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VITALI LEMMA APPROACH TO DIFFERENTIATION ON A TIME SCALE

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ABSTRACT. A new approach to differentiation on a time scale $\mathbb{T}$ is presented. We give a suitable generalization of the Vitali Lemma and apply it to prove that every increasing function $f : \mathbb{T} \to R$ has right derivative $f'_+(x)$ for μ_Δ-almost all $x \in \mathbb{T}$. Moreover, $\int_{[a,b)} f'_+(x)\, d\mu_\Delta \leq f(b) - f(a)$.

The theory of time scales has received much attention since Hilger's [4] initial paper introduced the unifying theory for continuous and discrete calculus. Subsequent developmental major works devoted to the calculus on time scales has been conducted by Agarwal and Bohner [1], Bohner and Peterson [2,3], and Kaymakgalan et al. [5].

In this paper, we are concerned with the covering theorem of Vitali on time scales. It is well-known that the Vitali lemma is the basic tool for the development of the differentiation theory in the Riemann and Lebegue integrals. A similar treatment of the differentiation theory on time scales is given.

Before introducing the problems of interest for this paper, we present some definitions and notations which are common to the recent literature.

Definition 1. *Let $\mathbb{T}$ be a nonempty closed subset of $\mathcal{R}$, and let $\mathbb{T}$ have the subspace topology inherited from the Euclidean topology on $\mathcal{R}$. Then $\mathbb{T}$ is called a time scale. For $t < \sup \mathbb{T}$ and $r > \inf \mathbb{T}$, define the forward jump operator, σ, and the backward jump operator, ρ, by*

$$\begin{aligned} \sigma(t) &= \inf\{\tau \in \mathbb{T} :\ \tau > t\} \in \mathbb{T}, \\ \rho(r) &= \sup\{\tau \in \mathbb{T} :\ \tau < r\} \in \mathbb{T}. \end{aligned}$$

If $\sigma(t) > t$, then t is said to be right-scattered, and if $\rho(r) < r$, then r is said to be left-scattered. If $\sigma(t) = t$, then t is said to be right-dense, and if $\rho(r) = r$, then r is said to be left-dense.

The sets of right-dense and left-dense points we shall denote, respectively by D_R and D_L.

Definition 2. *For $x : \mathbb{T} \to \mathcal{R}$ and $t \in \mathbb{T}$ (if $t = \sup \mathbb{T}$, assume t is not left-scattered), define the delta derivative of $x(t), x^\Delta(t)$, to be the number (when it exists) with the property that, for any $\epsilon > 0$, there is a neighborhood, U, of t such that*

$$|[x(\sigma(t)) - x(s)] - x^\Delta(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s|$$

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for all $s \in U$. If $F^{\Delta}(t) = h(t)$, then define the integral by

$$\int_a^t h(s)\Delta s = F(t) - F(a). \tag{0.1}$$

Definition 3. *For $a, b \in \mathbb{T}$, define the closed interval $[a,b]$, in $\mathbb{T}$ by*

$$[a,b] = \{t \in \mathbb{T} : \ a \le t \le b\}.$$

Open intervals and half-open intervals etc, are defined accordingly. For arbitrary interval $P \subset \mathbb{T}$ we denote its length by

$$l(P) = b - a.$$

Consider the family $\mathcal{F}_R = \{ \ [a,b) : \ a, b \in \mathbb{T}\}$ and for arbitrary set $A \subset \mathbb{T}$ put

$$m^*(A) = \inf \left\{ \sum_{i=1}^{\infty} l(P_i) : \ P_i \in \mathcal{F}_R \ \ and \ \ A \subset \bigcup_{i=1}^{\infty} P_i \right\}. \tag{0.2}$$

One can notice that m^* is an outer measure on $2^{\mathbb{T}}$. By Caratheodory Extension Theorem m^* defines a $\sigma - field$ $\mathcal{L} = \mathcal{L}_{\Delta}$ of measurable sets $A \in \mathcal{L}$ satisfying the condition

$$m^*(E) = m^*(E \cap A) + m^*(E \cap A')$$

for every $E \subset \mathbb{T}$, where $A' = \mathbb{T} \backslash A$. Denote by $\mu_{\Delta} = m^*|_{\mathcal{L}}$. Following Bohner & Peterson [2] the measure μ_{Δ} is called $\Delta-$ measure, while $\mathcal{L}$ is a family of $\Delta-$ measurable sets. Note that any $P \in \mathcal{F}_R$ is $\Delta-$ measurable with $\mu_{\Delta}(P) = l(P)$ and any compact set $K \subset \mathbb{T} \backslash \{\max \mathbb{T}\}$ is of finite $\Delta-$ measure. Moreover, any Borel set is $\Delta-$ measurable.

