\bar{x} = 0$ and $\bar{y} = 0$. Now, according to Proposition 2.1 (i) and Proposition 2.2, we can conclude that $(0, 0)$ is the unique equilibrium of (1.1) and it is unstable. This completes the proof. □

Two typical isoclines for $s < 0 < r$ with $p\lambda < 1$ and $p\lambda > 1$ are depicted in Figure 2.1 and Figure 2.2, respectively.

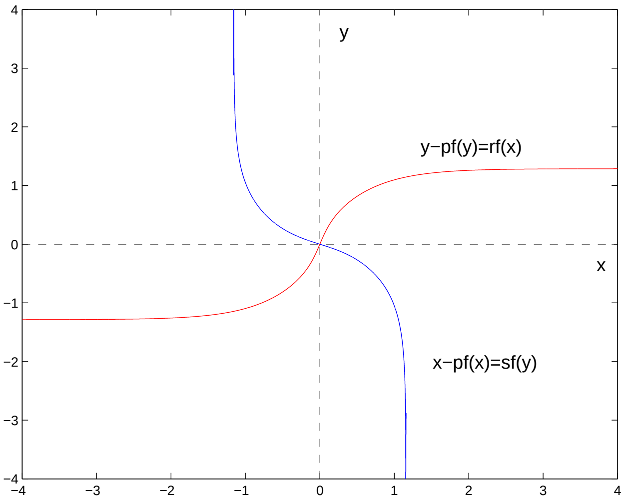

Figure 2.1: Isoclines of (1.1) for $s < 0 < r$ and $p\lambda < 1$

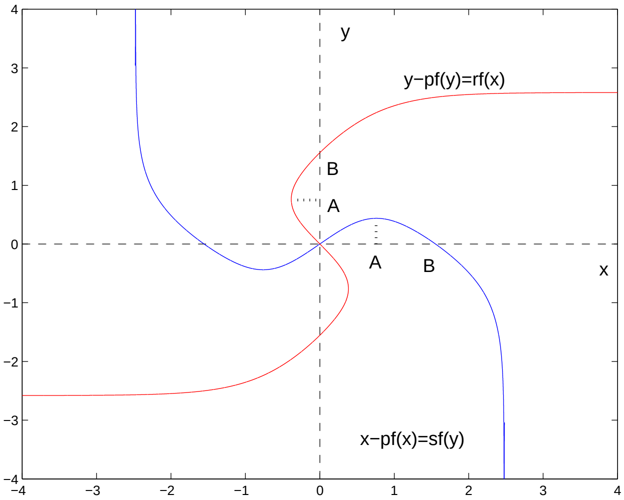

Figure 2.2: Isoclines of (1.1) for $s < 0 < r$ and $p\lambda > 1$

Corollary 2.1. *Let $I_i(t) \equiv 0$ for $i = 1, 2$, $s < 0 < r$ and $p\lambda > 1$. If*

$$\min\{r, -s\} \geq \frac{p\lambda - 1}{\lambda} \tag{2.8}$$

then $(0, 0)$ is the unique equilibrium of (1.1), and it is unstable.

Proof. By (2.2), we have $f(A) \leq \lambda A$ and

$$A - pf(A) - sf(A) \geq \frac{1}{\lambda} f(A) - pf(A) - sf(A) = \frac{f(A)}{\lambda}(1 - p\lambda - s\lambda) \geq 0,$$
$$pf(A) - A - rf(A) \leq pf(A) - \frac{1}{\lambda} f(A) - rf(A) = \frac{f(A)}{\lambda}(p\lambda - 1 - r\lambda) \leq 0.$$

Hence

$$\frac{A - pf(A)}{s} \leq f(A) \leq f(B) = \frac{B}{p},$$
$$\frac{pf(A) - A}{r} \leq f(A) \leq f(B) = \frac{B}{p}.$$

The assertion follows Theorem 2.1 and this completes the proof. □

Example 2.1. If $f(x) = \tanh x$, $p > 1$ and $s < 0 < r$ then one can check that $A = \ln(p + \sqrt{p-1})$ and $f(A) = \frac{\sqrt{p(p-1)}}{p}$. A condition stronger than (2.6) is obtained by replacing B with A since $A < B$. Therefore, if

$$\min\{r, -s\} \geq \frac{p(\sqrt{p(p-1)} - \ln(p + \sqrt{p-1}))}{\ln(p + \sqrt{p-1})},$$

then $(0, 0)$ is the unique equilibrium of (1.1), and it is unstable.

3. Existence of periodic solution

In section 2, in the absence of external input terms, we have proposed some sufficient conditions for ensuring that $(0,0)$ is the unique equilibrium of (1.1). More specifically, for $p\lambda > 1$ and $s < 0 < r$, we proved that $(0,0)$ is the unique unstable equilibrium whenever the connection strengths s and r are strong enough (cf. (2.6) or (2.8)). Along this direction, in this section, we are going to show that there exists an invariant square containing the unique unstable equilibrium $(0,0)$, and then the Poincaré-Bendixson theorem [7, 15] guarantees the existence of periodic solutions of (1.1). To this aim, we first define the square region $\Pi_{(r,p,s)}$ in the xy-plane by

$$\Pi_{(r,p,s)} = \{(x,y) \in \mathbb{R}^2 : \ |x| \leq (p-s)M \text{ and } |y| \leq (p+r)M\}, \tag{3.1}$$

where M is the bound of the sigmoidal activation function f described in (1.2). Then we have the following results.