Similarly consider the family $\mathcal{F}_L = \{(a,b] : \ a, b \in \mathbb{T}\}$ and define

$$m_*(A) = \inf \left\{ \sum_{i=1}^{\infty} l(P_i) : \ P_i \in \mathcal{F}_L \ \ and \ \ A \subset \bigcup_{i=1}^{\infty} P_i \right\}. \tag{0.3}$$

Now the Caratheodory procedure leeds to $\nabla-$ measure μ_{∇} and $\nabla-$ measurable sets $\mathcal{L}_{\nabla}$.

As usual we say that certain property is satisfied $\Delta-$almost everywhere ($\Delta - a.e.$) or for $\Delta-$almost all ($\Delta - a.a.$) points if the set of points where the property does not hold has $\Delta - measure$ equal to 0. Similarly we say $\nabla-$ almost everywhere ($\nabla - a.e.$) or for $\nabla-$almost all ($\nabla - a.a.$).

Let us also notice that the following relations

$$m_*(-A) = m^*(A) \ \ and \ \ m^*(-A) = m_*(A) \tag{0.4}$$

hold, where the outer measures on the right-hand sides are taken on the time scale $\mathbb{T}$, while on the left on $-\mathbb{T}$.

Our goal in this paper is to extend the formula (0.1) for the Lebesgue integrals $\int\limits_{[a,b)} f'_+(t)\, d\mu_{\Delta} \le f(b) - f(a)$ and $\int\limits_{(a,b]} f'_-(t)\, d\mu_{\nabla} \le f(b) - f(a)$, where derivatives $f'_+(t)$ and $f'_-(t)$ are understood a.e.. It requires new approach to differentiation on the time scale $\mathbb{T}$ by the use of a suitable version of the Vitali Lemma.

Before we formulate and prove our version of the the Vitali Lemma we need the following properties of outer measures m_* and m^*.

Proposition 1. (i) *For any* $A \subset D_R$ *we have* $m^*(A) \le m_*(A)$;

(ii) for any $A \subset D_L$ we have $m_*(A) \le m^*(A)$;
(iii) for any $A \subset D_L \cap D_R$ we have $m_*(A) = m^*(A)$.

Proof. We shall show just (i) since case (ii) follows from (i) by and (iii) is a simple conclusion from previous ones.

Fix $\varepsilon > 0$ and take $R_j = (a_j, b_j]$ such that

$$A \subset \bigcup_{j=1}^{\infty} R_j \quad and \quad \sum_{j=1}^{\infty} l(R_j) \le m_*(A) + \varepsilon.$$

Denote $N_1 = \{j : \ b_j \ is \ right - scattered\} \quad and \quad N_2 = \{j : \ b_j \ is \ right - dense\}$. For $j \in N_1$ take $P_j = [a_j, b_j)$, while for $j \in N_2$ choose $P_j = [a_j, c_j)$, where $c_j > b_j$ is such point in $\mathbb{T}$ that $c_j - b_j < \frac{\varepsilon}{2^j}$. Observe that

$$A \subset \bigcup_{j=1}^{\infty} P_j \quad and \quad \sum_{j=1}^{\infty} l(P_j) \le \sum_{j=1}^{\infty} l(R_j) + \varepsilon \le m_*(A) + 2\varepsilon.$$

Thus $m^*(A) \le m_*(A) + 2\varepsilon$ and since $\varepsilon > 0$ is arbitrary then $m^*(A) \le m_*(A)$, what ends the proof. □

Proposition 2. *For a sequence of intervals* $[a_i, b_i)$ *with* $a_i \in D_R$, $i = 1, 2, \ldots$ *and any* $A \subset T$ *the following relation*

$$m^*\left(A \cap \bigcup_{i=1}^{\infty} [a_i, b_i)\right) = m^*\left(A \cap \bigcup_{i=1}^{\infty} (a_i, b_i)\right) \tag{0.5}$$

holds. Similarly for intervals $(a_i, b_i]$ *with* $b_i \in D_L$, $i = 1, 2, \ldots$ *we have*

$$m_*\left(A \cap \bigcup_{i=1}^{\infty} (a_i, b_i]\right) = m_*\left(A \cap \bigcup_{i=1}^{\infty} (a_i, b_i)\right). \tag{0.6}$$

Proof. By (0.4) it is enough to show just (0.5). Notice that always

$$m^*\left(A \cap \bigcup_{i=1}^{\infty} [a_i, b_i)\right) \ge m^*\left(A \cap \bigcup_{i=1}^{\infty} (a_i, b_i)\right).$$