Lemma 3.1. $\Pi_{(r,p,s)}$ *is an invariant set of* (1.1) *and, moreover, it is also an attractor of* (1.1)*.*

Proof. For convenience, let $L_1 := (p-s)M$ and $L_2 := (p+r)M$. By system (1.1) with the set of properties (1.2) of f, we have

$$\begin{aligned} -x(t) - L_1 \ &\leq \ \frac{dx(t)}{dt} \ \leq \ -x(t) + L_1, \\ -y(t) - L_2 \ &\leq \ \frac{dy(t)}{dt} \ \leq \ -y(t) + L_2, \end{aligned}$$

which implies that

$$\begin{aligned} -L_1 + (x(0) + L_1)e^{-t} \ &\leq \ x(t) \ \leq \ L_1 + (x(0) - L_1)e^{-t}, \\ -L_2 + (y(0) + L_2)e^{-t} \ &\leq \ y(t) \ \leq \ L_2 + (y(0) - L_2)e^{-t}, \end{aligned}$$

where $(x(0), y(0))$ denotes the initial value of the solution $(x(t), y(t))$. From the above, it follows that $\Pi_{(r,p,s)}$ is an invariant set of (1.1) and it is also an attractor. This completes the proof. □

We are now in the position to state the results of existence of periodic solutions for (1.1) in the absence of external inputs.

Theorem 3.1. *Let* $I_i(t) \equiv 0$ *for* $i = 1,2$*,* $s < 0 < r$ *and* $p\lambda > 1$*. If either* (2.6) *or* (2.8) *is satisfied, then there exist periodic solutions of* (1.1)*.*

Proof. Since $\Pi_{(r,p,s)}$ is an invariant set of (1.1), any trajectories starting inside $\Pi_{(r,p,s)}$ will always remain in $\Pi_{(r,p,s)}$. However, $\Pi_{(r,p,s)}$ just contains one unstable equilibrium $(0,0)$. Thus, the Poincaré-Bendixson theorem ensures there is a periodic orbit lying in $\Pi_{(r,p,s)}$, and this completes the proof. □

Numerical simulation for the existence of periodic solutions of system (1.1) with typical sigmoidal activation functions $f(x) = \tanh x$ and $f(x) = \frac{1}{2}(|x+1| - |x-1|)$ are illustrated in Figure 3.1 and Figure 3.2, respectively.

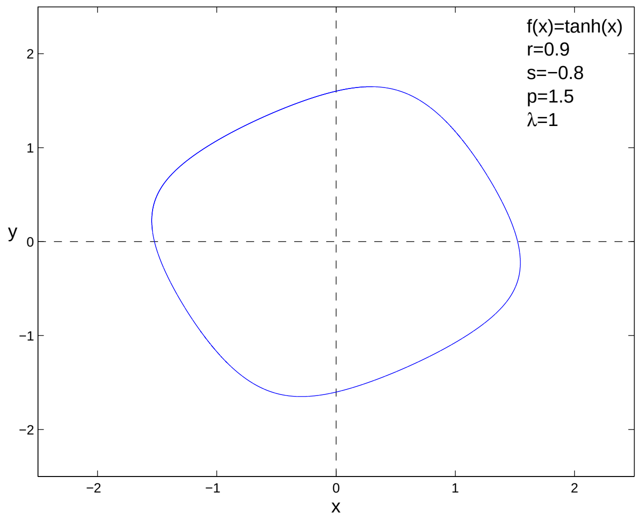

Figure 3.1: A periodic orbit of (1.1) with $f(x) = \tanh x$

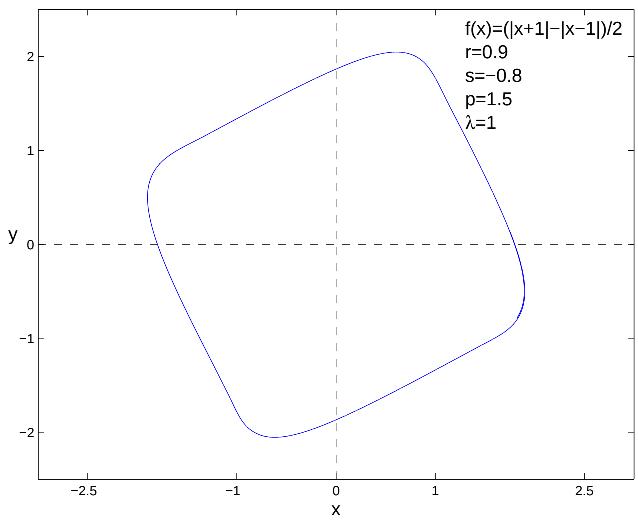

Figure 3.2: A periodic orbit of (1.1) with $f(x) = \frac{1}{2}(|x+1| - |x-1|)$

4. Uniqueness of periodic solution

In this section we give a criterion (see Theorem 4.1) to ensure the uniqueness of the periodic solution of (1.1) derived in Theorem 3.1. Throughout this section, we always assume that the sigmoidal activation function f is of class C^2. Define

$$E(x, y) = pf'(x) + pf'(y) - 2. \tag{4.1}$$

Then we have

Proposition 4.1. *Assume that* $1 < p\lambda < 2$. *Then* $\gamma_0 := \{(x, y)|\ E(x, y) = 0\}$ *is a simple closed curve in* $\mathbb{R}^2$.