The equality is obvious if $m^*\left(A \cap \bigcup_{i=1}^{\infty} (a_i, b_i)\right) = \infty$. If $m^*\left(A \cap \bigcup_{i=1}^{\infty} (a_i, b_i)\right) < \infty$ then for every $\varepsilon > 0$ there exist $R_j \in \mathcal{F}_{0R}$, $j = 1, 2, \ldots$ such that

$$A \cap \bigcup_{i=1}^{\infty} (a_i, b_i) \subset \bigcup_{j=1}^{\infty} R_j$$

and

$$\sum_{j=1}^{\infty} l(R_j) < m^*\left(A \cap \bigcup_{i=1}^{\infty} (a_i, b_i)\right) + \varepsilon.$$

Choose $P_i = [a_i, c_i) \in \mathcal{F}_{0R}$, $i = 1, 2, ..$ such that $l(P_i) < \frac{\varepsilon}{2^i}$. Then

$$A \cap \bigcup_{i=1}^{\infty} [a_i, b_i) \subset \bigcup_{j=1}^{\infty} R_j \cup \bigcup_{i=1}^{\infty} P_i$$

and hence

$$m^*\left(A\cap\bigcup_{i=1}^{\infty}[a_i,b_i)\right)\leq\sum_{j=1}^{\infty}l\left(R_j\right)+\sum_{i=1}^{\infty}l\left(P_i\right)<m^*\left(A\cap\bigcup_{i=1}^{\infty}(a_i,b_i)\right)+2\varepsilon.$$

Now passing with $\varepsilon\to 0$ we have (0.5) $\square$

Consider families $\mathcal{F}_{0R}=\{P\in\mathcal{F}_R:\ l(P)>0\}$ and $\mathcal{F}_{0L}=\{P\in\mathcal{F}_L:\ l(P)>0\}$. Let us observe that $\mathcal{F}_{0R}$ covers in the sense of Vitali the set D_R, while $\mathcal{F}_{0L}$ covers the set D_L. Indeed. If $x\in D_R$ then there is an decreasing sequence $\{t_n\}\subset\mathbb{T}$ such that $\lim t_n=x$ and hence the intervals $[x,t_n)\in\mathcal{F}_{0R}$ have property $\lim l([x,t_n))=0$. Similarly we handle with $\mathcal{F}_{0L}$.

We shall show that the following analogues of the Vitali Lemma hold:

Lemma 1. *a) (Vitali Lemma for $\Delta-$measure): Let $E\subset D_R$ be a set with $m^*(E)<\infty$. Then for every $\varepsilon>0$ there exist pairwise disjoint intervals $P_1,...,P_N\in\mathcal{F}_{0R}$ such that*

$$m^*\left(E\backslash\bigcup_{n=1}^{N}P_n\right)<\varepsilon;$$

b) (Vitali Lemma for $\nabla-$measure): Let $E\subset D_L$ be a set with $m_*(E)<\infty$. Then for every $\varepsilon>0$ there exist pairwise disjoint intervals $P_1,...,P_N\in\mathcal{F}_{0L}$ such that

$$m_*\left(E\backslash\bigcup_{n=1}^{N}P_n\right)<\varepsilon.$$

Proof. We shall discuss only the case a) because the arguments used can easily be adopted to b). Since $m^*(E)<\infty$ then there exists $W=\bigcup_{i=1}^{\infty}R_i\supset E$ with $R_i\in\mathcal{F}_{0R}$ and $\sum_{i=1}^{\infty}l(R_i)<\infty$. Therefore $\mu_\Delta(W)=m^*(W)\leq\sum_{i=1}^{\infty}l(R_i)<\infty$. Denote by $\mathcal{F}_W=\{P\in\mathcal{F}_{0R}:\ P\subset W\}$ and observe that $\mathcal{F}_W$ still covers E in the sense of Vitali. We choose by induction a family of pairwise disjoint intervals $\{P_n\}\subset\mathcal{F}_W$ as follows: Let $P_1\in\mathcal{F}_W$ be any interval and assume that $P_1,...,P_n\in\mathcal{F}_W$ have already been chosen.

If $E\subset\bigcup_{i=1}^{n}P_i$ we are done.

If $E\backslash\bigcup_{i=1}^{n}P_i\neq\varnothing$ then for any $x\in E\backslash\bigcup_{i=1}^{n}P_i$ there exists an interval $P\in\mathcal{F}_W$ such that

$$P\cap\left(\bigcup_{i=1}^{n}P_i\right)=\varnothing.$$

Thus the number

$$k_n=\sup\left\{l(P):\ P\cap\left(\bigcup_{i=1}^{n}P_i\right)=\varnothing,\ P\in\mathcal{F}_W\right\}. \tag{0.7}$$

satisfies the conditions

$$0<k_n\leq m^*(W)<\infty.$$

Therefore one can find $P_{n+1} \in \mathcal{F}_W$ such that

$$P_{n+1} \cap \left(\bigcup_{i=1}^{n} P_i \right) = \varnothing$$

and

$$\frac{1}{2} k_n < l \left(P_{n+1} \right) \leq k_n .$$

Notice that

$$\frac{1}{2} \sum_{n=1}^{\infty} k_n \leq \sum_{n=1}^{\infty} \mu_\Delta \left(P_n \right) = \mu_\Delta \left(\bigcup_{n=1}^{\infty} P_n \right) \leq \mu_\Delta \left(W \right) < \infty .$$

Therefore

$$k_n \to 0$$

as well as

$$l \left(P_n \right) \to 0.$$

Let $P_n = [a_n, b_n)$. Denote by

$$c_n = \inf \left\{ t \in \mathbb{T} : \ t \geq 3a_n - 2b_n \right\} , \tag{0.8}$$

$$d_n = \sup \left\{ t \in \mathbb{T} : \ t \leq 3b_n - 2a_n \right\} \tag{0.9}$$

and observe that c_n, $d_n \in \mathbb{T}$ with

$$c_n \geq a_n \quad and \quad b_n \leq d_n .$$

Thus

$$J_n = [c_n, d_n) \supset P_n$$

and

$$l \left(J_n \right) \leq \left(3b_n - 2a_n \right) - \left(3a_n - 2b_n \right) = 5 l \left(P_n \right) .$$

Now fix $\varepsilon > 0$ and take N such that

$$\sum_{n=N+1}^{\infty} l \left(P_n \right) < \frac{\varepsilon}{5} .$$

Hence

$$\sum_{n=N+1}^{\infty} l \left(J_n \right) < \varepsilon . \tag{0.10}$$

Let

$$X = E \backslash \left(\bigcup_{n=1}^{N} P_n \right) .$$

To establish the Lemma we need to show that

$$m^* \left(X \right) < \varepsilon . \tag{0.11}$$

Take an arbitrary point $x \in X$. Since $x \notin \bigcup_{i=1}^{N} P_i$ we can find $P \in \mathcal{F}_W$ with $x \in P$ in such way that

$$P \cap \left(\bigcup_{i=1}^{N} P_i \right) = \varnothing . \tag{0.12}$$

Let us notice that

$$P \cap \left(\bigcup_{i=1}^{\infty} P_i \right) \neq \varnothing. \tag{0.13}$$

Indeed. If $P \cap \left(\bigcup_{i=1}^{\infty} P_i \right) = \varnothing$ then for every $n = 1, 2, ...$ we have $P \cap \left(\bigcup_{i=1}^{n} P_i \right) = \varnothing$ and therefore $0 < l(P) \leq k_n$, what is a contradiction with $k_n \to 0$. From (0.12) and (0.13) we conclude that

$$P \cap \left(\bigcup_{i=N+1}^{\infty} P_i \right) \neq \varnothing.$$

Thus there is $i \geq N + 1$ such that

$$P \cap P_i \neq \varnothing. \tag{0.14}$$

Take n, the smallest possible integer exceeding $N + 1$ and satisfying (0.14). One can easily notice that

$$P \cap \left(\bigcup_{i=1}^{n-1} P_i \right) = \varnothing.$$

Hence $0 < l(P) \leq k_{n-1} < 2l(P_n)$ and if $P = [a, b)$ so we have

$$b - a < 2(b_n - a_n). \tag{0.15}$$

We claim that $x \in J_n$. To see this take $z \in P \cap P_n$ and observe that

$$\left| x - \frac{a_n + b_n}{2} \right| \leq |x - z| + \left| z - \frac{a_n + b_n}{2} \right| < (b - a) + \frac{b_n - a_n}{2} < \frac{5(b_n - a_n)}{2}.$$

Thus $3a_n - 2b_n < x < 3b_n - 2a_n$ and therefore

$$c_n \leq x \leq d_n.$$

To conclude our claim it remains to show that $x \neq d_n$. Assume to a contrary that $x = d_n$. Therefore we have

$$b > x = d_n \geq b_n.$$

From the latter we see that $b > 3b_n - 2a_n$ since if $b \leq 3b_n - 2a_n$ then $b \leq d_n$. We also have $b_n > a$, since $a \leq z < b_n$. So finally

$$b - a > (3b_n - 2a_n) - b_n = 2(b_n - a_n),$$

what contradicts with (0.15). Therefore $x \in J_n \subset \bigcup_{i=N+1}^{\infty} J_i$. But $x \in X$ was arbitrarily chosen, so

$$X \subset \bigcup_{i=N+1}^{\infty} J_i$$

and hence by (0.10) we have $m^*(X) \leq \sum_{n=N+1}^{\infty} l(J_n) = \sum_{n=N+1}^{\infty} \mu_\Delta(J_n) < \varepsilon$, what completes the proof. □