Proof. Since f' is an even function, it is obvious that $E(x, y)$ is symmetric about x-axis and y-axis. Therefore, we only need to show that the curve γ_0 in the first quadrant does not intersect itself. In fact, we claim that γ_0 in the the first quadrant is actually the graph of a function. Note that γ_0 passes through (A, A) in which $f'(A) = \frac{1}{p}$, and $E(x, y) = E(y, x)$. Therefore it suffices to show that for each $x \in [0, A]$ there is exactly one $y \geq 0$ satisfying $E(x, y) = 0$. By the properties of f, we have

$$\frac{1}{p} \leq f'(x) \leq \lambda \quad \text{for } x \in [0, A] \quad \text{and} \quad 0 < f'(x) \leq \frac{1}{p} \quad \text{for } x \in [A, \infty),$$

which implies

$$0 < \frac{2 - p\lambda}{p} \leq \frac{2 - pf'(x)}{p} \leq \frac{2 - p\frac{1}{p}}{p} = \frac{1}{p} \quad \text{for } x \in [0, A].$$

Therefore, by the intermediate value theorem, there exists $y \geq A$ such that

$$f'(y) = \frac{2 - pf'(x)}{p}.$$

The uniqueness follows from the fact that f' is strictly decreasing on $(0, \infty)$, since $f \in C^2$. This completes the proof. □

In the following, combining Proposition 4.1 with Dulac's criterion [7, 15], we prove the uniqueness and stability of periodic solutions of (1.1). Let $(x(t), y(t))$ be a solution of (1.1). Then the derivative of $E(x, y)$ with respect to t along the trajectory $(x(t), y(t))$ is given by

$$\frac{dE}{dt} = pf''(x)(-x + pf(x) + sf(y)) + pf''(y)(-y + pf(y) + rf(x)). \tag{4.2}$$

Theorem 4.1. *Let* $I_i(t) \equiv 0$ *for* $i = 1, 2$, $s < 0 < r$ *and* $1 < p\lambda < 2$. *Suppose that either* (2.6) *or* (2.8) *is satisfied, and* $\frac{dE}{dt}$ *given in* (4.2) *does not change sign on* γ_0. *Then there exists a unique periodic solution of* (1.1) *and it is globally asymptotically stable.*

Proof. Taking the divergence of the vector field of (1.1), we have

$$\nabla \cdot (-x + pf(x) + sf(y), -y + pf(y) + rf(x)) = E(x, y),$$

which is positive in the interior of γ_0 and negative in the exterior of γ_0, where $\gamma_0 := \{(x, y)|\ E(x, y) = 0\}$. Hence, by Dulac's criterion, (1.1) has no closed orbit lying entirely in the interior of γ_0. Moreover, since $\frac{dE}{dt}$ does not change sign on γ_0, it follows that there is no periodic orbit crossing γ_0. Therefore, the existence of periodic solutions ensured by Theorem 3.1 must lie in the exterior of γ_0 and in the interior of $\Pi_{(r,p,s)}$. Now, consider the annular region which is the intersection of the exterior of γ_0 with the interior of $\Pi_{(r,p,s)}$. Obviously, the divergence of the vector field of (1.1) does not change sign on this annular region. Combining Dulac's criterion with that fact that the unique equilibrium point $(0, 0)$ is unstable, we can conclude that there exists a unique periodic solution of (1.1) which is globally asymptotically stable. This completes the proof. □

In the rest of the section, we will concentrate on a concrete example with the sigmoidal activation function $f(x) = \tanh x$, and determine the feedback and connection strengths $[r, p, s]$ under which the two-neuron network (1.1) has a unique globally asymptotically stable periodic solution.

First, for $f(x) = \tanh x$, we have $\lambda = 1$, $f'(x) = 1 - (f(x))^2$, and $f''(x) = 2f(x)(f^2(x) - 1)$. Introducing the new variables (u, v) by

$$u = \tanh x \quad \text{and} \quad v = \tanh y,$$

then we have

$$E(x, y) = 0 \quad \text{if and only if} \quad u^2 + v^2 = \frac{2p - 2}{p}. \tag{4.3}$$

For such (u, v) lying on the circle $u^2 + v^2 = \frac{2p-2}{p}$, we have

$$\begin{aligned} \frac{1}{2p}\frac{dE}{dt} &= u(u^2 - 1)\Big(-\frac{1}{2}\ln(\frac{1+u}{1-u}) + pu + sv\Big) \\ &\quad + v(v^2 - 1)\Big(-\frac{1}{2}\ln(\frac{1+v}{1-v}) + pv + ru\Big) \\ &= p(u^4 + v^4) - (2p - 2) + su^3v + ruv^3 - (s + r)uv \\ &\quad + u(1 - u^2)\frac{1}{2}\ln(\frac{1+u}{1-u}) + v(1 - v^2)\frac{1}{2}\ln(\frac{1+v}{1-v}) \\ &= K(u, v) + G(u) + G(v) - (2p - 2), \end{aligned}$$

where K and G are defined by

$$\begin{aligned} K(u, v) &:= su^3v + ruv^3 - (s + r)uv, \\ G(u) &:= pu^4 + u(1 - u^2)\frac{1}{2}\ln(\frac{1+u}{1-u}). \end{aligned}$$

Now we are going to evaluate maximum values of $K(u,v)$ and $G(u)+G(v)$ on the circle $u^2+v^2=\frac{2p-2}{p}$, respectively. Applying the method of Lagrange multiplier, we find that a candidate point (u,v) for the maximum value of $K(u,v)$ subject to $u^2+v^2=\frac{2p-2}{p}$ satisfies:

$$\begin{aligned} rv^3+3su^2v-(s+r)v &= 2\mu u, \\ 3ruv^2+su^3-(s+r)u &= 2\mu v, \end{aligned}$$

where μ is the Lagrange multiplier. Therefore, we obtain

$$rv^4+3su^2v^2-(s+r)v^2=3ru^2v^2+su^4-(s+r)u^2,$$

or equivalently,

$$rv^4-su^4+3(s-r)u^2v^2+(s+r)(u^2-v^2)=0. \tag{4.4}$$

To find the root (u,v) of (4.4), since $u^2+v^2=\frac{2p-2}{p}$, we may denote (u,v) by

$$(u,v)=(\ell\cos\theta,\ell\sin\theta) \quad \text{for some } \theta,\ \text{where } \ell:=\sqrt{\frac{2p-2}{p}}.$$

With these notation, (4.4) can be written as

$$(r-s)\cos 4\theta-(1-\frac{2}{\ell^2})(r+s)\cos 2\theta=0,$$

which implies

$$\cos 2\theta=\frac{1}{4}\Big(\frac{-1}{p-1}\frac{r+s}{r-s}\pm\sqrt{(\frac{1}{p-1})^2(\frac{r+s}{r-s})^2+8}\Big), \tag{4.5}$$

and

$$\begin{aligned} \max_{u^2+v^2=\frac{2p-2}{p}} K(u,v) &= \frac{(p-1)^2}{2p^2}\Big((s-r)\sin 4\theta+\frac{1-2p}{(p-1)^2}(s+r)\sin 2\theta\Big) \\ &:= \Gamma(r,p,s). \end{aligned} \tag{4.6}$$

As a consequence of (4.6), if $r+s=0$ then we have $\Gamma(r,p,s)=\Gamma(r,p,-r)=r(\frac{p-1}{p})^2$.

As for the maximum value of $G(u)+G(v)$ on the circle $u^2+v^2=\frac{2p-2}{p}$, a candidate point (u,v) for the maximum value satisfies

$$G'(u)=2\mu u \quad \text{and} \quad G'(v)=2\mu v, \tag{4.7}$$

where μ is the Lagrange multiplier. We first assume that $u\neq 0$ and $v\neq 0$. In this case, (4.7) can be rewritten as

$$\frac{G'(u)}{u}=\frac{G'(v)}{v}.$$

It is obvious that $\frac{G'(u)}{u}$ is an even function. Next, we claim that $\frac{G'(u)}{u}$ is one-to-one for $u \in (0, \sqrt{\frac{2p-2}{p}})$ whenever $1 < p \le 3/2$. Applying Taylor's expansion, we have

$$\begin{aligned}
\frac{d}{du}\Big(\frac{G'(u)}{u}\Big) &= \frac{d}{du}\Big(4pu^2 + (u^{-1} - 3u)\frac{1}{2}\ln(\frac{1+u}{1-u})\Big) \\
&= 8pu - (u^{-2}+3)\frac{1}{2}\ln(\frac{1+u}{1-u}) + (u^{-1} - 3u)\frac{1}{1-u^2} \\
&= u\Big(8p - \sum_{k=1}^{\infty}(\frac{1}{2k+1} + \frac{3}{2k-1} + 2)u^{2k-2}\Big) \\
&> u\Big(8p - \frac{16}{3} - \frac{16}{5}\sum_{k=2}^{\infty}(\sqrt{\frac{2p-2}{p}})^{2k-2}\Big) \\
&= 8u\Big(p - \frac{2}{3} - \frac{4(p-1)}{5(2-p)}\Big).
\end{aligned}$$