Usually the Vitali Lemma is applied to problems concerning with the differentiation. The same is for the time scale $\mathbb{T}$, but in order to talk about derivatives of a function $f : \mathbb{T} \to R$ we have to distinguish between right-dense, left-dense, right-scattered and left-scattered points.

Let $x \in D_R$ be a right-dense point. Then quantities

$$D^+ f(x) = \limsup_{t \to x^+} \frac{f(t) - f(x)}{t - x}$$

and

$$D_+ f(x) = \liminf_{t \to x^+} \frac{f(t) - f(x)}{t - x}$$

we shall call, respectively, right upper and right lower derivative of f at x.

If x is right-scattered then we put $D^+ f(x) = D_+ f(x) = \frac{f(\sigma(x)) - f(x)}{\sigma(x) - x}$.

Similarly for $x \in D_L$ being a left-dense point we define

$$D^- f(x) = \limsup_{t \to x^-} \frac{f(t) - f(x)}{t - x}$$

and

$$D_- f(x) = \liminf_{t \to x^-} \frac{f(t) - f(x)}{t - x}$$

and if x is left-scattered then we put $D^- f(x) = D_- f(x) = \frac{f(\rho(x)) - f(x)}{\rho(x) - x}$.

One can easily observe that $D^+ f(x) \geq D_+ f(x)$ and $D^- f(x) \geq D_- f(x)$.

We say that f is right differentiable at x if $D^+ f(x) = D_+ f(x) \neq \pm\infty$. In this case that common value we shall call the right derivative of f at x and denote it by $f'_+(x)$. Analogously $f'_-(x) = D^- f(x) = D_- f(x)$ we shall call left derivative of f at x.

Finally, f is differentiable at x if $f'_+(x) = f'_-(x)$ and that common value we shall denote by $f'(x)$ and call it the derivative of f at x.

The derivatives just introduces have similar properties as usual ones. We just need to notice that for $g(x) = -f(-x)$ defined on the time scale $-\mathbb{T}$ we have

$$D^- g(x) = -D_+ f(-x) \quad \textit{and} \quad D_- g(x) = -D^+ f(-x)$$

and

$$g'_+(x) = -f'_-(-x) \quad \textit{and} \quad g'_+(x) = -f'_-(-x)\,. \tag{0.16}$$

In what follows we shall develope the theory of differentiation distinguishing right and left derivatives. We begin with the following observation:

Theorem 1. *Let $f : \mathbb{T} \to R$ be an increasing function. Then:*

(*i*) f is right differentiable $\Delta - a.e.$ on D_R;
(*ii*) f is left differentiable $\nabla - a.e.$ on D_L;
(*iii*) f is differentiable $\Delta - a.e.$ and $\nabla - a.e.$ on $D_L \cap D_R$.

The derivatives f'_+ and f'_- are measurable, nonnegative and for every $a, b \in \mathbb{T} \setminus \{\inf \mathbb{T}, \sup \mathbb{T}\}$ we have

$$\int_{[a,b)} f'_+(t)\, d\mu_\Delta \leq f(b) - f(a) \quad \textit{and} \quad \int_{(a,b]} f'_-(t)\, d\mu_\nabla \leq f(b) - f(a)\,. \tag{0.17}$$

Proof. We shall demonstrate just the case (iii), since (i) is similar, while (ii) can be concluded from (i) for $g(x) = -f(-x)$ defined on $-\mathbb{T}$. In that case we need to show that the set where both right and left derivatives are unequal has both Δ and $\bigtriangledown$ measures equal to 0, i.e.

$$m^*(E) = m_*(E) = 0. \tag{0.18}$$

We may consider only the set

$$E = \left\{x \in D_R \cap D_L : \ f'_+(x) > f'_-(x)\right\}$$

since the set arising from the opposite inequality can be similarly handled. Observe that E can be represented as the union of the sets

$$E_{n,\alpha,\beta} = \left\{x \in D_R \cap D_L \cap (m_n, M_n) : \ \ f'_+(x) > \alpha > \beta > f'_-(x)\right\}.$$

for all rationals $\alpha > \beta > 0$ and $n = 1, 2, ...$, where

$$m_n = \inf\{t \in \mathbb{T} : \ t \geq -n\} \ \ \textit{and} \ \ M_n = \sup\{t \in \mathbb{T} : \ t < n\}.$$

Moreover from Proposition 1 we know that $m^*(E_{n,\alpha,\beta}) = m_*(E_{n,\alpha,\beta})$. Denote that common value by

$$s = m^*(E_{n,\alpha,\beta}) = m_*(E_{n,\alpha,\beta}).$$

Hence to see (0.18) it is sufficient to check that

$$s = 0. \tag{0.19}$$

Pick arbitrary $\varepsilon > 0$ and choose $W = \bigcup_{i=1}^{\infty} R_i \supset E_{n,\alpha,\beta}$ with $R_i \in \mathcal{F}_{0L}$ and $\sum_{i=1}^{\infty} l(R_i) < s + \varepsilon$. Observe that for each point $x \in E_{n,\alpha,\beta}$ there is an arbitrarily small interval $P = (t, x] \subset W$ such that

$$f(x) - f(t) < \beta(x - t) = \beta l(P).$$

Thus the family

$$\mathcal{F}_W = \{P = (a, b] \in \mathcal{F}_{0L} : \ P \subset W, \ \ b \in D_L \ \ \textit{and} \ \ f(b) - f(a) < \beta l(P)\}$$

covers $E_{n,\alpha,\beta}$ in the sense of Vitali. Hence by *the Lemma* we can choose a finite collection $\{P_1, ...P_N\} \subset \mathcal{F}_W$ of pairwise disjoint intervals such that

$$m_*\left(E_{n,\alpha,\beta} \setminus \bigcup_{i=1}^{N} P_i\right) < \varepsilon.$$

Let $P_i = (a_i, b_i]$ and denote by $A = E_{n,\alpha,\beta} \cap \left(\bigcup_{i=1}^{N} (a_i, b_i)\right) \subset D_L \cap D_R$. Then by Proposition 1 and (0.6) we have

$$m^*(A) = m_*(A) > s - \varepsilon.$$

By the construction for each $i \in \{1, ..., N\}$ we have

$$f(b_i) - f(a_i) < \beta(b_i - a_i) = \beta l(P_i).$$

Summing up these inequalities we obtain

$$\sum_{i=1}^{N} (f(b_i) - f(a_i)) < \sum_{i=1}^{N} \beta l(P_i) < \beta(s + \varepsilon).$$

Now each point $z \in A \subset E_{n,\alpha,\beta}$ admits and arbitrarily small interval $I = [z, s) \subset (a_i, b_i)$, for some $i \in \{1, ..., N\}$, such that

$$f(s) - f(z) > \alpha(s - z) = \alpha l(I).$$

Therefore the family

$$\mathcal{F}_1 = \left\{ \begin{array}{c} I = [a, b) \in \mathcal{F}_W : \ I \subset (a_i, b_i) \ for\ some\ i \in \{1, ..., N\} \\ and\ f(b) - f(a) > \alpha l(I) \end{array} \right\}$$

covers A in the sense of Vitali. Now applying *the Lemma* again we can choose a finite collection $\{I_1, ...I_M\} \subset \mathcal{F}_1$ of pairwise disjoint intervals such that

$$m^*\left(A \backslash \bigcup_{k=1}^{M} I_k\right) < \varepsilon.$$

Let $I_k = [c_k, d_k)$ and denote by $B = A \cap \left(\bigcup_{k=1}^{M} I_k\right)$. Then

$$m^*(B) > m^*(A) - \varepsilon > s - 2\varepsilon.$$

and for each $k \in \{1, ..., M\}$ we have

$$f(d_k) - f(c_k) > \alpha(d_k - c_k) = \alpha l(I_k).$$

Hence

$$\sum_{k=1}^{M}(f(d_k) - f(c_k)) > \alpha\left(\sum_{k=1}^{M} l(I_k)\right) > \alpha m^*(B) > \alpha(s - 2\varepsilon).$$

But each interval I_k is contained in some interval P_i. Denote, for given i, by $N(i) = \{k : I_k \subset P_i\} = \{k_1, ..., k_m\}$, where the numeration is taken in such way that

$$a_i \leq c_{k_1} < d_{k_1} \leq c_{k_2} < d_{k_2} \leq ... \leq c_{k_m} < d_{k_m} \leq b_i.$$