Hence, if $1 < p \le 3/2$ then $\frac{G'(u)}{u}$ is strictly increasing and then it is an one-to-one function for $u \in (0, \sqrt{\frac{2p-2}{p}})$. Therefore, the nontrivial solutions of (4.7) are $u = \pm v = \pm\sqrt{\frac{p-1}{p}}$ and for these (u, v),

$$G(u) + G(v) = 2G(\sqrt{\frac{p-1}{p}}). \tag{4.8}$$

Next, if $u = 0$ or $v = 0$ then

$$\begin{aligned}
G(u) + G(v) &= G(\sqrt{\frac{2p-2}{p}}) \\
&= \frac{4(p-1)^2}{p} + \frac{2-p}{2p}\sqrt{\frac{2p-2}{p}}\ln\Big(\frac{\sqrt{p}+\sqrt{2p-2}}{\sqrt{p}-\sqrt{2p-2}}\Big). \qquad (4.9)
\end{aligned}$$

Finally, to determine the maximum value of $G(u)+G(v)$ on the circle $u^2+v^2 = \frac{2p-2}{p}$, we have the following lemma:

Lemma 4.1. *Assume that* $1 < p \le 3/2$. *Then we have*

$$\max_{u^2+v^2=\frac{2p-2}{p}}(G(u) + G(v)) = G(\sqrt{\frac{2p-2}{p}}). \tag{4.10}$$

Proof. It suffices to claim that

$$G(\sqrt{2}u) > 2G(u) \qquad \text{for } u \in (0, \sqrt{\frac{p-1}{p}}].$$

Since for $u \in (0, \sqrt{\frac{p-1}{p}})$, we have $0 < u < 1/\sqrt{3}$ and

$$\begin{aligned}
&\frac{d}{du}\Big(G(\sqrt{2}u) - 2G(u)\Big) \\
&= 8pu^3 + \sum_{k=1}^{\infty}\Big(\frac{2^{k+1}-2}{2k+1} - 6\frac{2^k-1}{2k-1}\Big)u^{2k+1} \\
&= 6u^3(p-1-u^2) + u^3(2p - \frac{42}{5}u^4) + \sum_{k=1}^{\infty}\Big(\frac{2^{k+1}-2}{2k+1} - 6\frac{2^{k+3}-1}{2k+5}u^6\Big)u^{2k+1} \\
&> 6u^3(p-1-u^2) + u^3(2p - \frac{42}{5}u^4) + \sum_{k=1}^{\infty}\Big(\frac{2^{k+1}-2}{2k+1} - 6\frac{2^{k+3}-1}{3^3(2k+5)}\Big)u^{2k+1} \\
&> 6\frac{(p-1)^2}{p}u^3 + (2p - \frac{14}{15})u^3 + \sum_{k=1}^{\infty}\Big(\frac{2^{k+1}-2}{2k+1} - 6\frac{2^{k+3}-1}{3^3(2k+5)}\Big)u^{2k+1} \\
&> 0,
\end{aligned}$$

where the last inequality is obtained by proving that

$$\frac{2^{k+1}-2}{2k+1} - 6\frac{2^{k+3}-1}{3^3(2k+5)} > 0 \qquad \text{for all } k \geq 1,$$

or equivalently,

$$(6k+111)2^{k+1} > 96k + 264 \qquad \text{for all } k \geq 1. \tag{4.11}$$

However, one can verify that the inequality (4.11) holds by mathematics induction. Since $G(\sqrt{2}u) - 2G(u)$ is strictly increasing and equal to zero at $u = 0$, we have $G(\sqrt{2}u) - 2G(u) > 0$ for $u = \sqrt{\frac{p-1}{p}}$, and this completes the proof. □

Finally, combining the above results with Theorem 4.1, we obtain

Theorem 4.2. *Let $I_i(t) \equiv 0$ for $i = 1, 2$, $f(x) = \tanh x$, $s < 0 < r$ and $1 < p \leq 3/2$. Suppose that (2.8) holds with $\lambda = 1$, and*

$$\frac{2(p-1)(p-2)}{p} + \frac{2-p}{2p}\sqrt{\frac{2p-2}{p}}\ln\Big(\frac{\sqrt{p}+\sqrt{2p-2}}{\sqrt{p}-\sqrt{2p-2}}\Big) + \Gamma(r,p,s) < 0,$$

where $\Gamma(r,p,s)$ defined in (4.6). Then there exists a unique periodic solution of (1.1) and it is globally asymptotically stable.

Proof. Since $\frac{dE}{dt} < 0$ on the circle γ_0, the assertion follows Theorem 4.1. □

A numerical example showing the unique periodic solution which is globally asymptotically stable is depicted in Figure 4.1, where O denotes the unique periodic orbit and O_i, $i = 1, 2, \cdots 5$, denote the orbits with initial values $(1, 2)$,

$(-2,1)$, $(-1,-2)$, $(2,-1)$, and $(0.5,0.5)$, respectively. One can observe the asymptotic behaviors of O_i as $t \to \infty$ that confirm our theoretical analysis.