Thus we obtain

$$\sum_{k \in N(i)}(f(d_k) - f(c_k)) =$$

$$= f(d_{k_m}) - f(c_{k_1}) + \sum_{r=1}^{m-1}\left(f\left(c_{k_{r+1}}\right) - f(d_{k_r})\right) \leq f(b_i) - f(a_i),$$

since f is increasing. Hence

$$\sum_{k=1}^{M}(f(d_k) - f(c_k)) = \sum_{i=1}^{N}\left(\sum_{k \in N(i)}(f(d_k) - f(c_k))\right) \leq \sum_{i=1}^{N}(f(b_i) - f(a_i))$$

and so

$$\alpha(s - 2\varepsilon) < \beta(s + \varepsilon).$$

Recall that $\varepsilon > 0$ was arbitrarily chosen, so we have $\beta s \geq \alpha s$. But $\beta < \alpha$, hence the latter inequality forces s to be 0. To show (0.17) it is enough to prove that

$$\int_{[a,b)} f'_+(x)\, d\mu_\Delta \leq f(b) - f(a) \tag{0.20}$$

since the second relation follows from (0.16) replacing f by $h(t) = -f(-t)$. Let us make an observation that $(i) - (iii)$ give the existence of sets $D_R^0 \subset D_R$ and $D^0 \subset D_R \cap D_L$ with

$$m^*\left(D_R \backslash D_R^0\right) = 0 \ \ and \ \ m^*\left(D_R \cap D_L \backslash D^0\right) = 0 \tag{0.21}$$

and such that the limits

$$\lim_{t\to x^+} \frac{f(t) - f(x)}{t - x} \quad and \quad \lim_{t\to x} \frac{f(t) - f(x)}{t - x}$$

exist (finite or not), respectively, for $x \in D_R^0$ and for $x \in D^0$ (for right-scattered points by definition). Denote by

$$D = \left[D_R \backslash D_R^0 \cup D_R \cap D_L \backslash D^0\right]' = D_R' \cup \left[D_R^0 \backslash D_L\right] \cup D_R^0 \cap D_L$$

and observe that by (0.21)

$$m^*(D') = 0.$$

Consider

$$g(x) = \begin{cases} \lim\limits_{t\to x^+} \frac{f(t)-f(x)}{t-x} & for \ \ x \in D_R^0 \\ +\infty & for \ \ x \in D_R \backslash D_R^0 \\ \frac{f(\sigma(x))-f(x)}{\sigma(x)-x} & for \ \ x \in T \backslash D_R \end{cases}.$$

Observe that f is right-differentiable whenever for $x \in D$ the function $g(x)$ is finite and then $g(x) = f_+'(x)$. We shall construct a sequence of nonnegative simple functions $g_n : [a, b) \to R$ with the following properties:

$a)$ g_n tends for $x \in [a, b) \cap D$ to $g(x)$;

$b)$ $\displaystyle\int_{[a,b)} g_n(x)\, d\mu_\Delta \leq f(b) - f(a)$.

The construction of such functions goes as follows. Take $\delta = \frac{b-a}{n}$ and consider a partition $\mathcal{P}_n$ of $[a, b)$ given by $a = t_0 < t_1 < ...t_N = b$ such that for each $i \in \{1, 2, ..., N\}$ either

$$t_i - t_{i-1} \leq \delta$$

or

$$t_i - t_{i-1} > \delta \quad and \quad \varrho(t_{i-1}) = t_i.$$

Such a partition $\mathcal{P}_n$ exists by Lemma 5.7 in Bohner &Peterson [3]. Define $g_n(x) = \frac{f(t_i)-f(t_{i-1})}{t_i - t_{i-1}}$ for $x \in [t_{i-1}, t_i)$. We shall show that $g_n(x)$ tends to $g(x)$ for $x \in D$. Denote by $\left[t_{i-1}^n(x), t_i^n(x)\right) \in \mathcal{P}_n$ the subinterval containing point x. We have to consider the following cases:

1. $x \in (\mathbb{T} \backslash D_R)$ (x is right-scattered). Then for sufficiently large n we have $\sigma(x) - x > \delta = \frac{b-a}{n}$. Therefore x and $\sigma(x)$ have to be consequtive division points of $\mathcal{P}_n$. But then $g_n(x) = \frac{f(\sigma(x))-f(x)}{\sigma(x)-x} = g(x)$;