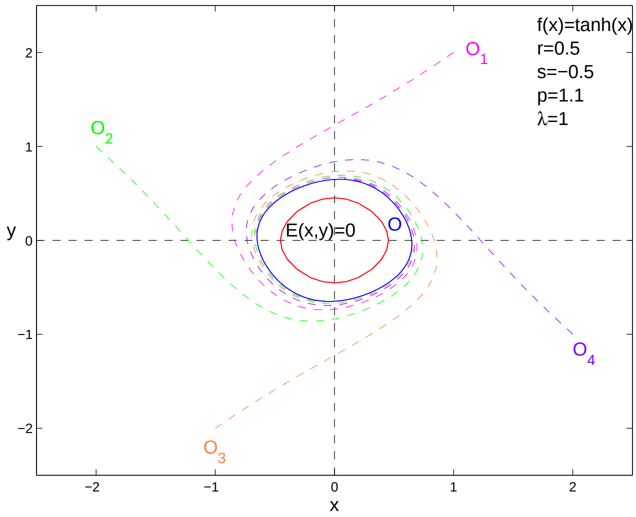

Figure 4.1: The unique periodic solution of (1.1)

Finally, for the specific case $r+s=0$, the numerical region Ω_2 and the theoretical region $\Omega_1 \subset \Omega_2$ corresponding to Theorem 4.1 and Theorem 4.2 for ensuring uniqueness of periodic solution are given in Figure 4.2, respectively.

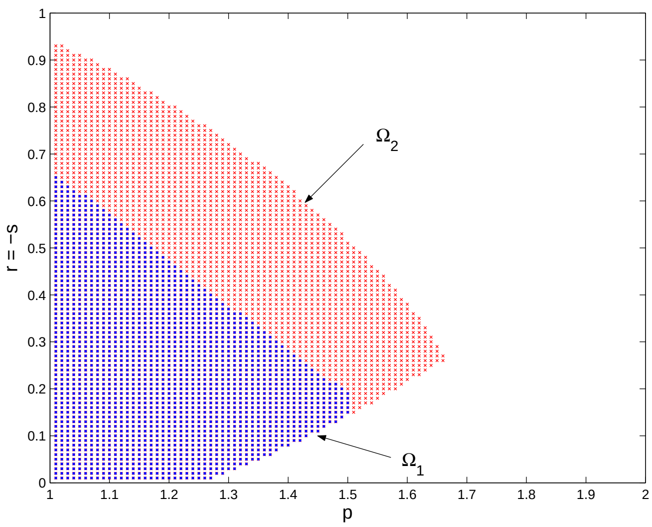

Figure 4.2: The regions for uniqueness of periodic solution

5. Periodic solutions for system with periodic external inputs

In this section, we study the periodic solution of the two-neuron system (1.1) with periodic external inputs. More specifically, we assume that $I_i : \mathbb{R}^+ \to \mathbb{R}$ are continuously periodic functions with the periods $\omega_i > 0$ for $i = 1, 2$, respectively. By using the ideas similar to that in [2] with a slight modification, we have the following results concerning the existence, uniqueness, and stability of periodic solutions of (1.1) based on the fixed point theory for contraction mappings.

Theorem 5.1. *Consider the two-neuron system* (1.1) *with ω_i-periodic external inputs $I_i(t)$, for $i = 1, 2$. Assume $\dfrac{\omega_1}{\omega_2} = \dfrac{n}{m}$ for some $m, n \in \mathbb{N}$ that are relatively prime. If there exist $\theta_1 > 0$ and $\theta_2 > 0$ such that*

$$2|p| + |s| + \frac{\theta_2}{\theta_1}|r| < \frac{2}{\lambda} \quad \text{and} \quad 2|p| + |r| + \frac{\theta_1}{\theta_2}|s| < \frac{2}{\lambda}, \tag{5.1}$$

then we have

(i) *The system* (1.1) *has a unique ω-periodic solution with $\omega := m\omega_1 = n\omega_2$.*

(ii) *All other solutions of* (1.1) *converge exponentially to the ω-periodic solution as $t \to +\infty$.*

Proof. Let $(x_1^0, y_1^0),\ (x_2^0, y_2^0) \in \mathbb{R}^2$. Denote the solutions of (1.1) with the initial conditions $(x_1^0, y_1^0),\ (x_2^0, y_2^0)$ by $(x_1(t), y_1(t)),\ (x_2(t), y_2(t))$, respectively. Then it follows that

$$\begin{aligned}
(x_2(t) - x_1(t))' &= -(x_2(t) - x_1(t)) + p(f(x_2(t)) - f(x_1(t))) \\
&\quad + s(f(y_2(t)) - f(y_1(t))), \\
(y_2(t) - y_1(t))' &= -(y_2(t) - y_1(t)) + p(f(y_2(t)) - f(y_1(t))) \\
&\quad + r(f(x_2(t)) - f(x_1(t))).
\end{aligned}$$

Due to (5.1), we can choose a sufficiently small $\varepsilon > 0$ such that

$$\begin{aligned}
&\frac{\varepsilon}{2} - 1 + \frac{\lambda}{2}\Big(2|p| + |s| + \frac{\theta_2}{\theta_1}|r|\Big) < 0, \\
&\frac{\varepsilon}{2} - 1 + \frac{\lambda}{2}\Big(2|p| + |r| + \frac{\theta_1}{\theta_2}|s|\Big) < 0.
\end{aligned} \tag{5.2}$$

We define the following Liapunov function,

$$U(t) = \frac{e^{\varepsilon t}}{2}\Big(\theta_1(x_2(t) - x_1(t))^2 + \theta_2(y_2(t) - y_1(t))^2\Big).$$

Calculating the rate of change of $U(t)$ along $(x_2(t) - x_1(t), y_2(t) - y_1(t))$, we have

$$\begin{aligned}
\frac{dU(t)}{dt} &= \theta_1\Big(\frac{1}{2}(x_2(t) - x_1(t))^2 \varepsilon e^{\varepsilon t} + (x_2(t) - x_1(t))(x_2(t) - x_1(t))' e^{\varepsilon t}\Big) \\
&\quad + \theta_2\Big(\frac{1}{2}(y_2(t) - y_1(t))^2 \varepsilon e^{\varepsilon t} + (y_2(t) - y_1(t))(y_2(t) - y_1(t))' e^{\varepsilon t}\Big).
\end{aligned}$$

Owing to the fact that f is Lipschitz continuous with the Lipschitz constant λ, we can verify that for $t > 0$

$$\begin{aligned}\frac{dU(t)}{dt} \leq\ & e^{\varepsilon t}\Big\{\theta_1\Big((\frac{\varepsilon}{2}-1)(x_2(t)-x_1(t))^2+\lambda|p|(x_2(t)-x_1(t))^2 \\ & +\lambda|s||x_2(t)-x_1(t)||y_2(t)-y_1(t)|\Big) \\ & +\theta_2\Big((\frac{\varepsilon}{2}-1)(y_2(t)-y_1(t))^2+\lambda|p|(y_2(t)-y_1(t))^2 \\ & +\lambda|r||y_2(t)-y_1(t)||x_2(t)-x_1(t)|\Big)\Big\}.\end{aligned}$$

Applying the elementary inequality, $2ab \leq a^2+b^2$ for all a, $b \in \mathbb{R}$, with (5.2), we further have

$$\begin{aligned}\frac{dU(t)}{dt} \leq\ & e^{\varepsilon t}\Big\{\theta_1\Big((\frac{\varepsilon}{2}-1)(x_2(t)-x_1(t))^2+\lambda|p|(x_2(t)-x_1(t))^2 \\ & +\frac{\lambda|s|}{2}(x_2(t)-x_1(t))^2+\frac{\lambda|s|}{2}(y_2(t)-y_1(t))^2\Big) \\ & +\theta_2\Big((\frac{\varepsilon}{2}-1)(y_2(t)-y_1(t))^2+\lambda|p|(y_2(t)-y_1(t))^2 \\ & +\frac{\lambda|r|}{2}(y_2(t)-y_1(t))^2+\frac{\lambda|r|}{2}(x_2(t)-x_1(t))^2\Big)\Big\}.\end{aligned}$$

That is,

$$\begin{aligned}\frac{dU(t)}{dt} \leq\ & e^{\varepsilon t}\Big\{\theta_1\Big(\frac{\varepsilon}{2}-1+\lambda|p|+\frac{\lambda|s|}{2}+\frac{\theta_2}{\theta_1}\frac{\lambda|r|}{2}\Big)(x_2(t)-x_1(t))^2 \\ & +\theta_2\Big(\frac{\varepsilon}{2}-1+\lambda|p|+\frac{\lambda|r|}{2}+\frac{\theta_1}{\theta_2}\frac{\lambda|s|}{2}\Big)(y_2(t)-y_1(t))^2\Big\} \\ \leq\ & 0,\end{aligned}$$

which implies that

$$U(t) \leq U(0) \qquad \text{for all } t \geq 0.$$

Since

$$\frac{e^{\varepsilon t}}{2}\min\{\theta_1,\theta_2\}\Big((x_2(t)-x_1(t))^2+(y_2(t)-y_1(t))^2\Big) \leq U(t),$$

for all $t \geq 0$ and

$$\begin{aligned}U(0) =\ & \frac{1}{2}\Big(\theta_1(x_2^0-x_1^0)^2+\theta_2(y_2^0-y_1^0)^2\Big) \\ \leq\ & \frac{1}{2}\max\{\theta_1,\theta_2\}\Big((x_2^0-x_1^0)^2+(y_2^0-y_1^0)^2\Big),\end{aligned}$$

Then we can easily get that

$$(x_2(t)-x_1(t))^2+(y_2(t)-y_1(t))^2 \leq Ce^{-\varepsilon t}\|(x_2^0-x_1^0, y_2^0-y_1^0)\|_2^2, \tag{5.3}$$

for all $t \geq 0$, where $\| \cdot \|_2$ is the usual Euclidean norm in $\mathbb{R}^2$ and $C \geq 1$ is a constant.

Now, letting $\omega := m\omega_1 = n\omega_2$, we can choose a positive integer k such that

$$Ce^{-\varepsilon k\omega} \leq \frac{1}{2},$$

and then define a Poincaré map $P : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$P(x_0, y_0) = (x(t), y(t)) \mid_{t=\omega}, \tag{5.4}$$

where $(x(t), y(t))$ is the solution of (1.1) with the initial value (x_0, y_0). Then, by (5.3), we can conclude that P^k is a contraction mapping. Hence, there exists a unique fixed point $(x_0^*, y_0^*) \in \mathbb{R}^2$ such that $P^k(x_0^*, y_0^*) = (x_0^*, y_0^*)$. Notice that

$$P^k(P(x_0^*, y_0^*)) = P(P^k(x_0^*, y_0^*)) = P(x_0^*, y_0^*),$$

which implies that $P(x_0^*, y_0^*)$ is also a fixed point of P^k. By the uniqueness of fixed point of P^k, we arrived at $P(x_0^*, y_0^*) = (x_0^*, y_0^*)$.