2. $x \in D_R^0 \backslash D_L$ (x is left scattered and right dense). Then for sufficiently large n we have $x - \varrho(x) > \delta = \frac{b-a}{n}$ and therefore $t_{i-1}^n(x) = x$ together with $t_i^n(x) \to x^+$. But then

$$g_n(x) = \frac{f(t_i^n) - f(x)}{t_i^n - x} \to g(x).$$

3. $x \in D_R^0 \cap D_L$ (x is left and right dense) Then $\lim\limits_{t\to x} \frac{f(t)-f(x)}{t-x} = g(x)$ and for each n we have $t_i^n(x) - t_{i-1}^n(x) < \delta = \frac{b-a}{n}$. Hence $t_{i-1}^n(x) \to x^-$ and $t_i^n(x) \to x^+$. But

then

$$\frac{f\left(t_{i-1}^{n}(x)\right)-f(x)}{t_{i-1}^{n}(x)-x}\rightarrow g(x)\quad \textit{and}\quad \frac{f\left(t_{i}^{n}(x)\right)-f(x)}{t_{i}^{n}(x)-x}\rightarrow g(x).$$

We claim that

$$\frac{f\left[t_{i}^{n}(x)\right]-f\left[t_{i-1}^{n}(x)\right]}{t_{i}^{n}(x)-t_{i-1}^{n}(x)}\rightarrow g(x).$$

Indeed.

a) *$g(x)$ is finite.* Then for every $\varepsilon>0$ and sufficietly large n we have

$$\left[g(x)-\varepsilon\right]\left[x-t_{i-1}^{n}(x)\right]\leq f(x)-f\left[t_{i-1}^{n}(x)\right]\leq\left[g(x)+\varepsilon\right]\left[x-t_{i-1}^{n}(x)\right]$$

and

$$\left[g(x)-\varepsilon\right]\left[t_{i}^{n}(x)-x\right]\leq f\left[t_{i}^{n}(x)\right]-f(x)\leq\left[g(x)+\varepsilon\right]\left[t_{i}^{n}(x)-x\right].$$

Adding those inequalities we obtain

$$\left[g(x)-\varepsilon\right]\left[t_{i}^{n}(x)-t_{i-1}^{n}(x)\right]\leq f\left[t_{i}^{n}(x)\right]-f\left[t_{i-1}^{n}(x)\right]\leq\left[g(x)+\varepsilon\right]\left[t_{i}^{n}(x)-t_{i-1}^{n}(x)\right]$$

or equivalently

$$g(x)-\varepsilon\leq\frac{f\left[t_{i}^{n}(x)\right]-f\left[t_{i-1}^{n}(x)\right]}{t_{i}^{n}(x)-t_{i-1}^{n}(x)}\leq g(x)+\varepsilon.$$

Thus

$$g(x)-\varepsilon\leq g_{n}(x)\leq g(x)+\varepsilon$$

and the latter shows that $g_{n}(x)\rightarrow g(x)$.

b) *$g(x)$ is infinite*, say $+\infty$. (for $-\infty$ we proceed similarly). Then for any $K>0$ and sufficietly large n we have

$$K\left[x-t_{i-1}^{n}(x)\right]\leq f(x)-f\left[t_{i-1}^{n}(x)\right]$$

and

$$K\left[t_{i}^{n}(x)-x\right]\leq f\left[t_{i}^{n}(x)\right]-f(x).$$

Adding again we obtain

$$K\leq\frac{f\left[t_{i}^{n}(x)\right]-f\left[t_{i-1}^{n}(x)\right]}{t_{i}^{n}(x)-t_{i-1}^{n}(x)}.$$

Thus

$$\liminf_{n\rightarrow\infty}\frac{f\left[t_{i}^{n}(x)\right]-f\left[t_{i-1}^{n}(x)\right]}{t_{i}^{n}(x)-t_{i-1}^{n}(x)}\geq K,$$

and this shows that also in this case $g_{n}(x)\rightarrow g(x)$.

Having constructed such functions from Fatou Lemma we conclude that

$$\int\limits_{[a,b)}g(t)\,d\mu_{\Delta}\leq\liminf_{n\rightarrow\infty}\int\limits_{[a,b)}g_{n}(t)\,d\mu_{\Delta}.$$

But

$$\int\limits_{[a,b)}g_{n}(t)\,d\mu_{\Delta}=\sum_{i=1}^{N}\int\limits_{[t_{i-1},t_{i})}\frac{f(t_{i})-f(t_{i-1})}{t_{i}-t_{i-1}}d\mu_{\Delta}=\sum_{i=1}^{N}\left[f(t_{i})-f(t_{i-1})\right]=f(b)-f(a)$$

and therefore

$$\int\limits_{[a,b)}g(t)\,d\mu_{\Delta}\leq f(b)-f(a).$$

Since $g(t) \geq 0$ for $\Delta - a.a.$ points in $[a, b)$, the latter shows what g is $\Delta - integrable$ and hence finite $\Delta - a.e.$ in $[a, b)$. Thus f is right differentiable a.e. in $[a, b)$ and $g = f'_+$ a.e. in $[a, b)$, what completes the proof. $\square$

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