Let $(x^*(t), y^*(t))$ be the solution of (1.1) with the initial value (x_0^*, y_0^*). Then, by (5.3) with (5.4), one can verify that $(x^*(t), y^*(t))$ is the unique periodic solution with period ω and, furthermore, all other solutions of (1.1) converge exponentially to it as $t \to +\infty$. This completes the proof. $\square$

Next, we give a numerical example in which the conditions in Theorem 5.1 are satisfied and especially the ratio of the periods of the external inputs is $\frac{\omega_1}{\omega_2} = 2 \in \mathbb{Q}$, while in the second example the condition (5.1) fails no matter what θ_1 and θ_2 are chosen. Both examples exhibit the uniqueness of globally exponential stable periodic solutions. See Figure 5.1 and Figure 5.2 below.

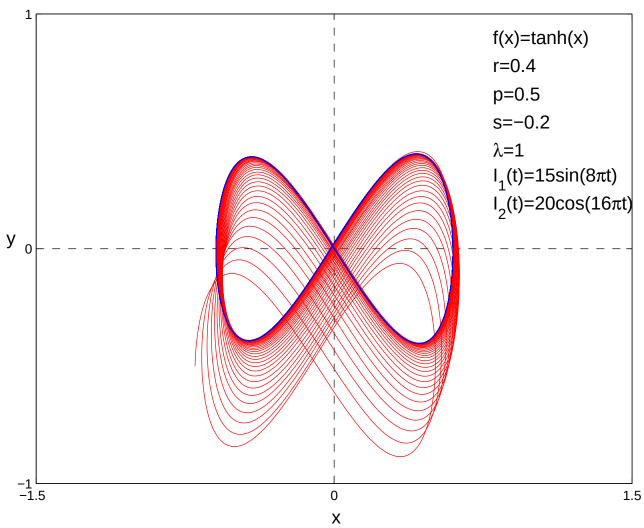

Figure 5.1: The unique periodic solution of (1.1) satisfying (5.1)

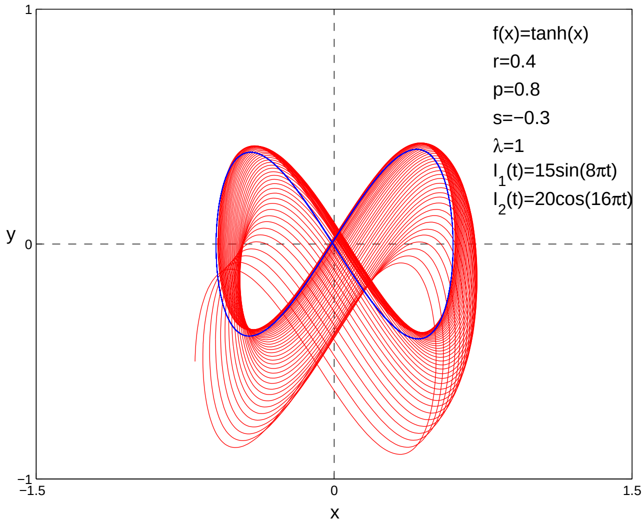

Figure 5.2: The unique periodic solution of (1.1) violating (5.1)

Finally, we conclude this section with a numerical example in which the ratio of the periods of the external inputs is $\frac{\omega_1}{\omega_2} = \frac{9\pi}{2} \in \mathbb{Q}^c$. It is interesting to note that, in this case, the so-called *quasi-periodic* solution seems to be observed. Unfortunately, in the present paper, we are not able to carry out a theoretical analysis on it.

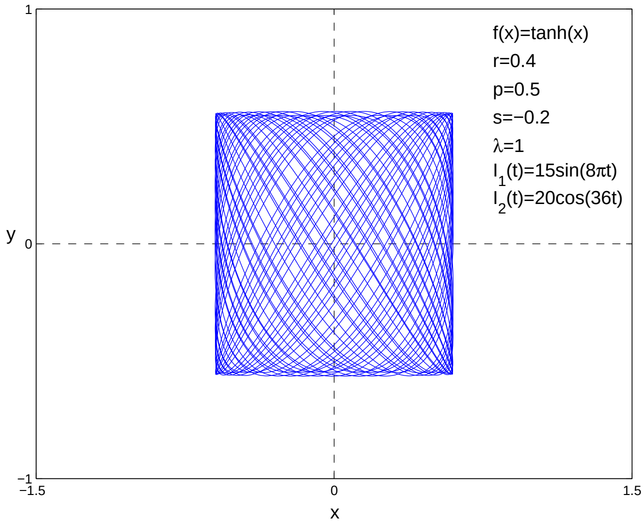

Figure 5.3: A *quasi-periodic* solution of (1.1) with irrational-ratio periodic external inputs

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